18.103 Fall 2013

1. Completeness of L^p.

For $1 \le p < \infty$, we define

$$L^p(X, \mu) = \{f : X \to \mathbf{C} : \ f \text{ is measurable and } \int_X |f(x)|^p d\mu(x) < \infty\},$$

but we identify two functions as equal if the differ on a set of zero measure. The norm on L^p is given by

$$\|f\|_p = \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p}.$$

One case of interest is the case in which X is the natural numbers $\mathbf{N} = \{1, 2, \dots\}$ and μ is the counting measure. Then

$$\|f\|_p = \left(\sum_{k=1}^{\infty} |f(k)|^p \right)^{1/p}.$$

Note that if f and g belong to $L^p(X, \mu)$,

$$\int_X |f(x) + g(x)|^p \, d\mu \le \int_X \max(|2f(x)|^p, |2g(x)|^p) \, d\mu \le 2^p \int_X (|f(x)|^p + |g(x)|^p) \, d\mu < \infty,$$

so that $f + g \in L^p(X, \mu)$, and we have

$$\|f + g\|_p^p \le 2^p \|f\|_p^p + 2^p \|g\|_p^p. \tag{1}$$

Let $1 < p < \infty$ and let q be the so-called dual exponent, defined by $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Hölder's inequality (Exercise 7, §3.1, p. 123) says that for every $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$, $fg \in L^1(X, \mu)$ and

$$\|fg\|_1 \le \|f\|_p \|g\|_q.$$

In particular, if μ is the counting measure on $\mathbf{N}$, we have

$$\sum_{k=1}^{\infty} |a_k b_k| \le \left(\sum_{k=1}^{\infty} |a_k|^p \right)^{1/p} \left(\sum_{k=1}^{\infty} |b_k|^q \right)^{1/q} \tag{2}$$

In the exercise that followed (Exercise 8) you deduced the triangle inequality

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

Thus $L^p(X, \mu)$ is a normed vector space.

Theorem 1. *For $1 \le p < \infty$, $L^p(X, \mu)$ is a Banach space.*

The fact that $\| \cdot \|_p$ is a norm follows from Exercise 8. Here we show that the space is complete. Consider a Cauchy sequence f_n, i. e.,

$$\|f_n - f_m\|_p \to 0 \quad \text{as} \quad m,\ n \to \infty$$

Choose $n_1 < n_2 < \cdots$ such that

$$\|f_n - f_m\|_p \le 2^{-2j}, \quad \text{for all } \ m,\ n \ge n_j$$

Let $g_j = f_{n_j}$ and $h_k = g_{k+1} - g_k$. Note that

$$\int_X |h_k|^p \, d\mu = \|h_k\|_p^p \le 2^{-2pk}$$

The only difference between this proof of completeness and the one in the text is the way we show that

$$\sum_{k=1}^{\infty} h_k(x)$$

converges almost everywhere. By (2) applied to $a_k = |h_k(x)| 2^{k/p}$, $b_k = 2^{k/p}$,

$$\sum_{k=1}^{\infty} |h_k(x)| = \sum_{k=1}^{\infty} a_k b_k \le \left(\sum_{k=1}^{\infty} 2^k |h_k(x)|^p \right)^{1/p} \left(\sum_{k=1}^{\infty} 2^{-kq/p} \right)^{1/q}$$

Let

$$C = \left(\sum_{k=1}^{\infty} 2^{-kq/p} \right)^{1/q} < \infty$$

It follows from the monotone convergence theorem that

$$\int_X \left(\sum_{k=1}^{\infty} |h_k(x)| \right)^p d\mu \le C^p \int_X \sum_{k=1}^{\infty} 2^k |h_k(x)|^p d\mu \le C^p \sum_{k=1}^{\infty} 2^k 2^{-2kp} < \infty$$

Therefore,

$$\left(\sum_{k=1}^{\infty} |h_k(x)| \right)^p < \infty$$

for almost every x. For such x, the series $\sum h_k(x)$ is absolutely convergent, and we can define

$$f(x) = g_1(x) + \sum_{k=1}^{\infty} h_k(x) = \lim_{n \to \infty} g_n(x)$$

Set $f(x) = 0$ on the exceptional set of measure 0 where the limit does not exist.

The remaining parts of the argument are nearly the same as in the case of L^1. By Fatou's lemma, for k fixed,

$$2^{-2kp} \geq \liminf_{j\to\infty} \int_X |g_j(x) - g_k(x)|^p \, d\mu \geq \int_X \liminf_{j\to\infty} |g_j(x) - g_k(x)|^p \, d\mu = \int_X |f(x) - g_k(x)|^p \, d\mu$$

In other words,

$$\|f - g_k\|_p \leq 2^{-2k}$$

In particular, for $k = 1$ we have $f - g_1 \in L^p(X, \mu)$ and hence $f = (f - g_1) + g_1 \in L^p(X, \mu)$. Finally, for all $n \geq n_k$,

$$\|f_n - f\|_p \leq \|f_n - g_k\|_p + \|g_k - f\|_p \leq 2^{-2k+1}$$

The space $L^\infty(X, \mu)$ is defined (with the usual equivalence) as the set of measurable functions such that

$$\|f\|_\infty = \text{ess sup}_X |f(x)| = \inf_E \sup_{x\in(X-E)} |f(x)| < \infty$$

where the infimum is taken over all sets E of measure zero. The expression on the right is known as the essential supremum (supremum ignoring sets of measure zero).

