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STABLE SPLITTINGS OF STIEFEL MANIFOLDS

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§1. INTRODUCTION

LET F be one of the skewfields $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and consider the Stiefel manifold $V_{n,q}$ of orthonormal q-frames in F^n. We regard this as the space of Hermitian inner-product preserving right F-linear maps from F^q to F^n. Pick a point $\phi_0 \in V_{n,q}$, and define a filtration of $V_{n,q}$ by closed subsets

$$F_k V_{n,q} = \{\phi\colon \dim_F \ker(\phi + \phi_0) \geqq q - k\}.$$

Thus $F_0 V_{n,q} = \{-\phi_0\}$, $F_1 V_{n,q}$ is the usual "generating complex," and $F_q V_{n,q} = V_{n,q}$. In this paper we will show that the strata $F_k - F_{k-1}$ of this filtration are vector bundles, and that the filtration splits stably, so that $V_{n,q}$ is stably equivalent to a wedge of the corresponding Thom spaces. Such a splitting was conjectured in case $q = n$ by C. A. McGibbon.

To describe these Thom spaces, let ad_k denote the adjoint representation of the relevant group G_k ($= O_k$, U_k, or Sp_k) on its Lie algebra. Let can_k denote the canonical representation of G_k on $\mathrm{Hom}_F(F^k, F)$. Let $G_{q,k} = G_q/G_k \times G_{q-k}$ denote the Grassmann manifold of k-planes in F^q. It is the base of a principal G_k-bundle with total space $V_{q,k}$, so for any representation ρ of G_k we may form the associated vector bundle $E(\rho)$ over $G_{q,k}$. Let $G_{q,k}^{\rho}$ denote the resulting Thom space.

THEOREM (A). *There are diffeomorphisms*

$$F_k V_{n,q} - F_{k-1} V_{n,q} \cong E(ad_k \oplus (n-q)\mathrm{can}_k)$$

compatible with the evident projections to $G_{q,k}$.

(B). *There are homeomorphisms*

$$F_k V_{n,q}/F_{k-1} V_{n,q} \cong G_{q,k}^{ad_k \oplus (n-q)\mathrm{can}_k}.$$

(C). *The filtrations split stably, so there are stable homotopy equivalences*

$$V_{n,q} \simeq \bigvee_{k=1}^{q} G_{q,k}^{ad_k \oplus (n-q)\mathrm{can}_k}.$$

When $k = 1$, the Thom space involved is a "stunted quasiprojective space" [2]. In particular, when F is commutative, G_1 is abelian, so ad_1 is trivial and

$$G_{q,1}^{ad_1 \oplus (n-q)\mathrm{can}_1} \cong \Sigma^{d-1} FP^{n-1}/FP^{n-2}$$

where $d = \dim_{\mathbb{R}} F$.

As special cases of Theorem C we mention

$$O_n,\ U_n,\ \text{or } Sp_n \simeq \bigvee_{k=1}^{n} G_{n,k}^{ad_k}$$

$$SO_n \text{ or } SU_n \simeq \bigvee_{k=1}^{n-1} G_{n-1,k}^{ad_k \oplus \mathrm{can}_k}$$

An addendum concerning naturality allows us to pass to a limit (keeping $r = n - q$ fixed). Write $G = \cup G_r$.

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Corollary D. *There are stable homotopy equivalences*

$$G/G_r \simeq \bigvee_{k \geqq 1} BG_r^{ad_k \oplus r\,\mathrm{can}_k}.$$

For example,

$$O \simeq \bigvee_{k \geqq 1} BO_k^{ad_k}$$
$$U \simeq \bigvee_{k \geqq 1} BU_k^{ad_k}$$
$$Sp \simeq \bigvee_{k \geqq 1} BSp_k^{ad_k}$$
$$SO \simeq \bigvee_{k \geqq 1} BO_k^{ad_k \oplus \mathrm{can}_k}$$
$$SU \simeq \bigvee_{k \geqq 1} BU_k^{ad_k \oplus \mathrm{can}_k}$$

