18.103 Fall 2013

Problem Set 8 (due 10am Fri, Nov 8)

1. Let $f \in L^1(\mathbb{R}/2\pi\mathbb{Z})$, and let σ_N denote the Cesaro mean of its Fourier series. Prove that if f has a left and right limit at x, then

$$\sigma_N(x) \to (f(x^+) + f(x^-))/2 \quad \text{as} \quad N \to \infty$$

(You may use the formula from lecture for F_N such that $\sigma_N(x) = f * F_N(x)$.)
Hint: Formulate and prove a variant of the "approximate identity" lemma, with stronger hypotheses on K_N in exchange for weaker properties of f, and confirm the stronger properties of F_N that you need.

2. Consider the Fourier series for f from 2a PS7 at $x = 0$ and $x = \pi$; g from 2b at $x = 0$; h from 2c at $x = \pi/2$. What are the consequences of the theorems in problems 3 PS7 and problem 1 above at these points?

3. Let R_N denote the 2^N dyadic intervals of $[0, 1)$ of length 2^{-N}, that is,

$$R_N = \{I = [(k-1)/2^N, k/2^N) : \ k = 1, 2, \dots, 2^N\}$$

Consider

$$V_N = \text{span } \{1_I : I \in R_N\}$$

Let $P_N : L^2([0, 1]) \to V_N$ be the orthogonal projection onto V_N, that is, the mapping such that $P_N f = f$ for all $f \in V_N$ and $P_N f \perp (f - P_N f)$ for all $f \in L^2([0, 1])$.

a) Find the formula for a_I (in terms of I and f) such that

$$P_N f = \sum_{I \in R_N} a_I 1_I$$

and show that $P_N f$ tends uniformly (on $[0, 1)$) to f for all $f \in C([0, 1])$.

b) Let $1 \leq p < \infty$. Show that $P_N f$ tends to f in $L^p([0, 1])$ for every $f \in L^p([0, 1])$.

c) For $f \in L^1([0, 1])$, find the formula for $P_0 f$ and $P_{N+1} f - P_N f$ in terms of $\langle f, H_{n,k} \rangle$ and $H_{n,k}$, the Haar functions defined in AG §3.3/11, pp. 136–137. *Warning: identify the misprint in part (a) p. 137.* Deduce that the Haar functions form a complete orthonormal system of $L^2([0, 1])$.

4. a) Do AG §3.3/9, p. 136 (Gram-Schmidt process).

b) Use power series to show that every function e^{inx} can be uniformly approximated on $[-\pi, \pi]$ by polynomials (ordinary polynomials in x).

c) Deduce from (b) that polynomials are dense in $L^2([-\pi, \pi])$.

d) Denote by ψ_0, ψ_1, $\ldots$, the functions obtained from the Gram-Schmidt process applied to the polynomials $f_0(x) = 1$, $f_1(x) = x$, $f_2(x) = x^2$, $\ldots$. Show that these form an orthonormal basis of $L^2([-\pi, \pi])$ and compute the first three. (The answers on $[-1, 1]$ are listed in AG §3.3/10 p. 136.)

Show further that the degree of ψ_n is n and that ψ_n is even if n is even and odd if n is odd.

e) Show by integration by parts that

$$R_n(x) = \frac{d^n}{dx^n}(x^2 - 1)^n$$

is orthogonal to 1, x, $\ldots$, x^{n-1} in $L^2([-1, 1])$ and $R_n(1) = 2^n n!$. (Hint: $x^2 - 1 = (x-1)(x+1)$.)

f) The Legendre polynomials are defined as the polynomials $P_n(x) = R_n(x)/2^n n!$.

In other words, they are normalized[1] so that $P_n(1) = 1$. Show how your formulas for ψ_n, $n = 0$, 1, 2 in (c) match this formula for P_n.

5. Define the Laplace operator $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ on the (x, y)-plane.

a) Show that in polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$),

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

b) Let $f \in C(\mathbb{R}/2\pi\mathbb{Z})$. Define u in polar coordinates by

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) e^{in\theta}, \quad 0 \le r < 1$$

Express u as a series in $z = x + iy$ and $\bar{z} = x - iy$. Confirm that u is infinitely differentiable in $x^2 + y^2 < 1$ and that $\Delta u = 0$ for $0 \le r < 1$. Solutions to $\Delta u = 0$ are known as harmonic functions.

[1] The functions φ_n of AG §3.3/10 p. 136 indexed starting from $n = 1$ and with the normalization that the L^2 norm on $[-1, 1]$ is 1 differ from the customary notation for Legendre polynomials P_n. Further properties (not assigned) are as follows.

$$\sum_{n=0}^{\infty} P_n(x) z^n = \frac{1}{\sqrt{1 - 2xz + z^2}} \quad \text{(generating function)}$$

Recurrence formula and L^2 norm:

$$(n-1)P_n(x) = (2n-1)xP_{n-1}(x) - nP_{n-2}; \qquad \int_{-1}^{1} P_n(x)^2 dx = 2/(2n+1).$$

Remark. One should think of $f(\theta)$ as a function on the unit circle $\{e^{i\theta} : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$ in the complex plane and u is a function of $z = re^{i\theta}$ in the unit disk. Then u is the harmonic function with boundary values f, as we now prove.

c) Compute the Poisson kernel P_r satisfying

$$u(r, \theta) = f * P_r(\theta)$$

Prove that if $f \in C(\mathbb{R}/2\pi\mathbb{Z})$, then

$$\max_{\theta} |u(r, \theta) - f(\theta)| \to 0 \quad \text{as } r \to 1^-$$

If $f \in L^1(\mathbb{R}/2\pi\mathbb{Z})$, then

$$\lim_{r \to 1^-} \int_{[-\pi,\pi]} |f * P_r(\theta) - f(\theta)| \, d\theta = 0$$

d) (Extra credit) Prove that if f is continuous, then u extends to a continuous function on the closed unit disk.[2] In other words,

$$u(r_j, \theta_j) \to f(\theta)$$

whenever $r_j \to 1^-$ and $\theta_j \to \theta$.

[2] Given that u is continuous in the closed disk, one can prove that u is unique using what is known as the maximum principle. The maximum principle (for the disk) says that if $v(z)$ is real-valued and continuous in $|z| \leq 1$ and harmonic in $|z| < 1$, then

$$\max_{|z| \leq 1} v(z) \leq \max_{|z|=1} v(z)$$

Let v be $\pm$ the difference of any two real-valued harmonic functions with the same boundary values, then by the maximum principle, $v = 0$ and the two functions are the same. Using uniqueness for continuous boundary values, one can deduce uniqueness of u with boundary values in the L^1 sense stated above.