18.103 Fall 2013

Fourier Series 2

Recall that if $f \in L^1(\mathbb{T})$, and we define the partial sum

$$s_N(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$$

then

$$s_N(x) = f * D_N(x) = \frac{1}{2\pi} \int_{\mathbb{T}} f(y) D_N(x-y)\, dy$$

where the Dirichlet kernel $D_N(x)$ is defined by

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}$$

Recall that if you sum the geometric series, you find the following closed formula.

$$D_N(x) = \frac{e^{i(N+1)x} - e^{-iNx}}{e^{ix} - 1} \tag{1}$$

Multiplying numerator and denominator by $e^{-ix/2}$, we obtain a second closed form formula for D_N, namely

$$D_N(x) = \frac{e^{i(2N+1)x/2} - e^{-i(2N+1)x/2}}{e^{ix/2} - e^{-ix/2}} = \frac{\sin[(2N+1)x/2]}{\sin(x/2)} \tag{2}$$

Taking the limit of the very last expression as $x \to 0$, we find $D_N(0) = 2N+1$ — a good double check, consistent with series for D_N with $2N+1$ terms, all equal to 1 at $x = 0$.

It took nearly a century from the time Fourier invented the series in the early 1800s to the proof of a general theorem about convergence. People got stuck because s_N and D_N are hard to work with.

The problem was (and is) that there are "bad" functions $f \in C(\mathbb{T})$ such that s_N diverges at some points. There are even uglier functions $f \in L^1(\mathbb{T})$ for which $s_N(x)$ diverges for *every* x. Today we won't discuss these pathologies. We will focus instead on the positive side.

The breakthrough took place in 1904, when L. Fejér showed that trigometric polynomials approximate all continuous periodic functions uniformly. The idea is to give up, temporarily, on trying to approximate functions using $s_N(x)$ and instead look at the Cesáro means

$$\sigma_N(x) = [s_0(x) + \cdots + s_{N-1}(x)]/N$$

This new sequence converges more readily, smoothing out some of the oscillations in the sequence $s_N(x)$.

Theorem 1. *(Fejér) Let f be continuous on $\mathbb{R}$ and periodic of period 2π. Then*

$$\max_x |\sigma_N(x) - f(x)| \to 0 \quad \text{as } N \to \infty$$

where

$$\sigma_N(x) = (s_0(x) + \cdots + s_{N-1}(x))/N$$

An immediate corollary is the density of the finite linear span of the functions e^{inx} in continuous periodic functions.

Corollary 1. *Trigonometric polyonomials are dense in continuous, periodic of 2π functions in the uniform norm.*

To prove Fejér's theorem, we first compute

$$\sigma_N = \frac{1}{N}[s_0 + \cdots + s_{N-1}] = f * \frac{1}{N}\sum_0^{N-1} D_N = f * F_N$$

where

$$F_N(x) = \frac{1}{N}(D_0 + D_1 + \cdots + D_{N-1})$$

F_N is known as Fejér's kernel. We claim that

$$F_N(x) = \frac{\sin^2(Nx/2)}{N\sin^2(x/2)} \tag{3}$$

To prove this, we use the representation (1).

$$\begin{aligned}
(e^{ix} - 1)^2 N F_N(x) &= (e^{ix} - 1)^2 \sum_0^{N-1} D_n(x) \\
&= (e^{ix} - 1)\left[\sum_{n=0}^{N-1} e^{i(n+1)x} - \sum_{n=0}^{N-1} e^{-inx}\right] \\
&= e^{i(N+1)x} - e^{ix} - e^{ix} + e^{-i(N-1)x} \\
&= e^{ix}[e^{iNx} - 2 + e^{-iNx}] = e^{ix}(e^{iNx/2} - e^{-iNx/2})^2
\end{aligned}$$

Therefore

$$F_N(x) = \frac{e^{ix}(e^{iNx/2} - e^{-iNx/2})^2}{N(e^{ix} - 1)^2} = \frac{(e^{iNx/2} - e^{-iNx/2})^2}{N(e^{ix/2} - e^{-ix/2})^2} = \frac{2i\sin^2(Nx/2)}{2iN\sin^2(x/2)}$$