Some of these results have been anticipated in the literature. Theorem A is due to T. Frankel [1] in case $q = n$ (i.e., $V_{n,q} = G_q$). He constructs a Morse–Bott function f on $V_{n,q}$ (for any $q \leqq n$) with critical submanifold diffeomorphic to a disjoint union of $G_{q,k}$, $0 \leqq k \leqq q$. It is not hard to see that the negative bundle over $G_{q,k}$ is $E(ad_k \oplus (n-q)\mathrm{can}_k)$, so on general principles a Riemannian metric on $V_{n,q}$ yields a decomposition into subspaces:

$$V_{n,q} \cong \coprod_{k=0}^{q} E(ad_k \oplus (n-q)\mathrm{can}_k).$$

These subspaces are the "stable submanifolds" of the gradient flow of f, associated to the connected components of the critical locus. In case $q = n$, Frankel notes that for any bi-invariant metric this decomposition is as we have described in A. Our proof of Theorem A in general is a modification of his argument, and B is an easy corollary.

We remark that the splitting result C may be expressed by saying that the attaching maps associated to Frankel's Morse–Bott function are stably trivial.

Results related to Theorem C exist in the literature also. I.M. James [2, Prop. 7.10, p. 50] proved that the stunted quasiprojective space $G_{q,1}^{ad_1 \oplus (n-q)\mathrm{can}_1}$ splits off from $V_{n,q}$ stably. There he also raised the question of the structure of the remaining factor. In [3], $\Sigma\, \mathbb{C}P_+^\infty$ is shown to split off from U stably, by a proof akin to the one given here.

Once A and B have been established, the splitting result C follows by extending a suitable suspension of the quotient map

$$h_k\colon F_k V_{n,q} \to G_{q,k}^{ad_k \oplus (n-q)\mathrm{can}_k}$$

to a map from that suspension of all of $V_{n,q}$, satisfying an evident compatibility condition. Not unexpectedly, this is done using a "transfer" or Pontrjagin–Thom construction. The whole proof is geometrical; no homology computations are called for.

Theorems A and B are proved in Section 2, and C is proved in Section 3, with certain lemmas whose proof uses Morse theory postponed to Section 4. Corollary D is checked at the end of Section 3.

I am indebted to Chuck McGibbon, who first brought the question of splitting U_n to my attention, and who proposed the form it might take; to Elias Micha and Bill Richter, for useful conversations; and to Martin Guest, for suggesting the relevance of Morse theory, and pointing out Frankel's work to me.

§2. THE FILTRATION

We fix a choice of ϕ_0: with respect to the standard bases, take

$$\phi_0 = \begin{bmatrix} 1_q \\ 0 \end{bmatrix}$$

where the subscript denotes the size of the matrix. We recall the filtration

$$F_k V_{n,q} = \{\phi\colon \dim\ker(\phi+\phi_0) \geqq q-k\}.$$

Our first step is to blow up $F_k V_{n,q}$ so as to get a manifold. The problem with $F_k V_{n,q}$ is that the $(n-q)$-dimensional subspace V on which ϕ is required to agree with $-\phi_0$ is not well-defined when $\phi \in F_{k-1} V_{n,q}$. To overcome this, we define

$$\Gamma_{n,q,k} = \{(\phi, V)\colon \phi|_V = -\phi_0|_V\} \subseteq V_{n,q} \times G_{q,q-k}. \tag{2.1}$$

This is a submanifold, and the obvious smooth map $\pi_1 : \Gamma_{n,q,k} \to V_{n,q}$ has image equal to $F_k V_{n,q}$. Moreover, if $\phi \in F_k V_{n,q} - F_{k-1} V_{n,q}$, then it has a unique preimage in $\Gamma_{n,q,k}$.

The projection $\pi_2 : \Gamma_{n,q,k} \to G_{q,q-k}$ is clearly a fiber bundle. To be specific, write $\phi \in V_{m,k}$ as $\begin{bmatrix}\phi' \\ \phi''\end{bmatrix}$ where ϕ' is a $k \times k$ matrix and ϕ'' is an $(m-k) \times k$ matrix. Let G_k act on $V_{m,k}$ from the left by means of the formula

$$\mu \cdot \phi = \begin{bmatrix}\mu\phi'\mu^{-1} \\ \phi''\mu^{-1}\end{bmatrix}. \tag{2.2}$$

Write $V^c_{m,k}$ for this G_k-space. Map $\Gamma_{n,q,k}$ to $G_{q,k}$ by composing π_2 with the diffeomorphism p: $G_{q,q-k} \to G_{q,k}$ sending V to $V^\perp$.

