Localization in homotopy theory
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Abstract. I will attempt to describe some of the ideas of chromatic homotopy theory, focusing on the role of periodicity. I will then describe "non-chromatic" periodicity operators that occur in motivic homotopy theory. All that is new here is joint work with Michael Andrews or work of his alone.

I want to thank Niko Naumann for this invitation: it's certainly an honor to give a lecture here in Regensburg honoring the memory of Johannes Kepler!

I have in mind a naive meaning for "localization," namely, the effect of inverting an operator. I want to talk about several local computations in homotopy theory. Some are quite old, and some are newer – motivic – and represent work of Michael Andrews and some joint work I've done with him.

At heart, homotopy theory is the study of "meso-scale" phenomena in geometry: perhaps high dimensional but certainly not infinite dimensional. And it has a computational inclination. For example we want to know $\pi_k(S^n)$. Homotopy theory is much more than merely a febrile attempt to compute these abelian groups, which is just as well because, for given $n > 1$, we never will know than a modest finite number of them in terms of generators and relations. But "chromatic homotopy theory" provides certain structural features of the homotopy theory of finite complexes that shed light on regularities among these groups, and it leads us to consider certain natural localizations.

Ideas of localization have pervaded homotopy theory for a long time. There are two preliminary localizations:

Stabilization. There is an operation on pointed spaces $(X, *)$ which preserves homology: Form the reduced cone on X and then divide by X. This is the "suspension" of X, and

$$\overline{H}_n(X) \xrightarrow{\cong} \overline{H}_{n+1}(\Sigma X)\,.$$

So for example $\Sigma S^{n-1} = S^n$. Suspension kills the cup product in cohomology; but otherwise the spaces X and ΣX look quite similar. There is a rather simple process by which we may invert the operator Σ acting on pointed spaces. We receive thereby the "Stable Homotopy Category" **S**, along with a functor

$$\Sigma^\infty : \mathrm{Ho}(\mathbf{Top}_*) \to \mathbf{S}$$

The objects in **S** are (unfortunately) called "spectra." **S** is an additive category, and the mapping cone construction provides it with a triangulation. It's a kind of non-abelian derived category. I might use curly brackets to denote Hom in this category. Because I've inverted suspension, there is a sphere S^n for every $n \in \mathbb{Z}$.

Since homology, or generalized homology, commutes with suspension, we may define the homology of a spectrum. For example $\pi_n(X) = \{S^n, X\}$, "stable homotopy," is a homology theory, with coefficient ring $\pi_*(S^0)$. Similarly with cohomology. "Brown representability" asserts that cohomology – in fact any generalized cohomology theory – is representable in this category. Also, the smash product (product modulo axes) descends to a symmetric monoidal structure – it's a tensor triangulated category, with unit given by S^0. Whitehead showed that any homology theory is of the form

$$X \mapsto \pi_*(E \wedge X)$$

for some spectrum E; for example, ordinary rational homology is

$$H_*(X; \mathbb{Q}) = \pi_*(H\mathbb{Q} \wedge X)$$

where $H\mathbb{Q}$ is an object called the (rational) Eilenberg Mac Lane spectrum.

Arithmetic localization. Once we've eliminated the fundamental group, the homotopy type of a space can be analyzed one prime at a time. Serre described a way to localize at a prime p on the algebra side, and later Sullivan, Quillen, and Bousfield pushed this construction into topology. In **S**, we can imitate one construction of $\mathbb{Z}_{(p)}$: form the sequence of spectra

$$X \xrightarrow{n_1} X \xrightarrow{n_2} \cdots$$

where the numbers $n_1, n_2, \ldots$ run through all the positive integers prime to p. Then we take the direct limit. (Actually, we want to form a homotopy theoretic version of the direct limit, making the construction homotopy invariant. Appropriately for this occasion, it's known as a *telescope*.) We are inverting all primes other than p, and we'll write $X_{(p)}$ for the result.

We will consider p-local spectra only; S^0 is shorthand for $S^0_{(p)}$, for example.

Here are two avatars of the kind of localization theorem I have in mind. First, what is

$$p^{-1}\pi_*(X) = \pi_*(X) \otimes \mathbb{Q}\,?$$

Serre gave an effective way of determining these "p-periodic" elements:

$$p^{-1}\pi_*(X) = \pi_*(X) \otimes \mathbb{Q} = H_*(X;\mathbb{Q})$$

This has a geometric version: the telescope $p^{-1}X$ of

$$X \xrightarrow{p} X \xrightarrow{p} X \longrightarrow \cdots \longrightarrow p^{-1}X$$

satisfies

$$\pi_*(p^{-1}X) = p^{-1}\pi_*(X)$$

[**Serre**] $p^{-1}X \xrightarrow{\simeq} L_{H\mathbb{Q}_*}X = H\mathbb{Q} \wedge X$.

Second, inverting p is the only interesting thing to do, by virtue of:

[**Nishida**] The nonzero elements of $\pi_0(S^0)$ exhaust the non-nilpotent elements in the graded ring $\pi_*(S^0)$.

Chromatic homotopy theory provides generalizations of these two facts.

Localization of the Moore spectrum. Let's look at low-dimensional stable homotopy of the sphere spectrum. Of course $\pi_0(S^0) = \mathbb{Z}$, given by degree. Next, the Hopf map $\eta : S^3 \to S^2$ stabilizes to an element of order 2 that generates $\pi_1(S^0)$. This famous element has a cousin at every prime p, less famous but no less important: α_1, a generator of the lowest-dimensional p-torsion. It is of order p for every prime p;

$$\pi_{2p-3}(S^0) \cong \mathbb{Z}/p\mathbb{Z}\,.$$

For large p this group would be very complicated if we had not tensored with $\mathbb{Z}_{(p)}$!

