# 18.155 Problem Set 1

## Due Wednesday, 9/17/08

Remarks and Hints: We moved the due date to Wednesday because we have not yet covered the Fourier transform. You can do Problem 1 and 3b already. We will carry out a special case of Problem 2 in lecture on Monday, in order to demonstrate that if $x_j u = 0$ for all $j$, then $u = c\delta$. Problem 3a requires the definition of a tempered distribution only. Problems 4–5 require you in places to use the ideas or statements from Problem 2. As a warm-up to problems 3, 4, 5, show that if $u$ is a distribution satisfying $(d/dx)u = 0$, then $u$ is constant. Problem 6 requires the definition of the Fourier transform on tempered distributions, covered in lecture on Mon, Sept 15.

**1.** a) Define

$$v_t(x) = \begin{cases} te^{itx}, & \text{if } x > 0; \\ 0, & \text{if } x \leq 0. \end{cases} \tag{1}$$

Evaluate $\lim_{t\to\infty} v_t$ in the sense of distributions directly (without using part (b)).

b) Define

$$w_t(x) = \begin{cases} e^{itx}, & \text{if } x > 0; \\ 0, & \text{if } x \leq 0. \end{cases} \tag{2}$$

Evaluate $\lim_{t\to\infty} w_t$ in the sense of distributions.

c) Find the distributional derivative $(d/dx)w_t$ and use parts (a) and (b) to confirm in this case that the derivative of the limit is the limit of the derivative.

**2.** a) Prove Taylor's formula in the form

$$f(1) = \sum_{k=0}^{N} f^{(k)}(0)/k! + R_N$$

where

$$R_N = \int_0^1 \frac{f^{N+1}(t)}{(N+1)!}\rho_N(t)dt \qquad \big(\text{where } \rho_N(t) = (N+1)(1-t)^N\big)$$

Note that the weight function $\rho_N$ has integral 1 on $[0,1]$, so the last formula shows that the remainder is an average of the next term in the series over that interval.

b) Recall the multinomial notation

$$\binom{m}{\alpha} = \frac{m!}{\alpha!} = \frac{(m)!}{\alpha_1!\alpha_2!\cdots\alpha_n!}$$

and the multinomial formula

$$(x_1 + \cdots + x_n)^m = \sum_{|\alpha|=m} \binom{m}{\alpha} x^\alpha$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$.

By considering $f(t) = g(tx)$, deduce Taylor's formula for functions of several variables

$$g(x) = \sum_{|\alpha|\leq N} \frac{\partial^\alpha g(0)x^\alpha}{\alpha!} + R_N(x)$$

where

$$R_N(x) = \int_0^1 \sum_{|\alpha|=N+1} \frac{\partial^\alpha g(tx) x^\alpha}{\alpha!} \rho_N(t)dt$$

As in part (a), the error term is an average of the next term in the expansion over the segment $tx$, $0 \le t \le 1$.

c) Deduce that if $g$ is $C^\infty$ and satisfies $\partial^\alpha g(0) = 0$ for all $|\alpha| \le N$, then there exist $C^\infty$ functions $g_\beta$ such that

$$g(x) = \sum_{|\beta|=N+1} x^\beta g_\beta(x)$$

d) Suppose that $g$ is a $C^\infty$ function in the unit ball $B_1$ satisfying the hypotheses of part (c). Show that $g$ is approximated in $C^N$ norm on $B_{1/2}$ by smooth functions that vanish in a neighborhood of the origin.

e) Suppose that $u$ is linear functional defined for every $f \in C_0^\infty(B_1)$, and that there exist $C < \infty$ and an integer $m$ such that

$$|u(f)| \le C \sup_{|\alpha| \le m} |\partial^\alpha f(x)|$$

Suppose further that if $f$ vanishes in a neighborhood of the origin, then $u(f) = 0$. (In short, $u$ is a distribution supported at the origin.)

Show that $u$ is a finite linear combination of derivatives of the delta function at the origin.

**3.** a) Show that the following defines a tempered distribution (known as the principle value of $1/x$, or p.v. $1/x$):

$$u(\phi) = \lim_{N\to\infty,\ \epsilon\to 0^+} \int_{\epsilon<|x|<N} \frac{\phi(x)}{x} dx$$

(Hint: subtract $\phi(0)$ from the numerator.)

b) Show that the distributional derivative $(d/dx) \log |x| = u$

**4.** (Hörmander, Exercise 3.1.20) Determine all distributions $u$ on $\mathbf{R}$, that is, all $u \in \mathcal{D}'(\mathbf{R})$ satisfying the following equations

a) $xu' + u = 0$; b) $(e^{2\pi i x} - 1)u = 0$; c) $x^2 u' + xu = \delta$

**5.** (Hörmander, Exercise 3.1.22) Determine all distributions on $\mathbf{R}^2$ satisfying both $(x_1^2 - x_2^2)u = 0$ and $x_1 x_2 u = 0$.

**6.** Compute the Fourier transforms of these tempered distributions on $\mathbf{R}$ and find the relationships among them and their Fourier transforms.

a) $\delta$, the Dirac delta function at the origin.

b) the principal value, p. v. $1/x$ (defined above)

c) $1/(x \pm 0i) = \lim_{t\to 0^+} 1/(x \pm it)$

d) $T_a(\phi) = \int_{-a}^{a} \frac{\phi(x) - \phi(0)}{|x|} dx + \int_{|x|>a} \frac{\phi(x)}{|x} dx$

e) $\log |x|$.

f) $\lim_{a\to -1} (x_\pm^a - c_a \delta)$, $a > -1$

(Find $c_a$ so that this distributional limit is well-defined. Notation: $x_+ = x$ for $x \ge 0$ and $x_+ = 0$ for $x < 0$; $x_- = -x$ for $x \le 0$ and $x_- = 0$ for $x > 0$.)