Fourier Series, Continued (part 3)

Proposition 1. *If f and g belong to $L^1(\mathbf{T})$, then $f * g \in L^1(\mathbf{T})$ and*

$$\|f * g\|_p \leq \|f\|_p \|g\|_1.$$

Proof. Fubini's theorem implies

$$\int_{\mathbf{T}} \left(\int_{\mathbf{T}} |f(x-y)g(y)| dy \right) dx = \int_{\mathbf{T}} \left(\int_{\mathbf{T}} |f(x-y)|g(y)| dx \right) dy$$

$$= \int_T 2\pi \|f\|_1 |g(y)| dy = (2\pi)^2 \|f\|_1 \|g\|_1 < \infty$$

It follows that

$$\int_{\mathbf{T}} |f(x-y)g(y)| dy < \infty \quad \text{a. e.}$$

For these values of x, $f(x-y)g(y)$ is integrable, and we may define

$$f * g(x) = \frac{1}{2\pi} \int_{\mathbf{T}} f(x-y)g(y) dy.$$

Moreover,

$$\|f * g\|_1 = \frac{1}{2\pi} \int_{\mathbf{T}} |f * g(x)| dx = \frac{1}{2\pi} \int_T \left| \frac{1}{2\pi} \int_{\mathbf{T}} f(x-y)g(y) dy \right| dx$$

$$\leq \frac{1}{(2\pi)^2} \int_T \int_{\mathbf{T}} |f(x-y)g(y)| dy \, dx = \|f\|_1 \|g\|_1.$$

$\square$

Exercise. Show that for $f \in L^\infty(\mathbf{T})$ and $g \in L^1(\mathbf{T})$, $f * g(x)$ is defined for every x and

$$\|f * g\|_\infty \leq \|f\|_\infty \|g\|_1.$$

Deduce, using a density argument, that $f * g$ is continuous. [1]

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[1] Recall that the space $L^\infty(X, \mu)$ is defined as the set of measurable functions for which the norm

$$\|f\|_\infty := \operatorname{esssup}_{x \in X} |f(x)|$$

Next, we introduce an operator notation for the Cesáro means of the Fourier series:

$$\sigma_N f(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \hat{f}(n) e^{inx} = f * F_N(x),$$

with F_N the Fejér kernel. Notice that this is a linear operation, $\sigma_N(af + bg) = a\sigma_N f + b\sigma_N g$ for complex numbers a and b.

Theorem 1. *Let $f \in L^1(\mathbf{T})$. Then*

$$\lim_{N \to \infty} \|f - \sigma_N f\|_1 = 0$$

In particular, trigonometric polynomials are dense in $L^1(\mathbf{T})$.

Proof. Take $\epsilon > 0$ and choose $g \in C(\mathbf{T})$ such that

$$\|f - g\|_1 \leq \epsilon$$

Then
$$\|\sigma_N f - f\|_1 \leq \|\sigma_N f - \sigma_N g\|_1 + \|\sigma_N g - g\|_1 + \|g - f\|_1, \tag{1}$$

and Proposition 1 implies

$$\|\sigma_N f - \sigma_N g\|_1 = \|\sigma_N(f - g)\|_1 = \|(f - g) * F_N\|_1 \leq \|f - g\|_1 \|F_N\|_1 = \|f - g\|_1 \leq \epsilon$$

Therefore, implies

$$\|\sigma_N f - f\|_1 \leq 2\epsilon + \|\sigma_N g - g\|_1$$

The main theorem from the preceding lecture was

$$\max_x |g(x) - \sigma_N g(x)| \to 0 \quad \text{as} \quad N \to \infty$$

This is the same as saying $\|g - \sigma_N g\|_\infty \to 0$. For any function h,

$$\|h\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(x)| dx \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{esssup}_x |h| dx = \|h\|_\infty$$

is finite. The essential supremum is defined by

$$\mathrm{esssup}_{x \in X} f(x) = \inf\{\sup_E f : \mu(X \setminus E) = 0\}$$

As with the other L^p spaces, we consider two functions in $L^\infty(X, \mu)$ to be equivalent if they are equal except on a set of measure zero.

