18.103 Fall 2013

1. Fourier Series, Part 1.

We will consider several function spaces during our study of Fourier series. When we talk about $L^p((-\pi,\pi))$, it will be convenient to include the factor $1/2\pi$ in the norm:

$$\|f\|_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^p\,dx\right)^{1/p}.$$

In particular, the Lebesgue space $L^2((-\pi,\pi))$ is a Hilbert space with inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx\,.$$

The starting place for the theory of Fourier series is that the family of functions $\{e^{inx}\}_{n=-\infty}^{\infty}$ is orthonormal, that is

$$\langle e^{inx}, e^{imx}\rangle = 0,\ n \neq m; \quad \langle e^{inx}, e^{inx}\rangle = 1, \quad n,\ m \in \mathbf{Z}.$$

The Fourier coefficients of f are defined by

$$\hat{f}(n) = \langle f, e^{inx}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx, \quad n \in \mathbf{Z}.$$

($\mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$ represents the integers.) The definition of Fourier the coefficients $\hat{f}(n)$ also makes sense for $f \in L^1((-\pi,\pi))$. The main issue is to find the ways in which the Fourier series

$$\sum \hat{f}(n)e^{inx}$$

represents the function f.

The first basic remark is that for all $f \in L^1((-\pi,\pi))$,

$$|\hat{f}(n)| \leq \|f\|_1 \tag{1}$$

This is proved by putting the absolute value inside the integral:

$$|\hat{f}(n)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|\,dx = \|f\|_1\,.$$

Let $C^k(\mathbf{R})$, $k = 0, 1, 2, \dots$, denote the complex-valued functions that are k times continously differentiable on $\mathbf{R}$. $C(\mathbf{R}) = C^0(\mathbf{R})$ denotes continuous functions on $\mathbf{R}$, and $f \in C^k(\mathbf{R})$ if and only if $f' \in C^{k-1}(\mathbf{R})$. Denote by $C^\infty(\mathbf{R})$ the infinitely differentiable functions on $\mathbf{R}$.

If the function f is periodic of period 2π ($f(x+2\pi) = f(x)$), then f defines a function on $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$, the quotient space of $\mathbf{R}$ under the equivalence relation $x \sim x'$ if $x - x' \in 2\pi\mathbf{Z}$. We will use the notation $C^k(\mathbf{T})$ for C^k functions on $\mathbf{T}$, which are identified with 2π-periodic functions in $C^k(\mathbf{R})$. We will identify functions in $L^p((-\pi,\pi))$ with 2π periodic functions on $\mathbf{R}$ and write $L^p(\mathbf{T})$.

The proof in the preceding set of lecture notes that $C_0^\infty(\mathbf{R})$ is dense in $L^p(\mathbf{R})$, $1 \le p < \infty$, can be modified in a routine way to show that $C^\infty(\mathbf{T})$ is dense in $L^p(\mathbf{T})$, $1 \le p < \infty$. Indeed, the density can be proved using C^∞ functions that are truncated to be zero in a small neighborhood of π (equivalent to $-\pi$).

Proposition 1. *If $f \in C^1(\mathbf{T})$, then*

$$|\hat{f}(n)| \le C/|n|$$

Proof. For $n \neq 0$,

$$\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \int_{-\pi}^{\pi} f(x)\frac{d}{dx}\left(\frac{e^{-inx}}{-in}\right)dx = -\int_{-\pi}^{\pi} f'(x)\frac{e^{-inx}}{-in}\,dx$$

Hence,

$$|\hat{f}(n)| \le \frac{1}{2\pi|n|}\int_{-\pi}^{\pi}|f'(x)|\,dx = \|f'\|_1/|n|$$

Exercise. Show that if $f \in C^k(\mathbf{T})$, then

$$|\hat{f}(n)| \le C/(1+|n|)^k$$

Lemma 1. *(Riemann-Lebesgue Lemma) Suppose that $h \in L^1(\mathbf{T})$. Then*

$$\hat{h}(n) \to 0 \quad \textit{as} \quad |n| \to \infty$$

Proof. Let $\epsilon > 0$, and choose $g \in C^1(\mathbf{T})$ so that

$$\|h - g\|_{L^1(\mathbf{T})} \le \epsilon.$$

By Proposition 1 $\hat{g}(n) \to 0$ as $|n| \to \infty$. Therefore,

$$\limsup_{n\to\infty}|\hat{h}(n)| \le \limsup_{n\to\infty}(|\hat{h}(n) - \hat{g}(n)| + |\hat{g}(n)|) = \limsup_{n\to\infty}|\hat{h}(n) - \hat{g}(n)|.$$

Next note that using (1),

$$|\hat{h}(n) - \hat{g}(n)| \le = \|h - g\|_{L^1(\mathbf{T})} \le \epsilon.$$

Thus we have shown

$$\limsup_{n\to\infty}|\hat{h}(n)| \le \epsilon.$$

And taking the limit as $\epsilon \to 0$ finishes the proof. $\square$

For any $f \in L^1(\mathbf{T})$, we define the partial sum of the Fourier series by

$$s_N(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}.$$

Substituting the formula for $\hat{f}(n)$ into this formula, we find

$$s_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \sum_{n=-N}^{N} e^{in(x-y)}\, dy,$$

which we also write

$$s_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) D_N(x-y)\, dy \quad \text{with} \quad D_N(t) = \sum_{n=-N}^{N} e^{int}.$$