LEMMA 2.3. *$\Gamma_{n,q,k}$ is diffeomorphic over $G_{q,k}$ to $V_{q,k} \times_{G_k} V^c_{k+n-q,k}$.*

Proof. Map $G_q \times V_{k+n-q,k} \to \Gamma_{n,q,k}$ by

$$(\alpha, \phi) \mapsto \left(\begin{bmatrix}\alpha & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}\phi' & 0 \\ 0 & -1 \\ \phi'' & 0\end{bmatrix}\alpha^{-1}, \alpha V_0\right)$$

where $V_0 \subseteq F^q$ is the subspace spanned by the first k standard basis vectors. This passes to a diffeomorphism

$$G_q \times_{G_k \times G_{q-k}} V_{k+n-q,k} \to \Gamma_{n,q,k}$$

where we let G_{q-k} act trivially on $V_{k+n-q,k}$. Dividing by G_{q-k} first, the result follows.

The filtration $F_\bullet$ on $V^c_{m,k}$ is preserved by the action of G_k, and consequently we have a filtration of $V_{q,k} \times_{G_k} V^c_{k+n-q,k} \cong \Gamma_{n,q,k}$. The projection π_1: $\Gamma_{n,q,k} \to V_{n,q}$ is filtration-preserving; and we have a relative diffeomorphism

$$V_{q,k} \times_{G_k} (V^c_{k+n-q,k}, F_{k-1} V^c_{k+n-q,k}) \xrightarrow{\cong} (F_k V_{n,q}, F_{k-1} V_{n,q}).$$

We now come to a key fact, whose proof we defer to Section 4. For any representation ρ, let $D(\rho)$ and $S(\rho)$ denote the unit disk and unit sphere, with respect to some invariant metric.

LEMMA 2.4. *There is a G_k-equivariant relative diffeomorphism*

$$(D(\rho_k), S(\rho_k)) \to (V^c_{m,k}, F_{k-1} V^c_{m,k})$$

where $\rho_k = ad_k \oplus (m-k)\mathrm{can}_k$.

We maintain this use of the symbol ρ_k for the rest of the paper.

Theorem A and B now follow from the composite relative diffeomorphism

$$V_{q,k} \times_{G_k} (D(\rho_k), S(\rho_k)) \to (F_k V_{n,q}, F_{k-1} V_{n,q}).$$

§3. THE SPLITTING MAPS

Notice that the homeomorphism

$$F_k V_{n,q} / F_{k-1} V_{n,q} \xrightarrow{\cong} G^{\rho_k}_{q,k} \tag{3.1}$$

may be construed as a Pontrjagin–Thom construction. For we have an embedding

$$i\colon G_{q,k} \to \Gamma_{n,q,k}$$

sending V to $(\phi, V^{\perp})$, where

$$\begin{aligned} \phi|_V &= \phi_0|_V \\ \phi|_{V^\perp} &= -\phi_0|_{V^\perp} . \end{aligned} \tag{3.2}$$

Composing with π_1, we obtain an embedding of $G_{q,k}$ into the submanifold $F_k - F_{k-1}$ of $V_{n,q}$. By Lemma 2.4, this submanifold is a tubular neighborhood of $G_{q,k}$, diffeomorphic to $E(\rho_k)$; and (3.1) is the corresponding collapse map.

Composing with the projection and adjoining a disjoint basepoint, we obtain a map

$$h_k\colon F_k V_{n,q}^+ \to G_{q,k}^{\rho_k};$$

and our next step is to show that a suitable suspension of this map extends over $V_{n,q}^+$. For this we claim:

PROPOSITION 3.3. *The stable normal bundle of* $\pi_1\colon \Gamma_{n,q,q-k} \to V_{n,q}$ *is* $\pi_2^* E(\rho_k)$.

