Toda’s realization theorem

Haynes Miller
June, 2007

$\mathcal{A}$ is the Steenrod algebra. Let $M$ be an $\mathcal{A}$ module which is bounded below and of finite type. I want to know whether there is a spectrum with this as its cohomology.

Let

\[ M \leftarrow F_0 \xleftarrow{d_0} F_1 \xleftarrow{d_1} \cdots \]

be a free resolution. We may assume that $F_s$ is trivial below dimension $s$ more than the connectivity of $M$. If $M$ is Bockstein-acyclic, the connectivity of $F_{s+1}$ can be chosen to be $2(p - 1)$ larger than the connectivity of $F_s$.

Let $K^s$ be the GEM with $H^*(K^s) = \Sigma^{1-s}F_s$. We have a diagram

\[ K_0 \xrightarrow{d^0} K_1 \xrightarrow{d^1} K_2 \xrightarrow{d^2} \cdots \]

where each $d$ has degree $-1$, which induces the resolution in cohomology. We wish to embed it into a diagram

\[
\begin{array}{cccccccc}
* & = & Y^0 & \xleftarrow{j^0} & Y^1 & \xleftarrow{j^1} & Y^2 & \xleftarrow{j^2} & Y^3 & \cdots \\
& & K^0 & \xleftarrow{d^0} & K^1 & \xleftarrow{d^1} & K^2 & \xleftarrow{d^2} & K^3
\end{array}
\]

where the arrows labelled $i$ and $d$ have degree $-1$. In cohomology, the maps $j$ will fit into a commutative diagram

\[
\begin{array}{cccccccc}
M & = & M & = & M & \cdots \\
H^*(Y^1) = F_0 & \xrightarrow{j_1} & H^*(Y^2) & \xrightarrow{j_2} & H^*(Y^3) & \cdots \\
& \xleftarrow{s_2} & \xrightarrow{p_2} & \xleftarrow{s_3} & \xrightarrow{p_3} & \xleftarrow{s_4}
\end{array}
\]

$Y^1 = \Sigma^{-1}K^0$, and the map $k^1 : Y^1 \rightarrow K^1$ is $d^0$. Let $Y^2$ be the fiber of $k^1$.

We want to factor $d^1 : K^1 \rightarrow \Sigma K^2$ through $i^1 : K^1 \rightarrow \Sigma Y^2$. This can be done since $k^1 : Y^1 \rightarrow K^1$ is just $d^0 : \Sigma^{-1}K^0 \rightarrow K^1$, and $d^1 d^0 = 0$. The map $k^2 : Y^2 \rightarrow K^2$ can be varied by adding a map of the form $Y^2 \xrightarrow{j^1} Y^1 \rightarrow K^2$.

Let $Y^3 \rightarrow Y^2$ be the fiber of $k^2$ and let $i^2 : K^2 \rightarrow \Sigma Y^3$ be the boundary homomorphism.

Next we want to factor $d^2 : K^2 \rightarrow \Sigma K^3$ through the map $i^2 : K^2 \rightarrow \Sigma Y^3$. So I want to know that $k^2$ can be chosen so that $Y^2 \xrightarrow{k^2} K^2 \xrightarrow{d^2} \Sigma K^3$ is null. Since the target is a GEM, it is equivalent to ask that this map be zero in cohomology.
Since \( \text{coker}(F_1 \to F_0) = M \), the long exact sequence for the cofibration sequence \( Y^2 \to Y^1 \to K^1 \) gives exactness of the top row in the diagram

\[
\begin{array}{cccccccc}
0 & \to & M & \xrightarrow{p} & H^*(Y^2) & \xrightarrow{\Sigma^{-1} \ker d_0} & 0 \\
\Sigma^{-1}F_4 & \xrightarrow{d_3} & \Sigma^{-1}F_3 & \xrightarrow{d_2} & H^*(K^2) & = & \Sigma^{-1}F_2 & \xrightarrow{d_1} & \Sigma^{-1}F_1 \\
\end{array}
\]

The composite \( d_1d_2 \) is zero, and the right vertical is a monomorphism, so the composite \( k_2d_2 \) factors through the inclusion \( p : M \to H^*(Y^2) \) by a map \( c : \Sigma^{-1}F_3 \to M \). Since \( pcd_3 = k_3d_2d_3 = 0 \), the map \( c \) is a cocycle representing a class in

\[ \text{Ext}^3_A(M, M) \]

If we assume that this group is zero, then \( c \) is a coboundary, which is to say that it factors through \( d_2 : \Sigma^{-1}F_3 \to H^*(K^2) = \Sigma^{-1}F_2 \) by a map \( b : H^*(K^2) \to M \). The map \( pb : \Sigma^{-1}F_2 \to H^*(Y^2) \) is the effect in cohomology of exactly the sort of map by which we are allowed to alter \( k^2 \); and \( pbd_2 = pc = k_2d_2 \), so if we replace \( k_2 \) by \( k_2 - pb \), then \( k_2d_2 = 0 \), as desired.

Notice that this choice of \( k_2 \) then factors through the surjection \( \Sigma^{-1}F_2 \to \Sigma^{-1}\ker d_0 \), and thus splits the top sequence in the diagram. Let \( s_2 : H^*(Y^2) \to M \) be the corresponding splitting of \( p \).

So \( d^2 \) factors as \( d^2 = k^3i^2 \). The map \( k^3 : Y^3 \to K^3 \) can be varied by any map of the form \( Y^3 \xrightarrow{j^3} Y^2 \xrightarrow{d^2} K^3 \).

Let \( j^3 : Y^4 \to Y^3 \) be the fiber of \( k^3 : Y^3 \to K^3 \).

Next we want to factor \( d^3 : K^3 \to \Sigma K^4 \) through \( i^3 : K^3 \to \Sigma Y^4 \); that is, we want to know that \( k^3 \) can be chosen so that \( Y^3 \xrightarrow{k^3} K^3 \xrightarrow{d^3} \Sigma K^4 \) is null.

For this we need to analyze the cohomology of \( Y^3 \). We have a diagram
\[ \Sigma^{-1} F_2 = H^*(K^2) \rightarrow H^*(Y^2) \rightarrow H^*(Y^3) \rightarrow \Sigma^{-2} F_2 = H^*(\Sigma^{-1} K^2) \rightarrow H^*(\Sigma^{-1} Y^2) \]

in which the straight lines are exact. Chasing it around, we find that the top row in the following sequence is exact.

\[ 0 \rightarrow M \rightarrow H^*(Y^3) \rightarrow \Sigma^{-2} \ker d_1 \rightarrow 0 \]

We are at the inductive step; the composite \( k_2 d_3 \) factors as \( \Sigma^{-1} F_4 \xrightarrow{c} M \xrightarrow{p} H^*(Y^3) \), and \( c \) is a cocycle, determining an element of

\[
\text{Ext}_{A^2}^4(M, M)
\]

If this cohomology class vanishes, the map \( k_3 \) can be altered so that \( k_3 d_3 = 0 \), and the process continues.