Problem Set 10 (due 10am Fri, Nov 22)

Do AG §3.5/ 9, 11, 12.

1. a) Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$ and $g \in L^1(\mathbb{R})$. Show that

$$\|f * g\|_p \leq \|f\|_p \|g\|_1$$

Hint: Let $1/p + 1/q = 1$, and apply Hölder's inequality to

$$|f(x-y)g(y)| = (|f(x-y)||g(y)|^{1/p})(|g(y)|^{1/q})$$

(See Prop 17, AG Appendix B.) Note that this inequality is also true for $p = 1$, using Fubini's theorem carried out in §3.5/9, and for $p = \infty$, using more elementary properties of the Lebesgue integral.

b) Deduce that if $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$ and $K \in L^1(\mathbb{R})$ with

$$\int_{\mathbb{R}} K(x)\,dx = 1; \quad K_\epsilon(x) = (1/\epsilon)K(x/\epsilon)$$

then

$$\lim_{\epsilon \to 0} \|f * K_\epsilon - f\|_p = 0$$

c) Show that if $f \in L^\infty(\mathbb{R})$ and $K \in L^1(\mathbb{R})$, then $f * K \in C_{ucb}(\mathbb{R})$, where $C_{ucb}(\mathbb{R})$ is the class of uniformly continuous functions bounded functions. (See also Fourier series notes 3, where the analogous statement on $\mathbb{T}$ is mentioned as an exercise, with a hint as to how it is proved.)

d) Give a counterexample to the statement in part b) in the case $p = \infty$.

2. We will solve the equation

$$\frac{\partial}{\partial t}u = \frac{\partial^2}{\partial x^2}u + a\frac{\partial}{\partial x}u \tag{1}$$

for a function $u(x,t)$ with initial value

$$u(x,0) = f(x).$$

This is interpreted as a heat equation or diffusion equation with drift (the $a(\partial/\partial x)$ term is the drift).

a) Denote by $\hat{u}(\xi,t)$ and $\hat{f}(\xi)$ the Fourier transform in the x variable of u and f. For each fixed ξ find the ordinary differential equation for $\hat{u}(\xi,t)$ formally (assuming the derivatives all make sense). Then solve the equation for $\hat{u}$ in terms of $\hat{f}$.

b) Take the inverse Fourier transform of your formula for $\hat{u}(\xi, t)$ in part (a), and find a proposed formula for u in terms of f in the form

$$u(x,t) = f * g_t(x)$$

(See §3.5/10 for the formula for g_t in the case $a = 0$.)

c) Show that the formula in part (b) solves the initial value problem. Namely, for $f \in L^1$, u given by the formula in (b) satisfies the differential equation (1) in $t > 0$, $x \in \mathbb{R}$, and satisfies the initial condition in the sense that

$$\lim_{t\to 0} \int_{-\infty}^{\infty} |u(x,t) - f(x)|dx = 0 \quad (t > 0).$$

3. (See also §3.5/6; SS Chap 5/ Exercise 23, p. 168–169.) Define

$$Tf(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-ixy}dx$$

a) Show that $T^4 = I$ the identity mapping on $\mathcal{S}$, the Schwartz class. (This extends by continuity to $L^2(\mathbb{R})$. Recall that we proved in lecture that T maps $\mathcal{S}$ to $\mathcal{S}$. The Plancherel formula says that T is an isometry in the $L^2(\mathbb{R})$ norm. We also showed in lecture that, since $\mathcal{S}$ is dense in L^2, one can extend T by continuity to the whole space $L^2(\mathbb{R})$, where it is again an isometry.)

b) Suppose that $h \in \mathcal{S}$ and $Th = ch$ (an eigenvector for T with eigenvalue c). Find the short list of possible values of c. (See SS, p. 163, 6.)

c) Consider the so-called annihilation and creation operators A and B defined by

$$A = \frac{d}{dx} + x; \quad B = -\frac{d}{dx} + x$$

and denote the $L^2(\mathbb{R})$ inner product by

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx$$

Show that for all f and g in $\mathcal{S}$, $\langle Af, g \rangle = \langle f, Bg \rangle$. This says that $B = A^*$, the adjoint of A, and $A^* = B$.

d) Find numbers a and b such that

$$TA = aAT; \quad TB = bBT$$

e) Let $h_0(x) = e^{-x^2/2}$ and $h_k = B^k h_0$. Show that $h_k(x) = H_k(x)e^{-x^2/2}$ with H_k a polynomial[1] of degree k and that

$$Th_k = \lambda_k h_k$$

[1] H_k is known as a Hermite polynomial. Its generating function and other closely related formulas can be found in SS p. 173.

for some λ_k. (Find λ_k explicitly.)

f) Show that the $h_k/\|h_k\|$ (with $\|\cdot\|$ the $L^2(\mathbb{R})$ norm) form a complete orthogonal system. (Hint: Consider $\langle B^k h_0, B^\ell h_0\rangle$. Use part (c) and the commutator formula $[A, B^n] = ncB^{n-1}$. Incidentally, the Hermite polynomials can also be obtained by applying the Gram-Schmidt process to the functions 1, x, x^2, x^3, etc, in $L^2(\mathbb{R}, e^{-x^2}dx)$.)