The Pontrjagin–Thom construction then yields the first map in the following composite of stable maps; the second is induced from π_2.

$$s_k\colon V_{n,q}^+ \to \Gamma_{n,q,q-k}^{\pi_2^*\rho_k} \to G_{q,k}^{\rho_k}$$

This will be our splitting map.

To establish (3.3), we will use the involution α of $V_{n,q}$ defined by sending ϕ to $-\phi$. We exploit the following fact, which will be proved in Section 4.

LEMMA 3.4. $F_{q-k} V_{n,q}$ *and* $\alpha F_k V_{n,q}$ *intersect transversely along* $G_{q,q-k}$ *(embedded via* (3.2)*).*

This makes sense since the intersection clearly lies in the manifold $F_{q-k} V_{n,q} - F_{q-k-1} V_{n,q}$. Consider the commutative diagram

$$\begin{array}{ccccc} & & \overset{=}{\frown\!\frown\!\frown} & & \\ G_{q,k} & \xrightarrow{\ \tilde{\imath}\ } & \Gamma_{n,q,q-k} & \xrightarrow{\pi_2} & G_{q,k} \\ \Big\downarrow{\scriptstyle \delta} & & \Big\downarrow{\scriptstyle j} & & \\ F_k V_{n,q} \times G_{q,k} & \xrightarrow{\ \alpha\times 1\ } & V_{n,q} \times G_{q,k} & & \end{array} \tag{3.5}$$

Here the map δ is the diagonal inclusion, defined using (3.2), and $\tilde{\imath} = ip$. Since the image of $\pi_1\colon \Gamma_{n,q,q-k} \to V_{n,q}$ is $F_{q-k} V_{n,q}$. Lemma 3.4 implies that the square is a transverse intersection. Thus

$$\nu(\delta) = \tilde{\imath}^* \nu(j).$$

We will prove the following lemma in a moment.

LEMMA 3.6. *There is a bundle* ξ *over* $G_{q,k}$ *such that* $\nu(j) \cong \pi_2^* \xi$.

We may then calculate ξ, using (2.4):

$$\xi = \tilde{\imath}^* \pi_2^* \xi = \tilde{\imath}^* \nu(j) = \nu(\delta) = E(\rho_k) \oplus \tau(G_{q,k}). \tag{3.7}$$

Pick an embedding

$$e\colon G_{q,k} \hookrightarrow \mathbb{R}^d.$$

The normal bundle of the resulting embedding

$$\hat{\pi}_1 = (1 \times e) \circ j \colon \Gamma_{n,q,q-k} \to V_{n,q} \times \mathbb{R}^d$$

is then, by (3.7),

$$\nu(\hat{\pi}_1) = \nu(j) \oplus \pi_2^* \nu(e) = \pi_2^*(E(\rho_k) \oplus \tau(G_{q,k}) \oplus \nu(e)) = \pi_2^* E(\rho_k) \oplus \underline{\underline{d}}.$$

This completes the proof of Proposition 3.3.

The compatibility diagram

$$\begin{array}{ccc} \Sigma^d F_k V_{n,q}^+ & & \\ \downarrow & \overset{\Sigma^d h_k}{\searrow} & \\ & & \Sigma^d G_{q,k}^{\rho_k} \\ & \underset{s_k}{\nearrow} & \\ \Sigma^d V_{n,q}^+ & & \end{array} \tag{3.8}$$

commutes by construction.

We proceed to the (standard) deduction of Theorem C. Filter the suspension spectrum

$$\bigvee_{k=0}^{q} G_{q,k}^{\rho_k}$$

by letting F_j truncate the wedge at $k = j$. The map

$$s \colon V_{n,q}^+ \to \bigvee_{k=0}^{q} G_{q,k}^{\rho_k}$$

with kth component s_k is then filtration preserving. When we pass to associated quotients, the diagram (3.8) shows that we obtain at each stage the stabilization of a homeomorphism. Thus by induction

$$F_j V_{n,q}^+ \xrightarrow{\simeq} \bigvee_{k=0}^{j} G_{q,k}^{\rho_k}.$$

Remark 3.9. The proof shows that this map exists after $\max\{d_k : 1 \leqq k \leqq j\}$ suspensions, where d_k is the embedding dimension of $G_{q,k}$. The Whitney embedding Theorem gives the estimate

