Some homological localization theorems

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UIUC, July 17, 2017
The challenge: explain this picture
The challenge: explain this picture

Conway’s Game of Life?
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Conway’s Game of Life? The rings of Saturn?
Slope-by-slope computation of Ext

The Adams $E_2$ term at $p = 3$:
Slope-by-slope computation of Ext

Slopes $1/4$ (old), $1/5$ (quite new)
Slope-by-slope computation of Ext

Slope $1/23$ (next up)

A “unit” map $S \to R$ in spectra determines a diagram

\[
\begin{array}{cccccc}
S & \to & \overline{R} & \to & \overline{R} \land \overline{R} & \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
R & \to & R \land \overline{R} & & & \\
\end{array}
\]

Apply $\pi_*(\_ \land X)$ to get an exact couple and a spectral sequence with

\[
E_1^s = R_*(\overline{R}^s \land X) \Longrightarrow \pi_*(X)
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R & \leftarrow & R \wedge R
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\]

Apply $\pi_*(- \wedge X)$ to get an exact couple and a spectral sequence with

\[
E_1^s = R_*(R^{\wedge s} \wedge X) \Longrightarrow \pi_*(X)
\]

If $R$ is a ring-spectrum such that $R_*R$ is flat over $R_*$, then $R_*R$ is a Hopf algebroid and

\[
E_1^* = C^*(R_*R; R_*X)
\]

– the cobar construction. So in this case

\[
E_2^* = H^*(R_*R; R_*X)
\]

and is determined by $R_*X$ as a comodule over $R_*R$. 
The Adams Spectral Sequence

Example: $R = Hk$, $k = \mathbb{F}_p$. Then $R_*R = A$, the dual Steenrod algebra. Plot filtration degree $s$ vertically and $t - s =$ topological dimension horizontally.
The Adams Spectral Sequence

**Example:** \( R = Hk, k = \mathbb{F}_p \). Then \( R_* R = A \), the dual Steenrod algebra. Plot filtration degree \( s \) vertically and \( t - s \) = topological dimension horizontally. With \( X = S, p = 3 \):
At least we know that there’s a vertical vanishing line: if $M_n = 0$ for $n < 0$ then $H^{s,t}(A; M) = 0$ for $t - s < 0$.

$H^{s,s}(A) = \langle v_0^s \rangle$, where $v_0$ represents $p\iota \in \pi_0(S)$. This acts on $H^*(A; M)$ for any $M$, and we may localize by inverting $v_0$. 
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**Theorem.**

$$H^*(A; M) \to v_0^{-1}H^*(A; M)$$

is iso for $s > c + \frac{t - s}{2p - 2}$, and

$$v_0^{-1}H^*(A; M) = k[v_0^{\pm 1}] \otimes H(M; \beta).$$
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In particular, $v_0^{-1}H^*(A; M)$ depends only on the action of $\beta$ on $M$. Using $A \to E[\tau_0]$, this can be written as

$$v_0^{-1}H^*(A; M) = v_0^{-1}H^*(E[\tau_0]; M).$$
Theorem. Above a line of slope \(1/(2p - 2)\), the Adams spectral sequence coincides with the mod \(p\) homology Bockstein spectral sequence.
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E.g. $H_5(X) = \mathbb{Z}/27$ and is 3-torsion free nearby. Then:
\textit{v}_1\text{-localization (1981)}

\textbf{Corollary.} If \( H(M; \beta) = 0 \), we get a vanishing line of slope \( 1/(2p - 2) \).

\textbf{Example:} (\( p \) odd) \( M = A \square_{A/\tau_0} N \) is Bockstein-acyclic.
For example, if \( N = H_*(X) \), this is \( H_*(V(0) \wedge X) \), where \( V(0) = S \cup_p e^1 \).
Then
\[
H^*(A; M) = H^*(A/\tau_0; N)
\]
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Now $\tau_1 \in A/\tau_0$ is primitive, and produces $v_1 \in H^1(A/\tau_0; N)$ acting on $H^*(A/\tau_0; N)$.