Lemma 1. *Approximate Identity Lemma. Let $f \in C(\mathbb{R}/2\pi\mathbb{Z})$, that is, f is a continuous function on $\mathbb{R}$ such that $f(x+2\pi) = f(x)$. Let $K_N(x)$ satisfy*

i) $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} K_N(x)dx = 1$

ii) $\displaystyle\sup_N \int_{-\pi}^{\pi} |K_N(x)|dx \le M$

iii) For any $\delta > 0$, $\displaystyle\int_{\delta \le |x| \le \pi} |K_N(x)|dx \to 0$ *as* $N \to \infty$.

Then

$$\max_x |f(x) - f * K_N(x)| \to 0 \quad \textit{as } N \to \infty$$

Proof. By property (i),

$$\begin{aligned} 2\pi[f * K_N(x) - f(x)] &= \int_{-\pi}^{\pi} K_N(y)(f(x-y) - f(x))dy \\ &= \int_{\delta \le |y| \le \pi} K_N(y)(f(x-y) - f(x))dy \\ &\quad + \int_{|y|<\delta} K_N(y)(f(x-y) - f(x))dy \end{aligned}$$

For any $\epsilon > 0$ choose $\delta > 0$ so that $|f(x-y) - f(x)| \le \epsilon$ for all $|y| \le \delta$. Note that there is such a $\delta > 0$ that works for all x simultaneously because a continuous function on a compact set is uniformly continuous.[1] Next, by property (iii),

$$\left| \int_{\delta \le |y| \le \pi} K_N(y)(f(x-y) - f(x))dy \right| \le 2 \max |f| \int_{\delta \le |y| \le \pi} |K_N(y)|dy \to 0$$

as $N \to \infty$. (The right side is independent of x, so the left side tends to zero uniformly in x.) Finally, using property (ii)

$$\left| \int_{|y|<\delta} K_N(y)(f(x-y) - f(x))dy \right| \le \int_{|y|<\delta} |K_N(y)|\epsilon dy \le M\epsilon$$

It follows that

$$\limsup_{N\to\infty} \max_x |f * K_N(x) - f(x)| \le M\epsilon/2\pi$$

[1] We need this uniform continuity on a larger compact interval than $-\pi \le x - y \le \pi$. It is this step that uses the property $f(-\pi) = f(\pi)$, or, equivalently, that f can be extended to a continuous periodic function on $\mathbb{R}$.

Since $\epsilon > 0$ is arbitrary, this concludes the proof of the lemma.

Fejér's theorem follows once we confirm that F_N satisfies the hypotheses of the lemma. Indeed,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)\,dx = 1$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)\,dx = \frac{1}{N}\sum_{0}^{N-1}\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)\,dx = 1,$$

which confirms (i). Formula (3) shows that $F_N \geq 0$, so

$$\int_{-\pi}^{\pi} |F_N(x)|\,dx = \int_{-\pi}^{\pi} F_N(x)\,dx = 2\pi < \infty$$

To prove (iii), fix $\delta > 0$. For $\delta \leq |x| \leq \pi$,

$$|F_N(x)| \leq \frac{1}{N\sin^2(x/2)} \leq \frac{1}{N\sin^2(\delta/2)} \leq C/N$$

for a constant C depending on δ. Thus the integral in (iii) tends to zero.

Final Remark. Later on, we be able to recognize the way in which F_N is better than D_N by looking at their Fourier series,

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}, \quad F_N(x) = \sum_{n=-N}^{N}\left(1 - \frac{|n|}{N}\right)e^{inx}$$

Let

$$h_1(s) = 1_{[-1,1]}, \quad h_2(s) = (1 - |s|)^+$$

The function h_1 is discontinuous, but h_2 has a bounded first derivative. The Dirichlet and Fejér kernels are

$$D_N(x) = \sum_{n=-\infty}^{\infty} h_1(n/N)e^{inx}, \quad F_N(x) = \sum_{n=-\infty}^{\infty} h_2(n/N)e^{inx},$$

and the fact that h_2 is smoother than h_1 accounts for the improved properties (ii) and (iii) of F_N that fail for D_N.