Boolean rings and Boolean algebra

The word ring as it is used measure theory corresponds to the notion of ring used elsewhere in mathematics, but I didn't give the correct correspondence in lecture. I will do so now.

A (commutative) ring is, by definition, a set with two commutative operations, addition and multiplication. The ring is a group under addition (has an additive identity, usually denoted 0, and additive inverses, namely elements $-x$ such that $x + (-x) = 0$). The multiplication satisfies the associative law and distributes over addition: $x(y+z) = xy+xz$. If there is a multiplicative identity it is usually denoted by 1, but it need not exist. Even if it does, there is no requirement that multiplicative inverses exist.

If S is a set, then consider the set 2^S of all functions $x : S \to \{0, 1\}$. We can identify 2^S with the set of all subsets of S as follows. A subset $A \subset S$ corresponds to the indicator function

$$1_A(s) = \begin{cases} 1 & s \in A \\ 0 & s \notin A \end{cases}$$

Give the set $\{0, 1\}$ the group law of $\mathbf{Z}/(2\mathbf{Z})$, namely,

$$1 + 1 = 0; \quad 1 + 0 = 0 + 1 = 1; \quad 0 + 0 = 0,$$

and use ordinary multiplication ($1 \cdot 1 = 1$, $0 \cdot 1 = 0 \cdot 0 = 0$). Then 2^S is a ring with

$$1_A + 1_B = 1_{S(A,B)}; \quad 1_A 1_B = 1_{A\cap B},$$

where $S(A, B) = (A - B) \cup (B - A)$, the symmetric difference. Thus addition is identified with symmetric difference of sets and multiplication with intersection of sets.

A Boolean ring is a ring with the additional property that $x^2 = x$ for all elements x. Indeed, in the situation above,

$$1_A 1_A = 1_A$$

so that the ring structure on sets described above is Boolean. The formulas for the operations we used in lecture to define rings, namely union and set difference, can be expressed in terms of the Boolean operations as follows.

$$1_{A\cup B} = 1_A + 1_B + 1_A 1_B; \quad 1_{A-B} = 1_A + 1_A 1_B$$

The additive identity is $1_\emptyset$ and 1_A is its own additive inverse. The multiplicative identity is 1_S. Note that we proved that the empty set is always an element of a ring of sets, but the total space S need not be. Likewise, a ring must have an additive identity, but is not required to have a multiplicative identity.

The algebraic structure that encodes the union and intersection (or, equivalantly, the "or" and "and" operations) as well as complementation (or, equivalently, negation) is usually called a Boolean algebra. Any Boolean algebra gives rise to a Boolean ring as follows. Define the operation $\vee$ (same as "or" or "union") on $\{0, 1\}$ as the ones used in a truth table in logic:

$$1 \vee 1 = 1 \vee 0 = 0 \vee 1 = 1; \quad 0 \vee 0 = 0.$$

Similarly, the operation $\wedge$ has the same rules as the truth table for "and" (or "intersection"):

$$1 \wedge 1 = 1; \quad 1 \wedge 0 = 0 \wedge 1 = 0 \wedge 0 = 0.$$

Thus $\wedge$ is the same as ordinary multiplication of 0 and 1.

Identify 1_A with the set A as above. Then

$$1_{A \cup B} = 1_A \vee 1_B; \quad 1_{A \cap B} = 1_A \wedge 1_B = 1_A 1_B.$$

Multiplication is distributive over the operation $\vee$:

$$1_{A \cap (B \cup C)} = 1_A(1_B \vee 1_C) = (1_A 1_B) \vee (1_A 1_C) = 1_{(A \cap B) \cup (A \cap C)}$$

The additive identity for the operation $\vee$ is $1_\emptyset$ as it was for addition modulo 2 above. But one cannot find additive inverses, and 2^S is not a group under the operation $\vee$. In other words, the Boolean algebra is not a ring under the operations $\wedge$ for multiplication and $\vee$ for addition. On the other hand, it is a ring under the operations $\wedge$ for multiplication and symmetric difference for addition. The symmetric difference $S(A, B) = (A - B) \cup (B - A)$ is expressed in terms of $\wedge$, $\vee$, and complementation by

$$1_A + 1_B = 1_{S(A,B)} = (1_A \wedge 1_{B^c}) \vee (1_B \wedge 1_{A^c}),$$

since $S(A, B) = (A \cap B^c) \cup (B \cap A^c)$.