**Theorem.** The localization map

$$H^*(A/\tau_0; N) \to v_1^{-1} H^*(A/\tau_0; N)$$

is an isomorphism above a line of slope $1/(p^2 - p - 1)$ (e.g. $1/5$ if $p = 3$).
(Still $p$ odd) There’s a formula for

$$\nu_1^{-1}H^*(A/\tau_0; N).$$

To state it, note the split Hopf algebra extension

$$E[\tau_1] \to A/\tau_0 \to A/(\tau_0, \tau_1)$$

A comodule over $E[\tau_1]$ is a graded vector space with a differential $\partial$ of degree $-|\tau_1| = 1 - 2p$. This makes $A/(\tau_0, \tau_1)$ into a differential Hopf algebra, and an $A/\tau_0$-comodule is the same thing as a differential $A/(\tau_0, \tau_1)$-comodule.
\( \nu_1 \)-localization (1981)

**Theorem.**

\[
\nu_1^{-1} H^*(A/\tau_0; N) = k[\nu_1^{\pm 1}] \otimes H^*(A/(\tau_0, \tau_1); N).
\]
\( \nu_1 \)-localization (1981)

**Theorem.**

\[
\nu_1^{-1} H^*(A/\tau_0; N) = k[\nu_1^{\pm 1}] \otimes \mathbb{H}^*(A/(\tau_0, \tau_1); N).
\]

There’s a spectral sequence converging to this hypercohomology:

\[
E_2 = H^*(H(A/(\tau_0, \tau_1)); H(N)) \Longrightarrow \mathbb{H}^*(A/(\tau_0, \tau_1); N)
\]

\[
H(A/(\tau_0, \tau_1)) = k[\xi_1, \xi_2, \ldots]/(\xi_1^p, \xi_2^p, \ldots)
\]

With \( N = k \), this spectral sequence collapses and we find that

\[
\nu_1^{-1} E_2^*(V(0)) = k[\nu_1^{\pm 1}] \otimes E[h_{1,0}, h_{2,0}, \ldots] \otimes k[b_{1,0}, b_{2,0}, \ldots]
\]
$\nu_1$-localization (1981)

\[
\nu_1^{-1}E_2^*(V(0)) = k[\nu_1^{\pm 1}] \otimes E[h_1,0, h_2,0, \ldots] \otimes k[b_1,0, b_2,0, \ldots]
\]

In the localized Adams spectral sequence,

\[
d_2 h_{n,0} = \nu_1 b_{n-1,0} + \cdots
\]

resulting in

\[
\nu_1^{-1}\pi_*(V(0)) = k[\nu_1^{\pm 1}] \otimes E[h_{1,0}] .
\]
The $\nu_1$ wedge (2015)

There’s a Bockstein spectral sequence relating $E_2(S \cup_p e^1)$ to $E_2(S)$, and Michael Andrews has worked it out – explaining this picture above a line of slope $1/(p^2 - p - 1)$, or $1/5$ when $p = 3$. 
The \( \nu_1 \) wedge (2015)

Here are Andrews’s Bockstein differentials: Let \( p^{[n]} = \frac{p^n - 1}{p - 1} \).

\[
d_{p^{[n]}} \nu_1^{p^{n-1}} = \nu_1^{-p^{[n-1]}} h_{n,0}
\]

\[
d_{p^n-1}(\nu_1^{-p^{[n]}} h_{n,0}) = \nu_1^{-p \cdot p^{[n]}} b_{n,0}
\]
“$\nu_1$-periodic” $E_2$ term, $p = 7$: slope $1/41$
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From this calculation Andrews deduces Adams differentials above the $1/(p^2 - p - 1)$ line, differentials accounting for the order of $\text{Im} \ J$. To my knowledge these are the first examples with $d_r \neq 0$ for arbitrarily large $r$ in the Adams spectral sequence for the sphere.
Novikov observed that when \( p > 2 \), the dual Steenrod algebra admits a second grading, giving \( \tau_n \) “weight” 1. The result is that the extension spectral sequence for

\[
P \to A \to E[\tau_0, \tau_1, \ldots]
\]
collapses to an isomorphism

\[
H^*(A) = H^*(P; Q)
\]

with

\[
Q = H^*(E[\tau_0, \tau_1, \ldots]) = k[v_0, v_1, \ldots]
\]

\( E_2(S) \) splits into a sum

\[
H^*(A) = \bigoplus_n H^*(P; Q^n)
\]
Reduced powers vanishing line (1981)

For $M$ bounded below, $H^*(P; M)$ exhibits a vanishing line of slope

$$\frac{1}{(p^2 - p - 1)}.$$ 

With $p = 3$ this is $\frac{2}{10} = \frac{1}{5}$.

