18.103 Fall 2013

Problem Set 11 (due 10am Fri, Dec 6)

1. Consider $f : \mathbb{R} \to \mathbb{R}$ monotone increasing, i. e. $x \le y \implies f(x) \le f(y)$.
For $-\infty < a < b < \infty$, define

$$\nu_f((a,b)) = \lim_{\epsilon \to 0^+} (f(b-\epsilon) - f(a+\epsilon))$$

The same proof as in the construction of Lebesgue measure shows that ν_f is a measure on the ring of finite unions of intervals and extends to a measure on Borel sets. ν_f is sometimes denoted df. (Thus, in the special case $f(x) = x$, we get Lebesgue measure $\nu_x = dx$.)

a) Show that, conversely, for any measure μ on $\mathbb{R}$ that assigns a finite (nonnegative) number to each compact set there is a *left continuous* function f such that $\mu = \nu_f$. Left continuous means

$$\lim_{x \to a^-} f(x) = f(a)$$

and $x \to a^-$ means $x \to a$ with $x < a$.

b) Suppose that f is monotone increasing. Show that f is continuous at all but (at most) countably many points x_j. Find a continuous monotone increasing function g such that

$$\nu_f = \nu_g + \sum c_j \delta(x - x_j), \quad c_j = f(x_j^+) - f(x_j^-)$$

The sum of delta functions is called the pure point or atomic part of the measure ν_f, and ν_g is the continuous part.

c) Consider the Cantor function defined for each sequence $a_k \in \{0,1\}$ by

$$C(x) = \sum_{k=1}^{N-1} a_k 2^{-k} + 2^{-N}, \quad 3^{-N} \le x - 2\sum_{k=1}^{N-1} a_k 3^{-k} \le 2 \cdot 3^{-N}$$

Show that C can be extended uniquely to a continuous monotone increasing function on $\mathbb{R}$ satisfying $C(x) = 0$ for all $x \le 0$ and $C(x) = 1$ for all $x \ge 1$. Denote the corresponding Cantor measure by $dC = \mu_C$. Show that μ_C is supported on the Cantor set. In other words $\mu_C(E) = 0$ for any $E \subset \mathbb{R}$ such that E is disjoint from the standard middle third Cantor set.

d) Let

$$\mu_1 * \mu_2 * \cdots * \mu_n; \quad \text{with} \;\; \mu_k = (1/2)[\delta(x) + \delta(x - 2/3^k)]$$

From part (a) there is a monotone, left continuous function C_n such that $C_n(x) = 0$ for $x \le 0$ and $dC_n = \mu_1 * \cdots * \mu_n$. Show that

$$\lim_{n \to \infty} C_n(x) = C(x)$$

e) Deduce that dC_n tends weakly to μ_C and establish the Fourier transform formula

$$\hat{\mu}_C(y) = e^{ay} \prod_{k=1}^{\infty} \cos(y/3^k).$$

In the process identify the complex number a and show that the infinite product converges. Moreover, by considering the values at $y = 2\pi 3^n$, show that $\hat{\mu}_C(y)$ does not tend to zero as $y \to \infty$. In other words, we have constructed a continuous measure whose Fourier transform does not tend to zero at infinity.

2. We say a function $f : \mathbb{R} \to \mathbb{R}$ has *bounded variation* or f is a BV function, if there is $M < \infty$ for which

$$\sum_{k=1}^{N} |f(x_k) - f(x_{k-1})| \leq M$$

for every sequence $x_0 < x_1 < x_2 < \cdots < x_N$ and every N.

a) Show that a function f of bounded variation on $\mathbb{R}$ is continuous except at countably many points, and identify a left continuous function g that agrees with f at all but countably many points. Give an example showing that g may have smaller total variation than f.

b) Show that a left continuous function g of bounded variation can be written

$$g(x) = h_1(x) - h_2(x)$$

with h_1 and h_2 left continuous, monotone increasing, and bounded. Hint: For $x > 0$, define

$$g_+(x) = \sup\{\sum_{j=1}^{n} [g(x_j) - g(x_{j-1})]^+ : 0 < x_0 < x_1 < \cdots x_n \leq x\};$$

$$g_-(x) = \sup\{\sum_{j=1}^{n} [g(x_j) - g(x_{j-1})]^- : 0 < x_0 < x_1 < \cdots x_n \leq x\}$$

where $a^+ = \max(a, 0)$ and $a^- = \max(-a, 0)$.

c) Consider g as in part (b). Show that for every $\varphi \in \mathcal{S}$,

$$-\int_{\mathbb{R}} \varphi'(x) g(x)\, dx = \int_{R} \varphi\, dh_1 - \int_{\mathbb{R}} \varphi\, dh_2$$

This says, by definition, that the generalized derivative of g is $dh_1 - dh_2$.

d) The procedure in parts (b) and (c) works for all functions of bounded variation, not just ones that are left continuous. Write down explicit monotone increasing functions h_1 and h_2 such that $f(x) = h_1(x) - h_2(x)$ for

$$f(x) = \begin{cases} 0 & x < 0 \\ 2 & x = 0 \\ 1 & x > 0 \end{cases}$$

and calculate the generalized derivative f'. Alternatively, find g left continuous that agrees with f except at one point and apply (b) and (c) to g. Compare the generalized derivative g' with f'. Are they the same?