The formula for s_N can be written in more compact form using an important operation $*$ known as convolution.

$$s_N(x) = f * D_N(x) \tag{2}$$

Convolution. In general, for f and g in $L^1(\mathbf{T})$, we define the operation of *convolution* by

$$f * g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x-y)\, dy = \frac{1}{2\pi} \int_{a}^{a+2\pi} f(y) g(x-y)\, dy$$

For such f and g Fubini's theorem implies that $f * g$ defines an integrable function. In particular, $f * g(x)$ is defined and finite for almost every x (and periodic of period 2π). It's easy to see that convolution satisfies the distributive law, $f * (g + h) = f * g + f * h$. One can also confirm, using a change of variable, that the operation is commutative. In other words,

$$f * g(x) = g * f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(y) f(x-y)\, dy$$

There will be more about convolution later.

Theorem 1. *(Dini Test) If $f \in L^1(\mathbf{T})$, and for some fixed x*

$$\int_{-\pi}^{\pi} \frac{|f(x+y) - f(x)|}{|y|} dy < \infty,$$

then $s_N(x) \to f(x)$ as $N \to \infty$.

Proof. (Note that although f is merely an L^1 function, the hypothesis specifies the value of $f(x)$ uniquely.) To prove the theorem observe first that

$$\int_{-\pi}^{\pi} D_N(y)\,dy = \int_{-\pi}^{\pi}\left(\sum_{n=-N}^{N} e^{iny}\right)dy = \int_{-\pi}^{\pi} dy = 2\pi$$

Therefore,

$$\begin{aligned} s_N(x) - f(x) = D_N * f(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(y) f(x-y)\,dy - \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(y) f(x)\,dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x-y) - f(x))D_N(y)\,dy \end{aligned}$$

Furthermore,

$$D_N(y) = \sum_{n=-N}^{N} e^{iny} = \frac{e^{i(N+1)y} - e^{-iNy}}{e^{iy} - 1}$$

Thus

$$s_N(x) - f(x) = \hat{h}_x(N+1) - \hat{h}_x(-N)$$

with

$$h_x(y) = \frac{f(x-y) - f(x)}{e^{iy} - 1}.$$

Since $|e^{iy} - 1| \geq 2|y|/\pi$ for all $|y| \leq \pi$, the hypothesis implies

$$\int_{-\pi}^{\pi} |h_x(y)|\,dy \leq \frac{\pi}{2}\int_{-\pi}^{\pi}\frac{|f(x-y) - f(x)|}{|y|}dy < \infty$$

Therefore, by the Riemann-Lebesgue lemma (Lemma 1)

$$\lim_{N\to\infty} \hat{h}_x(N+1) - \hat{h}_x(-N) = 0$$

and the theorem is proved. □

Corollary 1. *If $f \in C^1(\mathbf{T})$, then*

a) $s_N(x) \to f(x)$ as $N \to \infty$ for all $x \in \mathbf{T}$.

b) $\|s_N - f\|_p \to 0$ as $N \to \infty$, $1 \leq p < \infty$.

Proof. Let $M = \max|f'|$. Then $|f(x-y) - f(x)| \leq M|y|$ so that

$$|h_x(y)| \leq \left|\frac{f(x-y) - f(x)}{e^{iy} - 1}\right| \leq \pi M/2$$

In particular, by the Dini test (Theorem 1), $s_N(x) \to f(x)$. Furthermore, by (1), we have

$$|s_N(x)| \le |\hat{h}_x(N+1)| + |\hat{h}_x(-N)| \le 2\|h_x\|_1 \le M\pi$$

so that $|s_N(x) - f(x)|^p \le (M\pi + |f(x)|)^p$ is a majorant. By the dominated convergence theorem,

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} |s_N(x) - f(x)|^p \, dx = 0$$

□

Exercise. For each α, $0 < \alpha < 1$, define $C^\alpha(\mathbf{T})$ as the collection of 2π periodic functions on $\mathbf{R}$ satisfying

$$|f(x) - f(y)| \le C|x-y|^\alpha, \quad \text{for all} \quad x,\ y \in \mathbf{R}$$

Show that the conclusion of Corollary 1 holds for all $f \in C^\alpha(\mathbf{T})$.

Corollary 2. *The functions e^{inx}, $n \in \mathbf{Z}$ form an orthonormal basis for $L^2(\mathbf{T})$. In particular, for all $f \in L^2(\mathbf{T})$,*

$$\lim_{N\to\infty} \|s_N - f\|_2 = 0, \quad \textit{and} \quad \|f\|_2^2 = \sum_{n\in\mathbf{Z}} |\hat{f}(n)|^2.$$

Proof. Corollary 1 shows that the closure of V in the $L^2(\mathbf{T})$ distance includes all functions in $C^1(\mathbf{T})$. Our density theorem says, in particular, that $C^1(\mathbf{T})$ is dense in $L^2(\mathbf{T})$. Thus V is dense in $L^2(\mathbf{T})$, and this is condition (a) of our theorem characterizing orthonormal bases. □