It follows that as $N \to \infty$,

$$\|g - \sigma_N g\|_1 \leq \|g - \sigma_N g\|_\infty \to 0$$

Thus,

$$\lim_{N \to \infty} \|\sigma_N f - f\|_1 \leq 2\epsilon$$

and since $\epsilon > 0$ was arbitrary, we have proved the theorem. $\square$

Corollary 1. *(Uniqueness of Fourier Series) If $f \in L^1(\mathbf{T})$ and $\hat{f}(n) = 0$ for all n, then $f(x) = 0$ for almost every x.*

Proof. By the theorem, $\|\sigma_N f - f\|_1 \to 0$ as $N \to \infty$. But the fact that $\hat{f}(n) = 0$ for all n implies $\sigma_N f \equiv 0$ for all N, so we have $\|f\|_1 = 0$. $\square$

Further Results. We mention without proof several negative and positive results about convergence that we won't have time to prove in this class. To state these, we will also use operator notation for the partial sums $s_N(x)$ as follows.

$$S_N f(x) := \sum_{n=-N}^{N} \hat{f}(n) e^{inx} = f * D_N(x)$$

where D_N is the Dirichlet kernel.

1. There exists $f \in C(\mathbf{T})$ such that $S_N f(0) \to \infty$ as $N \to \infty$. (Pointwise convergence of $S_N f$ can fail for continuous functions.)

2. There exists $f \in L^1(\mathbf{T})$ such that $\|S_N f\|_1 \to \infty$ as $N \to \infty$. (Norm convergence of $S_N f$ can fail for L^1 functions.)

3. On the other hand, if $f \in L^p(\mathbf{T})$ for some p, $1 < p < \infty$, then

$$\|S_N f - f\|_p \to 0, \qquad N \to \infty.$$

(Norm convergence of $S_N f$ succeeds for L^p functions, $1 < p < \infty$. The main step in the proof is a theorem of Marcel Riesz that $\|S_N f\|_p \leq C_p \|f\|_p$, independent of N. In this class, we will only prove the weaker statement that this works for $\sigma_N f$. This depends on the inequality $\|\sigma_N f\|_p \leq \|f\|_p$ which is relatively easy.)

4. If $f \in L^p(\mathbf{T})$ for some p, $p \geq 1$, then

$$\lim_{N \to \infty} \sigma_N f(x) = f(x), \quad \text{a. e. } x.$$

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(Pointwise convergence of $\sigma_N f$ succeeds for L^p functions, for all $p \geq 1$. This follows from what are known as maximal function estimates. This type of estimate also plays the central in what is known as the Lebesgue differentiation theorem, which says that the fundamental theorem of calculus works for integrals of L^1 functions.)

5. If $f \in L^p(\mathbf{T})$ for some p, $1 < p < \infty$, then

$$\lim_{N \to \infty} S_N f(x) = f(x), \quad \text{a. e. } x.$$

This last result is due to Lennart Carleson (1965) for $p \geq 2$ and to Richard Hunt (1967) for $1 < p < 2$, and the proof is difficult.

Rather than prove these more detailed results about ordinary and Cesáro convergence, we prefer to talk about applications. The text by Stein and Shakarchi features two lovely, illustrative applications of Fourier analysis, which we now present. They are a proof of the isoperimetric inequality and a proof of Weyl's equidistribution theorem.

Applications

The fundamental idea motivating Fourier is that differentiation can be understood using the Fourier basis. The linear operator d/dx can be *diagonalized* in the basis e^{inx}. Formally

$$\frac{d}{dx} \sum_n a_n e^{inx} = \sum_n i n a_n e^{inx}$$

In analogy with finite dimensions, we say that that d/dx is represented by the matrix with diagonal entries 0, $\pm i$, $\pm 2i$, $\pm 3i$, etc.[2]

If one assumes that $\sum |n a_n| < \infty$, then one can justify this formula pointwise for each x. Here is a Fourier coefficient version of the differentiation formula above.

Proposition 2. *If $f \in C^1(\mathbf{R}/2\pi\mathbf{Z})$, then (proved in class by integration by parts)*

$$\widehat{f'}(n) = in\hat{f}(n)$$

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[2]Mathematicians have found that this important formula gives a consistent way to **define** d/dx even when differentiation in the ordinary sense does not work and the sums don't converge in any ordinary sense. As we will explain later in the class, this formula for d/dx is true in the sense of distributions.