3. (Shannon sampling theorem; Stein-Shakarchi Problem 20) A function f is called *band-limited* if its Fourier transform is supported on an interval of length L. We show that band-limited functions whose frequencies come from an interval of length L can be recovered from values spaced by $2\pi/L$. By dilation and translation we may assume that $L = 2\pi$ and center the interval of frequencies around 0.

a) Let $f \in L^2(\mathbb{R})$ be such that $\hat{f}$ is supported on $[-\pi, \pi]$. Show that f is continuous. (More precisely, f has a continuous representative which we will use from now on.)

b) Show that f from part (a) satisfies

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \sum_{-\infty}^{\infty} |f(n)|^2$$

and

$$\hat{f}(\xi) = 1_{[-\pi,\pi]}(\xi) \sum_{n=-\infty}^{\infty} f(n) e^{-in\xi}$$

in the sense of $L^2(\mathbb{R})$-norm convergence.

c) Show that

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) K_1(x-n)$$

for K_1 from optional Problem 5 below. In the course of the proof, explain why the series converges for every x. (Warning: You may carry out the computation formally first. But when you justify the appropriate exchange of limits, remember that you only have norm convergence in L^2.)

d) One can also recover f from more densely spaced samples. Show that

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{\lambda} f\left(\frac{x}{\lambda}\right) K_\lambda\left(x - \frac{n}{\lambda}\right),$$

with K_λ from Problem 5. Note that $K_\lambda(y) = O(y^{-2})$ so that this series converges faster than the one in (c).

4. In this problem we deduce the Fourier series/inversion formula from its discrete analogue on $\mathbb{Z}/N\mathbb{Z}$.

a) Consider $f \in C(\mathbb{R}/2\pi\mathbb{Z})$ and $x_N(j) = 2\pi j/N$. Define

$$c_N(n) = \frac{1}{N} \sum_{j=1}^{N} f(x_N(j)) e^{-ix_N(j)n}.$$

Show that for any integer M,

$$f(x_N(j)) = \sum_{n=-M}^{N-M-1} c_N(n) e^{i x_N(j) n}. \tag{1}$$

(Later, we'll use the case $M = N/2$ if N is odd or $M = (N+1)/2$ if N is even.)

b) Show that for continuous f,

$$\lim_{N\to\infty} c_N(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx}\, dx$$

c) Prove directly the following elementary version of the dominated convergence theorem.

$$\text{If } \ |a_N(n)| \le g(n), \qquad \sum_{n=-\infty}^{\infty} g(n) < \infty, \text{ and } \quad \lim_{N\to\infty} a_N(n) = a(n),$$

then

$$\lim_{N\to\infty} \sum_{n=-\infty}^{\infty} a_N(n) = \sum_{n=-\infty}^{\infty} a(n)$$

d) Suppose that $f \in C^2(\mathbb{R}/2\pi\mathbb{Z})$. Carry out the following steps to prove

$$|c_N(n)| \le \max |f''|/n^2 \tag{2}$$

i) Let $g : \mathbb{Z} \to \mathbb{C}$ satisfying $g(j+N) = g(j)$. For $n \in \mathbb{Z}$, let $\omega = e^{2\pi i n/N}$ and define

$$F(\omega) = \frac{1}{N}\sum_{j=1}^{N} g(j)\omega^{-j}$$

Show that

$$(\omega + \omega^{-1} - 2)F(\omega) = \frac{1}{N}\sum_{k=1}^{N} (g(j+1) + g(j-1) - 2g(j))\omega^{-j}$$

and deduce that for $|n| \le 3N/4$,

$$|F(\omega)| \le \frac{N^2}{n^2} \max_j |g(j+1) + g(j-1) - 2g(j)|$$

ii) Show that for $f \in C^2(\mathbb{R}/2\pi\mathbb{Z})$,

$$|f(x+h) + f(x-h) - 2f(x)| \le h^2 \max |f''|$$

using the Taylor formula

$$m(1) = m(0) + m'(0) + \int_0^1 m''(t)(1-t)\,dt$$

applied to $m(t) = f(x+th) + f(x-th) - 2f(x)$.

iii) Deduce (2) from (i) and (ii).

e) Explain how (a)–(d) yield the Fourier series formula for every function $f \in C^2(\mathbb{R}/2\pi\mathbb{Z})$.

