18.103 Fall 2013

Old Hour Test Solutions.

4. a) (T/F) If A_k are measurable subsets of $\mathbf{R}$, then $\lim_{N\to\infty} \mu\left(\bigcap_{k=1}^{N} A_k\right) = \mu\left(\bigcap_{k=1}^{\infty} A_k\right)$

False. (Only works when one of the measures is finite.) Let $A_k = [k, \infty)$, then the limit is infinity, whereas

$$\emptyset = \bigcap_{k=1}^{\infty} A_k,$$

so that the right side is zero.

b) (T/F) If $f(x, y) \geq 0$ is measurable on $\mathbf{R} \times \mathbf{R}$, and $\int_{\mathbf{R}} \left(\int_{\mathbf{R}} f(x, y) d\mu(x)\right) d\mu(y) < \infty$, then $\dfrac{xyf(x, y)}{x^2 + y^2}$ is integrable on $\mathbf{R} \times \mathbf{R}$.

True. Note that $\dfrac{xyf(x, y)}{x^2 + y^2}$ is measurable. By the version of Fubini's theorem on a problem set, f is integrable on $\mathbf{R}^2$ with respect to $\mu \times \mu$. Finally, because $x^2 - 2xy + y^2 = (x - y)^2 \geq 0$,

$$\left|\frac{xy}{x^2 + y^2}\right| \leq \frac{1}{2}$$

Therefore,

$$\int_{\mathbf{R}\times\mathbf{R}} \left|\frac{xyf(x, y)}{x^2 + y^2}\right| d(\mu \times \mu) \leq \frac{1}{2} \int_{\mathbf{R}\times\mathbf{R}} f(x, y) d(\mu \times \mu) < \infty$$

Thus the function is integrable.

5. If f_n is a sequence of measurable functions on $[0, 1]$ such that $0 \leq f_n(x) \leq 1$. Then

$$\limsup_{n\to\infty} \int_0^1 f_n(x)\, d\mu(x) \leq \int_0^1 \limsup_{n\to\infty} f_n(x)\, d\mu(x),$$

This is proved by applying Fatou's lemma to the functions $g_n(x) = 1 - f_n(x)$. The inequality may be strict as in this example with LHS $= 1/2$; RHS $= 1$.

$$f_{2n}(x) = \begin{cases} 0 & 0 \leq x \leq 1/2 \\ 1 & 1/2 < x \leq 1 \end{cases}; \qquad f_{2n+1}(x) = \begin{cases} 1 & 0 \leq x \leq 1/2 \\ 0 & 1/2 < x \leq 1 \end{cases}.$$