The formula $\widehat{f'}(n) = in\hat{f}(n)$ is of central importance, just like its counterpart in summation form above.

It follows from Proposition 2, that if $f \in C^1(\mathbf{R}/2\pi\mathbf{Z})$, then continuous function $f' \in C(\mathbf{T}) \subset L^2(\mathbf{T})$. Hence by our result showing that e^{inx} is an orthormal basis of $L^2(\mathbf{T})$, f' is represented by its series, and the Parseval formula says

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(x)|^2 \, dx = \sum_{n \in \mathbf{Z}} |in\hat{f}(n)|^2.$$

(In particular, the series on the right side is finite.)

Application 1. The Isoperimetric Inequality.

Let D be a region of the plane enclosed by a simple[3] C^1 curve $\Gamma : (x(t), y(t))$. The isoperimetric inequality[4]

$$A(D) \leq \ell(\Gamma)^2/4\pi$$

where $A(D)$ denotes the area of D and $\ell(\Gamma)$ denotes the length of Γ. Moreover, the case of equality occurs if and only if Γ is a circle.

The idea is to convert this inequality into one concerning Fourier coefficients of $x(t)$ and $y(t)$.

We begin with a standard 18.02 formula for area,

$$A(D) = \frac{1}{2} \int_{\Gamma} x\,dy - y\,dx,$$

which follows from Green's theorem,

$$\int_{\Gamma} M\,dx + N\,dy = \int\int_{D} [(\partial N/\partial x) - (\partial M/\partial y)] \, dx\,dy$$

with $M = -y/2$, $N = x/2$. Thus

$$A(D) = \frac{1}{2} \int_{a}^{b} [x(t)y'(t) - x'(t)y(t)]dt$$

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[3] "Simple" means that the curve does not cross itself.
[4] We follow Stein-Shakarchi, although we treat the C^1 case, a bit more general a hypothesis than in that text.

Moreover, the length of Γ is given by

$$\ell(\Gamma) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt.$$

Step 1. By rescaling, we may assume $\ell(\Gamma) = 2\pi$. Then our goal is to prove that

$$A(D) \leq (2\pi)^2/4\pi = \pi$$

We can also change variables so that the parametrization has unit speed:

$$x'(t)^2 + y'(t)^2 = 1$$

which implies $b - a = \ell(\Gamma) = 2\pi$. Thus, we may suppose $a = -\pi$, $b = \pi$ and that $x(t)$ and $y(t)$ are in $C^1(\mathbf{R}/2\pi\mathbf{Z})$.

Step 2. Next we relax the constraint from the unit speed condition to the constraint.

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} [x'(t)^2 + y'(t)^2] dt = 1 \tag{2}$$

In the case $x'(t)^2 + y'(t)^2 = 1$, this constraint is obviously true, so if we succeed in proving $A(D) \leq \pi$ under the constraint (2), then we have proved the isoperimetric inequality.

What is less obvious, is why we did this and why we can get away with it. We will answer these questions before proceeding further. The reason why we did this is that the constraint on the integral of $\sqrt{x'(t)^2 + y'(t)^2}$ can't be written in any useful way in terms of Fourier coefficients. Neither can the constraint $x'(t)^2 + y'(t)^2 = 1$. On the other hand, the constraint (2) can be rewritten using Parseval's formula (see Step 3).

There remains the question why we can get away with this relaxation of the constraint. The answer is that the Cauchy-Schwarz inequality implies

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sqrt{x'(t)^2 + y'(t)^2} dt \leq \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} (x'(t)^2 + y'(t)^2) dt \right)^{1/2} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} 1^2 \, dt \right)^{1/2} = 1$$

In other words, all curves $(x(t), y(t))$ satisfying (2) also have length less than or equal to 2π.

It looks peculiar the first time you see it, but replacing L^1 norm of the speed $|(x'(t), y'(t))|$ by the L^2 norm is a standard device in the theory of geodesics (curves that minimize the distance between points in Riemannian manifolds). The curves minimizing the quadratic integral have constant speed, which has the further advantage of eliminating the non-uniqueness in the parametric representation of a shortest length curve.

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Step 3. Reformulation in terms of Fourier series.