5. This **optional** problem for no credit is here so that you can use the formulas from it to carry out the Shannon sampling theorem above in Problem 3. We begin by showing that the convolution of L^2 functions is compatible with the Fourier transform, then compute the explicit example used for Shannon's theorem.

a) Let f and g belong to $L^2(\mathbb{R})$. Denote

$$g_N(x) = \frac{1}{2\pi}\int_{-N}^{N} \hat{g}(\xi)e^{ix\xi}\,d\xi$$

Show that

$$\widehat{(fg)} = \frac{1}{2\pi}\hat{f} * \hat{g}$$

by evaluating

$$\lim_{N\to\infty}\int_{-\infty}^{\infty} f(x)g_N(x)e^{-ix\xi}\,dx$$

in two ways.

b) Let $I_a(y) = 1_{[-a,a]}(y)$. For $0 < h \le a$, find $J_{a,h}(x)$ such that

$$\hat{J}_{a,h}(\xi) = I_a * I_h(\xi)$$

c) Deduce from part (b) the formula for the function $K_\lambda(x)$ of the form

$$K_\lambda(y) = \frac{C\sin(Ay)\sin(By)}{y^2}, \quad \lambda > 1$$

whose Fourier transform has the trapezoidal shape

$$\hat{K}_\lambda(\xi) = \begin{cases} 1, & |\xi| \le \pi \\ (\lambda\pi - |\xi|)/\pi(\lambda-1), & \pi \le |\xi| \le \pi\lambda \\ 0, & \pi\lambda \le |\xi| \end{cases}$$

Indeed, $A = \pi(\lambda-1)$, $B = \pi(\lambda+1)$, $C = 2/\pi^2(\lambda-1)$. (See Stein-Shakarchi Problem 20, p. 167-168. But be warned that our convention for the Fourier transform is different, and the formula there is written with cosines, not sines.)

d) Evaluate $K_\lambda(0)$ and show that

$$K_1(y) = \lim_{\lambda \to 1} K_\lambda(y).$$

6. This **optional** exercise shows how to construct a countable family of independent random variables. It's taken from the book Probability Theory, by D. W. Stroock. Consider a countable sequence of probability spaces $(X_n, \mathcal{F}_n, \mu_n)$. Define the ring of so-called cylinder sets, namely sets of the form $B \times X_{n+1} \times X_{n+2} \times \cdots$ with $B \in \mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \mathcal{F}_n$.

Theorem. There exists a unique measure μ on the cylinder ring satisfying

$$\mu(B \times X_{n+1} \times X_{n+2} \times \cdots) = \mu_1 \times \mu_2 \times \cdots \times \mu_n(B)$$

Follow this outline to prove the theorem. Define $X = \prod_{j=1}^{\infty} X_j$, and define the projection

$$\pi_n : X \to X_1 \times X_2 \times \cdots \times X_n, \quad \text{by} \quad \pi_n(x) = (x_1, x_2, \ldots, x_n)$$

a) Show that the theorem follows if one shows that any nested sequence $A_n \supset A_{n+1}$ in the cylinder ring, for which

$$\liminf_{n \to \infty} \mu(A_n) \geq \epsilon > 0$$

also satisfies

$$\bigcap_{n=1}^{\infty} A_n \neq \emptyset$$

b) Show that it suffices to consider the situation in which for some $B_n \in \mathcal{F}_1 \times \cdots \times \mathcal{F}_n$, the nested sequence satisfies

$$A_n = \pi_n^{-1}(B_n); \quad B_n \times X_{n+1} \supset B_{n+1}$$

c) Define $g_{m,m}(x_1, x_2, \ldots, x_m) = 1_{B_m}(x_1, x_2, \ldots, x_m)$. For all $n > m$, define $g_{m,n} : X_1 \times X_2 \times \cdots \times X_m \to [0, 1]$ by

$$g_{m,n}(x_1, x_2, \ldots, x_m) = \int_{X_{m+1} \times \cdots \times X_n} 1_{B_n}(x_1, x_2, \ldots x_n) d(\mu_{m+1} \times \cdots \times \mu_n)$$

Show that the limit

$$g_m(x) := \lim_{n \to \infty} g_{m,n}(x), \quad x = (x_1, \ldots, x_m)$$

exists and that

$$g_m(x_1, \ldots, x_m) = \int_{X_{m+1}} g_{m+1}(x_1, \ldots, x_{m+1}) d\mu_{m+1}$$

d) Use induction to find $x = (x_1, x_2, \dots) \in X$ such that for every m,

$$g_m(x_1, x_2, \dots, x_m) \geq \epsilon$$

and deduce that

$$x \in \bigcap_{m=1}^{\infty} A_m$$

This establishes (a) and concludes the construction of the infinite product measure.

e) The monotone class theorem says that if $\mathcal{M}$ is a collection of subsets of a set Z that is closed under nested countable union and nested countable intersection and contains a ring $\mathcal{A}$, then $\mathcal{M}$ contains the sigma-ring generated by $\mathcal{A}$. (The theorem does not say whether the whole set Z is in $\mathcal{M}$, but in our situation $Z \in \mathcal{A}$, so that the collection of subsets will be a sigma-field.) Show that this result implies that the measure constructed above is the unique measure on the sigma field generated by cylinder sets that agrees with the finite product measures. (You may take the monotone class theorem for granted or prove it. It's a bit easier than the $\pi - \lambda$ theorem, but in the same spirit.)