Exercise. Show that $L^\infty(X, \mu)$ is a Banach space. (This does not require an accelerated Cauchy sequence. The main issue is to identify the exceptional set of measure zero on which the convergence may fail.)

2. Density in L^p

The space $C_0^\infty(\mathbf{R}^n)$ denotes all infinitely differentiable functions on $\mathbf{R}^n$ that are zero outside a compact set.

Theorem 2. *$C_0^\infty(\mathbf{R}^n)$ is dense in $L^p(\mathbf{R}^n)$ for $1 \leq p < \infty$.*

Proof. Step 1. Approximation of $1_{[0,1]}$. To accomplish this we will find for each ϵ, $0 < \epsilon < 1/2$, a function $h_\epsilon \in C_0^\infty(\mathbf{R})$ satisfying $0 \leq h(x) \leq 1$ for all x, $h_\epsilon(x) = 1$ for $\epsilon \leq x \leq 1 - \epsilon$, and $h_\epsilon(x) = 0$ for all $x \notin [0, 1]$. It follows that

$$\|1_{[0,1]} - h_\epsilon\|_p^p = \int_{\mathbf{R}} |1_{[0,1]} - h_\epsilon(x)|^p dx \leq 2\epsilon$$

Start by defining

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Then f is infinitely differentiable and $f(x) \to 1$ as $x \to \infty$. The function $g(x) = f(x)f(1-x)$ belongs to $C_0^\infty(\mathbf{R})$ is zero outside $[0,1]$ and satisfies $0 < g(x) < 1$ in $0 < x < 1$. Denote

$$c = \int_0^1 g(x)\, dx$$

and define

$$G(x) = \frac{1}{c} \int_0^x g(t)\, dt.$$

Then $G \in C^\infty(\mathbf{R})$, $0 \le G(x) \le 1$ for all x, $G(x) = 0$ for all $x \le 0$, $G(x) = 1$ for all $x \ge 1$. Finally, let

$$h_\epsilon(x) = G(x/\epsilon)G((1-x)/\epsilon).$$

Then $1_{[\epsilon,1-\epsilon]} \le h_\epsilon \le 1_{[0,1]}$, and hence $\|1_{[0,1]} - h_\epsilon\|_p \le (2\epsilon)^{1/p} \to 0$ as $\epsilon \to 0$.

Step 2. Approximate 1_R for rectangles $R = I_1 \times I_2 \times \cdots \times I_n$, $I_j = [a_j, b_j]$ by

$$\prod_{j=1}^n h_\epsilon((x - a_j)/(b_j - a_j))$$

Step 3. Approximate 1_E in case E is a measurable subset of $\mathbf{R}^n$ of finite measure.

Taking sums of functions from Step 2, one can approximate 1_R by functions in $C_0^\infty(\mathbf{R}^n)$ for any R in the rectangle ring (finite union of rectangles). By Theorem 20 (§1.3, p. 34 of the textbook), $\mu(E) < \infty$ implies $E \in \mathcal{M}_F$. Hence there is a sequence R_k in the rectangle ring such that

$$\mu(S(E, R_k)) \to 0 \quad \text{as } k \to \infty$$

where $S(A,B) = (A - B) \cup (B - A)$, the set-theoretical symmetric difference. Moreover, $\|1_E - 1_{R_k}\|_p^p = \mu(S(E, R_k))$, so 1_{R_k} tends to 1_E in $L^p(\mathbf{R}^n)$ for any p, $1 \le p < \infty$.

Step 4. From Step 3, we can approximate any finite linear combination of functions of the form 1_E with $\mu(E) < \infty$ in $L^p(\mathbf{R}^n)$ norm by functions in $C_0^\infty(\mathbf{R}^n)$. Finally, consider any measurable $f : \mathbf{R}^n \to \mathbf{C}$. Then $f = u + iv = (u^+ - u^-) + i(v^+ - v^-)$, and we may apply Theorem 6 (§2.2, page 62) to each of the functions $u^\pm$ and $v^\pm$ to find a sequence of simple functions s_k such that

$$\lim_{k\to\infty} s_k(x) = f(x), \quad |s_k(x)| \le |f(x)|.$$

Note that if $0 \le s \le u^+$ and s is simple, then for any $c > 0$,

$$\mu(\{x \in \mathbf{R}^n : s(x) = c\}) \le \mu(\{x \in \mathbf{R}^n : |f(x)| \ge c\}) \le \frac{1}{c^p} \int_{\mathbf{R}^n} |f|^p\, d\mu < \infty.$$

for $f \in L^p(\mathbf{R}^n)$. Thus s_k is a linear combination of indicator functions 1_E with $\mu(E) < \infty$, and hence each s_k can be approximated, (Thanks to S. M. for pointing out the gap in the preceding version in which we forgot to check this finiteness property of s_k.) Finally, $|s_k(x) - f(x)|^p \leq (2|f(x)|)^p$ is a majorant, and the dominated convergence theorem implies

$$\lim_{k\to\infty} \int_{R^n} |f(x) - s_k(x)|^p dx = 0.$$

This concludes the proof that $C_0^\infty(\mathbf{R}^n)$ is dense in $L^p(\mathbf{R}^n)$. $\square$