The Fourier series of x and y are given by

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}; \quad y(t) = \sum_{n=-\infty}^{\infty} b_n e^{int}$$

Since x and y are real-valued, $a_{-n} = \overline{a_n}$ and $b_{-n} = \overline{b_n}$. Moreover, Proposition 2 says that

$$x'(t) = \sum_{n=-\infty}^{\infty} ina_n e^{int}; \quad y'(t) = \sum_{n=-\infty}^{\infty} inb_n e^{int}$$

with convergence in L^2 norm. Parseval's formula implies that (2) can be written

$$1 = \|x'\|^2 + \|y'\|^2 = \sum_{n=-\infty}^{\infty} |ina_n|^2 + |inb_n|^2 = \sum_{n=-\infty}^{\infty} n^2[|a_n|^2 + |b_n|^2]$$

Next, the scalar product formula (polarization of the Parseval formula) implies

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$$

Thus,

$$A(D) = \frac{1}{2} \int_{-\pi}^{\pi} [x(t)y'(t) - x'(t)y(t)]dt = \pi[\langle x, y' \rangle - \langle y, x' \rangle] = \pi \sum_{n=-\infty}^{\infty} [a_n\overline{inb_n} - b_n\overline{ina_n}]$$

Step 4. Recall that we want to prove that $A(D) \leq \pi$. Note that for real numbers a and b, $2ab \leq a^2 + b^2$. Thus

$$|a_n\overline{b_n} - b_n\overline{a_n}| \leq 2|a_n||b_n| \leq |a_n|^2 + |b_n|^2$$

and hence

$$A(D) \leq \pi \sum_{n=-\infty}^{\infty} |a_n\overline{inb_n} - b_n\overline{ina_n}| \leq \pi \sum_{n=-\infty}^{\infty} |n|[|a_n|^2 + |b_n|^2] \leq \pi \sum_{n=-\infty}^{\infty} n^2[|a_n|^2 + |b_n|^2] = \pi$$

This ends the proof of the isoperimetric inequality.

Step 5. It remains to prove that in the case of equality in the isoperimetric inequality, Γ is a circle. To prove this note that if equality holds, each of the inequalities in the proof is an equation. The last one says

$$\sum_{n=-\infty}^{\infty} |n|[|a_n|^2 + |b_n|^2] = \sum_{n=-\infty}^{\infty} n^2[|a_n|^2 + |b_n|^2]$$

Since $|n| < n^2$ for all $|n| \geq 2$, we have $|a_n|^2 + |b_n|^2 = 0$ for all $|n| \geq 2$. Thus

$$x(t) = a_0 + a_1 e^{it} + \bar{a}_1 e^{-it}; \quad y(t) = b_0 + b_1 e^{it} + \bar{b}_1 e^{-it};$$

Moreover,

$$1 = \sum_{n=-1}^{1} n^2[|a_n|^2 + |b_n|^2] = 2|a_1|^2 + 2|b_1|^2$$

Furthermore, $a \geq 0$, $b \geq 0$,

$$2ab = a^2 + b^2 \implies (a-b)^2 = 0 \implies a = b$$

From this and the equality $2|a_1 b_1| = |a_1|^2 + |b_1|^2$, we conclude that $|a_1| = |b_1|$. Thus,

$$|a_1|^2 = |b_1|^2 = 1/4$$

Therefore we may write

$$a_1 = e^{i\alpha}/2; \quad b_1 = e^{i\beta}/2$$

and

$$x(t) = a_0 + \cos(\alpha + t); \quad y(t) = b_0 + \cos(\beta + t)$$

Finally, substitute into the equality

$$|a_1 \bar{b}_1 - \bar{a}_1 b_1| = 2|a_1||b_1| = 1/2$$

to find

$$(1/4)|e^{i(\alpha - \beta)} - e^{i(\beta - \alpha)}| = (1/2)|\sin(\alpha - \beta)| = 1/2$$

Finally, this yields $\alpha - \beta = \pm\pi/2 \mod 2\pi$, so that

$$\cos(\beta + t) = \pm \sin(\alpha + t)$$

This finishes the proof that Γ is a unit circle (parametrized counterclockwise or clockwise) centered at (a_0, b_0).

Application 2. Weyl Equidistribution Theorem

For $x \in \mathbf{R}$, let $\{x\}$ denote the fractional part, that is, $x - \{x\}$ is the largest integer that is less than or equal to x.