$$d_k \leqq 2dk(q-k), \quad d = \dim_{\mathbb{R}} F.$$

We now return to a proof of Lemma 3.6. For this we consider the diagram

$$\begin{array}{ccc} \Gamma_{n,q,q-k} & \xrightarrow{\pi_2} & G_{q,k} \\ \Big\downarrow j & & \Big\downarrow l \\ V_{n,q} \times G_{q,k} & \xrightarrow{\pi} & V_{n,q,q-k} \end{array}$$

in which

$$V_{n,q,q-k} = \{F^q \supseteq V \xrightarrow{\psi} F^n \colon \dim V = q-k,\ \psi \text{ is inner-product preserving}\}$$

$$\pi(\phi, W) = (F^q \supseteq W^\perp \xrightarrow{\phi|W^\perp} F^n)$$

$$l(W) = (F^q \supseteq W^\perp \xrightarrow{-\phi_0|W^\perp} F^n).$$

The map π is clearly a fiber-bundle projection, and the diagram is a pull-back. It follows that $\nu(j) = \pi_2^* \nu(l)$.

Remark 3.10. Let G be a compact Lie group and P a compact principal G-space with orbit-space B. Let ad denote the adjoint representation of G. Let $\mathrm{End}_G(P)$ denote the space of continuous equivariant endomorphisms of P. Then in [3] ideas of Becker and Schultz are shown to yield a stable map from $\mathrm{End}_G(P)_+$ to B^{ad}. In particular, take $G = G_k$, $P = V_{q,k}$, $B = G_{q,k}$; then we have a stable map

$$\mathrm{End}_{G_k}(V_{q,k})_+ \to G_{q,k}^{ad_k}.$$

Since G_q acts from the left G_k-equivariantly on $V_{q,k}$, we have by composition a stable map $G_q^+ \to G_{q,k}^{ad_k}$. This is precisely the map s_k constructed here, in the special case when $q = n$.

Finally, we turn to the "naturality" condition needed to establish Corollary D. Let $V_{n,q} \to V_{n+1,q+1}$ carry ϕ to $\phi \oplus 1$. Let $G_{q,k} \to G_{q+1,k}$ apply the map $\begin{bmatrix} 1_q \\ 0 \end{bmatrix}: F^q \to F^{q+1}$. This is covered by a G_k-bundle map $V_{q,k} \to V_{q+1,k}$ sending ϕ to $\begin{bmatrix} \phi \\ 0 \end{bmatrix}$, so we get a map $G_{q,k}^{\rho_k} \to G_{q+1,k}^{\rho_k}$ of Thom spaces. We require:

PROPOSITION 3.11. *The diagram*

$$\begin{array}{ccc} V_{n,q}^+ & \longrightarrow & V_{n+1,q+1}^+ \\ \downarrow s_k & & \downarrow s_k \\ G_{q,k}^{\rho_k} & \longrightarrow & G_{q+1,k}^{\rho_k} \end{array}$$

of suspension spectra is homotopy-commutative.

Proof. We consider the diagram

$$\begin{array}{ccc} G_{q,k} & \longrightarrow & G_{q+1,k} \\ \uparrow \pi_2 & & \uparrow \pi_2 \\ \Gamma_{n,q,q-k} & \longrightarrow & \Gamma_{n+1,q+1,q+1-k} \\ \downarrow j & & \downarrow j \\ V_{n,q} \times G_{q,k} & \longrightarrow & V_{n+1,q+1} \times G_{q+1,k} \end{array}$$

We leave it to the reader to check that the bottom square is a transverse intersection. Thus [3] the corresponding diagram involving Pontrjagin–Thom collapses commutes, and (3.11) follows.

Remark 3.12. There are many other canonical, maps relating Stiefel manifolds—composition maps, the James intrinsic maps, direct sums, bundle projections, ... The expression of these maps in terms of our splitting presents an entertaining exercise.