The primitive element $\xi_1 \in P$ produces

$$h \in H^{1,2(p-1)}(P)$$

and its “transpotence” class

$$b \in H^{2,2p(p-1)}(P)$$

$b$ is non-nilpotent. It acts along the vanishing edge, and

$$H^*(P; M) \to b^{-1}H^*(P; M)$$

is an iso above a line of slope $\frac{1}{(p^3 - p - 1)}$, e.g. $1/23$ for $p = 3$. 
So if we can understand $b^{-1}H^*(P; M)$, at least for $M = Q^n$, we will understand the Adams $E_2$ term above a line of slope $1/(p^3 - p - 1)$, or $1/23$ for $p = 3$: a big improvement. This is just the odd-primary analogue of understanding $v_0^{-1}H^*(A)$ at $p = 2!$
$H^{*}(P)$ for $p = 3$: 
Harvey Margolis (1983) and John Palmieri (2001) set up a stable homotopy category of chain complexes of comodules over a Hopf algebra $P$.

Analogies:

<table>
<thead>
<tr>
<th>Spectra</th>
<th>Comodules</th>
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</thead>
<tbody>
<tr>
<td>$S^0$</td>
<td>$k$</td>
</tr>
<tr>
<td>$Hk$</td>
<td>$P$</td>
</tr>
<tr>
<td>$\wedge$</td>
<td>$\otimes \Delta$</td>
</tr>
<tr>
<td>$\pi_*(X)$</td>
<td>$H^*(P; M)$</td>
</tr>
<tr>
<td>$R_*(X)$</td>
<td>$H^*(P; R \otimes M)$</td>
</tr>
</tbody>
</table>
Margolis-Palmieri Adams spectral sequence

Suppose $R$ is a ring-spectrum; for example a $P$-comodule algebra. We can form

\[
\begin{array}{ccc}
\kappa & \leftarrow & R \\
R & \downarrow & R \\
R \otimes R & \downarrow & R \\
& \leftarrow & R \\
& & R \otimes R
\end{array}
\]

Apply $\pi_\ast(- \otimes M)$ to get an exact couple and a spectral sequence with

\[
E_1^s = \mathbb{H}_\ast(P; R \otimes \overline{R}^{\otimes s} \otimes M) = R_\ast(\overline{R}^{\otimes s} \otimes M) \Longrightarrow H_\ast(P; M)
\]

This replaces the Cartan-Eilenberg spectral sequence

\[
H_\ast(R; H_\ast(D; M)) \Longrightarrow H_\ast(P; M)
\]

which makes sense only when $R$ is the Hopf kernel of a normal map $P \to D$. 
If $R$ is a ring-spectrum such that

$$R_* R = H^*(P; R \otimes R)$$

is flat over

$$R_* = H^*(P; R)$$

then $H^*(P; R \otimes R)$ is a Hopf algebroid and

$$E_1^* = C^*(R_* R; R_* X)$$

– the cobar construction. So in this case

$$E_2^* = H^*(R_* R; R_* X)$$

and is determined by $R_* X$ as a comodule over $R_* R$. 

Try this with the dual reduced powers

\[ P = k[\xi_1, \xi_2, \ldots] \]

and the \( P \)-comodule algebra

\[ R = k[\xi_1^p, \xi_2, \ldots]. \]

That is,

\[ R = P \square_D k \]

where

\[ D = k[\xi_1]/\xi_1^p \]

(\( R \) is the analogue of \( H_*(H\mathbb{Z}) \) as an \( A \)-comodule when \( p = 2 \)).

Then

\[ R_* M = H^*(P; R \otimes M) = H^*(D; M) \]

\[ R_* = H^*(P; R) = H^*(D) = E[h] \otimes k[b]. \]
$R_* M = H^*(D; M)$ is rarely flat over $R_*$, certainly not if $M = R$. But we’re interested in $b^{-1}H^*(P)$, so let’s invert $b$ on $R$. We can invert $b$ on the level of “spectra”: replace $R$ by a fibrant object, represent $b$ by a map $\Sigma^2 R \rightarrow R$, and take the colimit to form a new “2-periodic” ring spectrum $b^{-1}R$ with “homotopy”

$$\mathbb{H}^*(P; b^{-1}R) = b^{-1}H^*(D) = E[h] \otimes k[b^{\pm1}]$$

Its self-homology is

$$b^{-1}R_* R = b^{-1}H^*(D; P)$$

This is still not flat over $b^{-1}R_* = b^{-1}H^*(D) \ldots$
\[ b^{-1}R \]

\[ R_* M = H^*(D; M) \] is rarely flat over \( R_* \), certainly not if \( M = R \). But we’re interested in \( b^{-1}H^*(P) \), so let’s invert \( b \) on \( R \). We can invert \( b \) on the level of “spectra”: replace \( R \) by a fibrant object, represent \( b \) by a map \( \Sigma^2 R \to R \), and take the colimit to form a new “2-periodic” ring spectrum \( b^{-1}R \) with “homotopy”

\[ H^*(P; b^{-1}R) = b^{-1}H^*(D) = E[h] \otimes k[b^{\pm 1}] \]

