

18.103 Fall 2013

Midterm Review

The main topics so far have been

1. The construction of Lebesgue measure
2. The Lebesgue integral, convergence theorems and Fubini's theorem
3. Applications
 - a) Probability theory
 - b) Lebesgue $L^p(X, \mu)$ spaces are complete.
 - c) $C_0^\infty(\mathbf{R}^n)$ is dense in $L^p(\mathbf{R}^n)$, $1 \leq p < \infty$.
 - d) Riemann-Lebesgue Lemma (follows from density).
 - e) Fourier representation of 2π -periodic L^2 functions
(i. e., e^{inx} , $n \in \mathbf{Z}$ is a complete orthonormal system).

The key to constructing Lebesgue measure is to confirm that volume is countably additive on the rectangle ring. See Theorem 7, §1.3, and Exercise §1.1/21.

Note that for measurable sets E and F ,

$$\|1_E - 1_F\|_1 = \int_X |1_E - 1_F| d\mu = \mu(S(E, F))$$

where $S(E, F) = (E - F) \cup (F - E)$ is the set-theoretical symmetric difference. Thus for measurable sets, the distance $d(E, F) = \mu^*(S(E, F))$ is the same as the $L^1(X, \mu)$ distance between the indicator functions 1_E and 1_F . After the basic key consistency property mentioned in the last paragraph, the rest of the process we went through to construct Lebesgue measure is essentially the same as the process of completing any metric space. Thus an alternative definition of L^1 is as the completion under the L^1 distance of step functions (or Riemann integrable functions or continuous functions). In the abstract, the completion process yields equivalence classes of Cauchy sequences. The advantage of our present approach is that we can identify the limits as actual sets and functions.

Although in the Lebesgue approach the limiting objects are actual functions, not just equivalence classes of Cauchy sequences, we still have to get used to some ambiguity. The measurable sets and integrable functions we obtain as limits are certainly more satisfying than equivalence classes, but we still pay the price that they are only defined up to a set of measure zero. Since Lebesgue's time, we have figured out many ways to deal with that inconvenience and get back to "classical" statements involving continuous functions and functions with definite values at points of interest. We will learn some of these in the second half of the course.

Hour Test

The 50-minute test in class on Friday, October 25, covers Sections 1.1 through 3.3 of the text by Adams and Guillemin, except Section 2.8. It also covers the notes that have been posted. Here is a rough description of the types of problems, followed by last year's test. In all cases of proofs you should be ready to state carefully not only the theorem you are proving but also the ingredients in the proof.

1. Establish the main step in the Lebesgue measure construction, that is, confirm that μ^* equals the volume measure on the rectangle ring.
2. Prove one of these convergence theorems. (I will pick one.)
 - a) The monotone convergence theorem
 - b) Fatou's lemma
 - c) The dominated convergence theorem
3. Prove or establish a step or two in the proof of one of the following theorems.
 - a) Borel-Cantelli Lemma 1 or 2
 - b) Bounded measurable functions are uniform limits of simple functions.
 - c) Bounded Riemann integrable functions are Lebesgue integrable with the same value.
 - d) $L^1(X, \mu)$ is complete.
- 4 + 5. Questions in which you have to decide what's true and why. You will be given statements related to limit theorems, Fubini's theorem, or issues of integrability or measurability. Most students find these to be the trickiest questions because of the uncertainty. They are not designed to be devious, but you need to come to the test armed with examples related to where the hypotheses of the theorems do and don't work.
6. The last question is up for grabs. It may include a computation or the evaluation of a limit as an application of any of our theorems. For example,
 - a) A computation with Rademacher functions.
 - b) Use of $\mathbf{E}(f_1 f_2 \cdots f_n) = \mathbf{E}(f_1) \mathbf{E}(f_2) \cdots \mathbf{E}(f_n)$ for independent random variables.
 - c) Computation of the probability distribution μ_f given an explicit function f .
 - d) Evaluation of a limit using a convergence theorem, Fubini's theorem, or density of smooth functions in L^p functions, $1 \leq p < \infty$.
 - e) Evaluation of a probability using a Borel-Cantelli lemma

Last year's test, on the next page, follows the rubric above.

PRACTICE HOUR TEST (Throughout, μ denotes Lebesgue measure on \mathbf{R} .)

1. (20 pts) If J is a finite union of bounded intervals, define $\ell(J)$ as the length of J .

a) Give a definition of Lebesgue outer measure μ^* on \mathbf{R} in terms of ℓ .

b) Prove directly from this definition that $\mu^*([0, 1]) = 1$. Use without proof that ℓ is finitely subadditive and finitely additive on finite unions of intervals. You may not, however, use the fact that ℓ is countably subadditive or countably additive.

2. (20 pts) Deduce the dominated convergence theorem from Fatou's lemma. (Your answer must include a careful statement of both the theorem and the lemma.)

3.

a) (10 pts) Show that every Cauchy sequence in $L^1(I, \mu)$ has a subsequence that converges pointwise almost everywhere.

b) (6 pts) Find a Cauchy sequence as in part (a) that does not converge pointwise almost everywhere.

4. (16 pts) Decide if the following statements true or false and give a **reason** if true and a **counterexample** if false. (4 points for each correct answer; 4 points for the reason or counterexample.) As in all parts of the test, μ denotes Lebesgue measure on \mathbf{R} .

a) (T/F) If A_k are measurable subsets of \mathbf{R} , then $\lim_{N \rightarrow \infty} \mu \left(\bigcap_{k=1}^N A_k \right) = \mu \left(\bigcap_{k=1}^{\infty} A_k \right)$

b) (T/F) If $f(x, y) \geq 0$ is measurable on $\mathbf{R} \times \mathbf{R}$, and $\int_{\mathbf{R}} \left(\int_{\mathbf{R}} f(x, y) d\mu(x) \right) d\mu(y) < \infty$, then $\frac{xyf(x, y)}{x^2 + y^2}$ is integrable on $\mathbf{R} \times \mathbf{R}$.

5. (16 pts) Let f_n be a sequence of measurable functions on $[0, 1]$ such that $0 \leq f_n(x) \leq 1$. Find the relationship between

$$\limsup_{n \rightarrow \infty} \int_0^1 f_n(x) d\mu(x) \quad \text{and} \quad \int_0^1 \limsup_{n \rightarrow \infty} f_n(x) d\mu(x),$$

In other words, decide if they are equal or if one is necessarily less than or equal to the other. Prove your answer and give an example if they can be unequal.

6. (12 pts) Show that for all $f \in L^1(\mathbf{R}, \mu)$,

$$\lim_{t \rightarrow 0} \int_{\mathbf{R}} |f(x) - f(x+t)| d\mu(x) = 0$$

One way to prove this (*not* the fastest) is to start with 1_E .