§4. MORSE THEORY

Recall that a smooth real-valued function f on a compact manifold M is called a *Morse–Bott* function when the critical locus C forms a submanifold of M, and the null-space of the Hessian H of f at any point $c \in C$ coincides with the tangent space to C at c. The normal bundle of C in M then splits as $P \oplus N$, where $H|_P$ is positive-definite and $H|_N$ is negative-definite. Standard Morse theory shows that M is homotopy-equivalent to an identification space formed from the bundle N (or dually from the bundle P).

In the presence of a Riemannian metric we may say more, however. For we may then form the gradient ∇f of the Morse function. The set of zeros of this vector-field is exactly C. Let φ denote the associated flow. The *stable submanifold* associated to a critical point c is

$$S(c) = S_f(c) = \left\{ x \in M : \lim_{t \to \infty} \varphi_t x = c \right\}.$$

If $C = \amalg C_i$ is the decomposition into connected components, we let

$$S_i = \cup \{ S(c) : c \in C_i \}.$$

It projects to C_i, and is diffeomorphic over C_i to the vector-bundle $N|C_i$. Dually, the *unstable submanifold* associated to $c \in C$ is $U(c) = S_{-f}(c)$; and we let

$$U_i = \cup \{U(c): c \in C_i\}.$$

It maps to C_i, and is diffeomorphic over C_i to the vector-bundle $P|C_i$. Thus M decomposes into a disjoint union of vector-bundles, in two complementary ways. Note that S_i and U_i intersect transversely along C_i.

Consider the Stiefel manifold $V_{n,q}$. Decompose $\phi \in V_{n,q}$ into $\begin{bmatrix} \phi' \\ \phi'' \end{bmatrix}$, with ϕ' a $q \times q$ matrix and ϕ'' an $(n-q) \times q$ matrix. Let

$$f(\phi) = \operatorname{Re} \operatorname{tr} \phi'.$$

In [1], Frankel shows that this is a Morse–Bott function, and that its critical locus is

$$C = \coprod_{k=0}^{q} G_{q,k}$$

embedded in $V_{n,q}$ via (3.2).

In case $q = n$, we may choose a bi-invariant metric on $V_{n,q} = G_q$. Frankel then shows that the stable submanifold S_k associated to $G_{q,k}$ is, in our notation, $F_k - F_{k-1}$. We must generalize this result.

As usual, we embed $V_{n,q}$ into the space of all $n \times q$ matrices over F. This vector space has a natural Hermitian inner product over F, given by

$$\langle A, B \rangle = \operatorname{tr} A^* B,$$

where A^* is the transpose-conjugate of A, and so a natural inner product over $\mathbb{R}$, given by taking the real part. We give $V_{n,q}$ the induced Riemannian metric.

PROPOSITION 4.1. *The stable submanifold S_k associated to $G_{q,k}$ is*

$$F_k V_{n,q} - F_{k-1} V_{n,q} = \{\phi: \dim \ker(\phi + \phi_0) = q - k\}.$$

This proposition leads immediately to a proof of Lemma 2.4, since the isotropy representation of G_k on the tangent space to $V^c_{n,k}$ at ϕ_0 is clearly ρ_k. In fact, (4.1) gives an alternate proof of Theorems A and B. We have chosen this organization because alternate proofs of (2.4) are sometimes possible (see (4.6) below), and so as to introduce the manifolds $\Gamma_{n,q,k}$.

Since $f(\alpha\phi) = -f(\phi)$, we see also that

$$U_k = \alpha(F_{q-k} V_{n,q} - F_{q-k-1} V_{n,q}) = \{\phi: \dim \ker(\phi - \phi_0) = k\}.$$

Lemma 3.4 follows immediately.

The proof of (4.1) is a matrix calculation, more elementary than the Lie group techniques of [1]. If we let G_q act on $V_{n,q}$ as in (2.2), then our metric is invariant. If we also let G_k act trivially on $\mathbb{R}$, then f is equivariant. Thus ∇f is an equivariant vector field, and the action of G carries stable submanifolds to stable submanifolds. Moreover, each component of the critical locus is an orbit. Thus we may assume $\phi \in G_{q,k} \subset V_{n,q}$ has the special form

$$\phi_{q-k} = \begin{bmatrix} 1_k & 0 \\ 0 & -1_{q-k} \\ 0 & 0 \end{bmatrix}.$$

We claim that

$$S(\phi_{q-k}) = \left\{ \phi = \begin{bmatrix} \phi' & 0 \\ 0 & -1 \\ \phi'' & 0 \end{bmatrix} : \phi \notin F_{k-1} V_{n,q} \right\} \tag{4.2}$$

Write T_k for the right hand side here. Since this lies in $F_k V_{n,q} - F_{k-1} V_{n,q}$, (4.1) follows.