Its self-homology is

\[ b^{-1}R_* R = b^{-1}H^*(D; P) \]

This is still not flat over \( b^{-1}R_* = b^{-1}H^*(D) \) … unless \( p = 3 \).
For this reason (and others) we’ll take \( p = 3 \) now. So \( |h| = (1, 4) \) and \( |b| = (2, 12) \).

Then the self-homology

\[
b^{-1}H^*(D; P)
\]

is a Hopf algebroid over

\[
b^{-1}H^*(D) = E[h] \otimes k[b^{\pm 1}].
\]
Here’s a wonderful surprise (still for $p = 3$):

**Theorem (Belmont)** There are primitives

$$e_n \in H^{1,2(3^n+1)}(D; P)$$

such that

$$b^{-1}H^*(D; P) = b^{-1}H^*(D) \otimes E[e_2, e_3, \ldots]$$

as Hopf algebras.
Consequently in the localized Margolis-Palmieri Adams spectral sequence

\[ E_2 = b^{-1}H^*(D) \otimes k[w_2, w_3, \ldots] \implies b^{-1}H^*(P). \]
Consequently in the localized Margolis-Palmieri Adams spectral sequence

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E_2 = b^{-1}H^*(D) \otimes k[w_2, w_3, \ldots] \Longrightarrow b^{-1}H^*(P).
\]

If we draw the MPASS in the standard Adams way, \(E_2\) looks like this; \(s - t\) is total cohomological degree.

\[
\begin{array}{cccccc}
& & & & & \\
& & & & \swarrow & \\
& & & w_n & \\
& & b & h & 1 & \\
& -2 & -1 & 0 & 1 & t - s
\end{array}
\]
Comparison with data

The class $w_n$ contributes a $b$-tower in $H^*(P)$ starting in degree $(0, 2(3^n - 5))$: $(0, 8), (0, 44), (0, 154), \ldots$. Here's the polynomial subalgebra generated by $bw_n$'s (marked as $w_n$).
$b^{-1}H^*(P)$

This doesn’t correspond well to our picture of $H^*(P)$; there are differentials in this MPASS. We are still working on this, but we think we know what they are.
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**Conjecture.** Only \( d_4 \) and \( d_8 \) are nontrivial, and

\[
d_4 w_n = h w_2^2 w_{n-1}^3
\]
$E_4$
$E_8$
\section*{D-comodules}

A $D$-comodule structure is a graded vector space $M$ with an operator $\partial : M \to M$ of degree $-4$ such that $\partial^3 = 0$. Define

$$W = k[w_2, w_3, \ldots], \quad |w_n| = 2(3^n - 5),$$

with $D$-comodule-algebra structure determined by

$$\partial w_n = w_2^2 w_{n-1}^3$$

extended as a derivation.

\textbf{Conjecture}

$$b^{-1}H^*(P) = b^{-1}H^*(D; W).$$
This fits the data. For example, it implies that $b^{-1}H^*(P)$ is free over the exterior algebra $E[h]$. We have a sketch of an argument.
This fits the data. For example, it implies that $b^{-1}H^*(P)$ is free over the exterior algebra $E[h]$. We have a sketch of an argument. Moreover, it seems that the MPASS coincides under this isomorphism with the spectral sequence associated with the weight filtration on $W$, putting each $w_n$ in degree 1. Then

$$d_4 x = h \partial x.$$ 

The only remaining nonzero differential is $d_8$, and

$$d_8 (hx) = b \partial^2 x.$$
Acknowledgements

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www.nullhomotopie.de

and Hood Chatham for his spectral sequence package,

www.ctan.org/pkg/spectralsequences
And two announcements

with editorial board including
Benoit Fresse, Sadok Kallel, Haynes Miller, Said Zarati
is open for business, using EditFlow.
Happy birthday, Paul!