Theorem 2. *(Weyl equidistribution theorem) If α is irrational, and $0 \le a < b \le 1$, then*

$$\lim_{N \to \infty} \frac{\#\{m : 0 \le m \le N-1, \quad a \le \{m\alpha\} \le b\}}{N} = b - a$$

Proof. The conclusion can be rewritten

$$\lim_{N \to \infty} \frac{1}{N} \sum_{m=0}^{N-1} f(\{m\alpha\}) = \int_0^1 f(x)dx \tag{3}$$

with $f = 1_{[a,b]}$. Extend f to be periodic of period 1. For any $\epsilon > 0$ there are functions f_1 and f_2 continuous and periodic of period 1 such that $f_1 \le f \le f_2$ and

$$\int_0^1 f_1(x) \ge (b-a) - \epsilon; \quad \int_0^1 f_2(x)dx \le (b-a) - \epsilon$$

Thus if we can prove (3) for f_1 and f_2 we have

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{m=0}^{N-1} 1_{[a,b]}(\{m\alpha\}) \le \lim_{N \to \infty} \frac{1}{N} \sum_{m=0}^{N-1} f_2(\{m\alpha\}) = \int_0^1 f_2(x)dx \le (b-1) + \epsilon$$

and similarly the liminf is greater than $(b-a) - \epsilon$. Since $\epsilon > 0$ is arbitrary, Theorem 2 follows.

To prove (3) for continuous functions with period 1, recall that they can be uniformly approximated by trigonometric polynomials with period 1. In other words, for $\epsilon > 0$, and any continuous periodic f, we can find a trigonometric polynomial g such that

$$\left| \frac{1}{N} \sum_{m=0}^{N-1} f(\{m\alpha\}) - \frac{1}{N} \sum_{m=0}^{N-1} g(\{m\alpha\}) \right| \le \max |f - g| \le \epsilon.$$

So it suffices to confirm (3) for trigonometric polynomials, and hence for single exponentials, $f = \varphi_n(x)$, with
$$\varphi_n(x) = e^{2\pi i n x}, \quad n = 0, \pm 1, \pm 2, \ldots .$$

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The case $n = 0$ is immediate since

$$\frac{1}{N}\sum_{m=0}^{N-1}\varphi_0(\{m\alpha\}) = \frac{1}{N}\sum_{m=0}^{N-1}1 = 1 = \int_0^1 \varphi_0(x)\,dx.$$

For $n \in \mathbf{Z}$, $n \neq 0$, we have

$$\varphi_n(\{m\alpha\}) = e^{2\pi i\{m\alpha\}} = e^{2\pi im\alpha} = \varphi_n(m\alpha),$$

and

$$\frac{1}{N}\sum_{m=0}^{N-1}\varphi_n(\{m\alpha\}) = \frac{1}{N}\sum_{m=0}^{N-1}e^{2\pi inm\alpha} = \frac{e^{2\pi inN\alpha} - 1}{N(e^{2\pi in\alpha} - 1)}$$

Here, we used the fact that α is irrational in order to know that $e^{2\pi in\alpha} - 1 \neq 0$. Letting N tend to infinity we see that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{m=0}^{N-1}\varphi_n(\{m\alpha\}) = 0 = \int_0^1 e^{2\pi inx}dx = \int_0^1 \varphi_n(x)dx$$

$\square$

Exercise. For $x \in \mathbf{R}^n$, denote the fractional parts of its components by

$$\{x\} = (\{x_1\}, \ldots, \{x_n\}),$$

(Put another way, $\{\cdot\} : \mathbf{R}^n \to \mathbf{R}^n/\mathbf{Z}^n$ is the quotient mapping.) Let R be a rectangle (multi-interval) in $[0,1]^n$. Let

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

be such that 1, α_1, $\ldots$, α_n are linearly independent over $\mathbf{Q}$, the rational numbers. Show that

$$\lim_{N\to\infty}\frac{\#\{m : 0 \leq m \leq N - 1, \{m\alpha\} \in R\}}{N} = \text{vol}(R)$$

(Hint: Formulate and prove the appropriate density theorem for trigonometric polynomials on $\mathbf{R}^n/\mathbf{Z}^n$.)