We will prove in a moment that ∇f is tangent to T_k. This implies (4.2). Here is one argument for this, which I owe to W. Richter. By induction on n, we may assume (4.1) for smaller n. Suppose first $k < q$. Notice that $\bar{T}_k$ is a submanifold of $V_{n,q}$ diffeomorphic to $V_{n-q+k,k}$, and that the metric and the Morse function on $V_{n,q}$ restrict to the corresponding structures on $V_{n-q+k,k}$. By our inductive assumption, we know that the stable manifold for $f|\bar{T}_k$, associated to $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \in V_{n-q+k,k}$, is T_k. Since ∇f is tangent to $\bar{T}_k$, $\nabla(f|\bar{T}_k) = (\nabla f)|\bar{T}_k$; so $T_k = S(\phi_{q-k}) \cap \bar{T}_k$. But Frankel shows that the index of ϕ_{q-k} is $\dim \bar{T}_k$, so we conclude that $T_k = S(\phi_{q-k})$. The case $k = q$ now follows, since T_q is the complement of the union of the stable submanifolds associated to the nonmaximal critical points.

So consider $\phi \in T_k$, and let β be a tangent vector to $V_{n,q}$ at ϕ. Since $V_{n,q}$ is a submanifold of the vector space of $n \times q$ matrices over F, β is such a matrix. The defining equation for $V_{n,q}$ yields the equation

$$\beta^* \phi + \phi^* \beta = 0. \tag{4.3}$$

If we write

$$\beta = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix}$$

using the same decomposition of $n \times q$ matrices as used above in defining ϕ_k, then (4.3) results in four equations, two of which are

$$\beta_{11}^* \phi' + \beta_{31}^* \phi'' + \phi'^* \beta_{11} + \phi''^* \beta_{31} = 0 \tag{4.4}$$

$$\beta_{22}^* + \beta_{22} = 0. \tag{4.5}$$

Now assume β is orthogonal to T_k at ϕ. Then we want to show that

$$\mathrm{d}f(\beta) = 0;$$

that is,

$$\operatorname{Re} \operatorname{tr} \beta_{11} + \operatorname{Re} \operatorname{tr} \beta_{22} = 0.$$

The second term is zero by (4.5). As to the first term, we claim that in fact $\beta_{11} = 0$. To see this, take γ tangent to T_k at ϕ. Then γ must have the form

$$\gamma = \begin{bmatrix} \gamma_{11} & 0 \\ 0 & 0 \\ \gamma_{31} & 0 \end{bmatrix},$$

and the pair $(\gamma_{11}, \gamma_{31})$ is subject only to (4.4) (with γ replacing β). Since β is orthogonal to any such γ, it is orthogonal in particular to

$$\gamma = \begin{bmatrix} \beta_{11} & 0 \\ 0 & 0 \\ \beta_{31} & 0 \end{bmatrix}.$$

This forces $\beta_{11} = 0$ and $\beta_{31} = 0$, and completes the proof.

Remark 4.6. One may hope to prove Lemma 2.4 by showing that $F_{k-1} V_{m,k}$ is the cut locus of ϕ_0 with respect to a suitable Riemannian metric. While this does not seem to be known in general, it is easy to see in case $m = k$, so $V_{m,k} = G_k$. Take $F = \mathbb{C}$, for instance. Give the Lie algebra u_k of $G_k = U_k$ the invariant inner product $\langle A, B \rangle = \operatorname{tr} A^* B$. This defines an invariant Riemannian metric on U_k, and $F_{k-1} U_k$ is the cut locus of 1 with respect to this metric. Indeed, it is easy to see that the set of matrices $A \in u_k$ all of whose eigenvalues are of modulus less than π maps diffeomorphically under the exponential map to the complement of $F_{k-1} U_k$.

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