Suspend α_1 once: $\alpha_1 : S^{2p-2} \to S^1$. Saying that $p\alpha_1 = 0$ is the same as saying that the diagonal factorization occurs in

$$\begin{array}{ccc} S^{2p-2}/p & \overset{v_1}{\dashrightarrow} & S^0/p \\ {\scriptstyle\iota}\uparrow & \searrow & \downarrow{\scriptstyle\pi} \\ S^{2p-2} & \xrightarrow{\alpha_1} & S^1 \end{array}$$

where S^{2p-2}/p denotes the effect of "coning off" the map $p : S^{2p-2} \to S^{2p-2}$:

$$S^{2p-2}/p = S^{2p-2} \cup_p e^{2p-1} \,.$$

For p odd this new map also has order p, and we receive a further factorization through a self-map of S^0/p, of degree $2p-2$, written v_1.

The map v_1 is certainly essential, since α_1 is. The interesting thing is that it is *non-nilpotent*: All of the composites

$$S^{2p-2}/p \longrightarrow S^0/p \longrightarrow S^{-(2p-2)}/p \longrightarrow S^{-2(2p-2)}/p \longrightarrow \cdots$$

are essential. You can't see this using homology! But there are other homology theories, and in fact v_1 induces an isomorphism in complex K-theory.

A principle interest in non-nilpotent self-maps is that they give rise to infinite families of elements in $\pi_*(S^0)$:

$$\begin{array}{ccc} S^{(2p-2)k}/p & \xrightarrow{v_1^k} & S^0/p \\ \uparrow & & \downarrow \\ S^{(2p-2)k} & \xrightarrow{\alpha_k} & S^1 \end{array}$$

each of which which turns out to be essential.

We got an estimate of the size of the collection of "p-periodic" families by forming the telescope $p^{-1}X$. Just so, here I can form the "mapping telescope" of this sequence to obtain a spectrum

$$v_1^{-1}S^0/p \,.$$

This is a *periodic* spectrum: it is isomorphic to its own $(2p-2)$-fold suspension. It has trivial homology but the map from S^0/p to it is a K-theory isomorphism so it's certainly nontrivial. Here are a couple of homotopy classes in $\pi_*(S^0/p)$:

$$\begin{array}{ccc} S^0 & \xrightarrow{\iota} & S^0/p \\ {\scriptstyle \alpha_1}\uparrow & \nearrow {\scriptstyle \iota\alpha_1} & \\ S^{2p-3} & & \end{array}$$

Computation. $v_1^{-1}\pi_*(S^0/p) = \pi_*(v_1^{-1}S^0/p) = \mathbb{F}_p[v_1^{\pm 1}]\langle \iota, \iota\alpha_1\rangle$.

Since $S^0/p \to v_1^{-1}S^0/p$ is a K-theory isomorphism, we get a canonical map, which turns out to be an equivalence:

Theorem. $v_1^{-1}S^0/p \xrightarrow{\simeq} L_K S^0/p$.

Question. Is there a more general construction, analogous to our expression of Serre's theorem, valid for all spectra? We shall see.

Adams, Smith, Toda. Something similar happens at $p = 2$; it's just a little more complicated. Adams constructed a K-theory isomorphism

$$v_1^4 : S^8/2 \to S^0/2\,,$$

and Mahowald computed $v_1^{-1}\pi_*(S^0/2)$. Adams also considered Moore spectra with larger cyclic groups, using the J-homomorphism.

Now let's see: $p : S^0 \to S^0$ and then (if $p > 2$) $v_1 : S^{2p-2}/p \to S^0/p$. This suggests that the the mapping cone

$$S^0/p, v_1$$

might have an interesting self-map of it. Larry Smith found one, for $p > 3$:

$$v_2 : S^{2p^2-2}/p, v_1 \to S^0/p, v_1\,.$$

Both sides have trivial K-theory now, so he used MU to detect non-nilpotence; v_2 acts non-nilpotently in MU_*. Today we could equally well say that it is an isomorphism in a Morava K-theory. Morava K-theories form a family of ring spectra associated with complex bordism and hence controlled by the theory of one-dimensional formal groups. There's one for each n, with

$$K(n)_*(S^0) = \mathbb{F}_p[v_n^{\pm 1}]\,, \quad |v_n| = 2p^n - 2\,.$$

In any case, this self-map produced elements $\beta_n \in \pi_*(S^0)$ occuring every $2(p^2-1)$ dimensions. They are the surface of an extremely complex family of elements which are still far from understood. At this second level, the theories known as "elliptic cohomomology" come into play, and the spectrum of topological modular forms.

And we can consider the telescope $v_2^{-1}S^0/p, v_1$, and its homotopy

$$v_2^{-1}\pi_*(S^0/p, v_1).$$

We do not know what this module is.

This was hard work. It was continued by Toda and has been taken up by others to the present day – Behrens, Hopkins, Mahowald, Hill, ... It seemed that these finite complexes were very special; they enjoyed an extra symmetry in the form of an interesting non-nilpotent self-map.

Ravenel's conjectures; Hopkins-Smith. In the late 1970s Doug Ravenel formulated a series of conjectures, most of which were soon verified in work by Mike Hopkins and Jeff [unrelated to Larry] Smith. First of all,

Theorem. If X is finite and $f : \Sigma^? X \to X$ is nilpotent in MU_*, then f is nilpotent.

This generalizes Nishida's theorem. (Niko Naumann and collaborators have recently proven a different generalization of Nishida's theorem.) But there's more. Say that a finite spectrum X has "type n" if

$$K(i)_*(X) = 0 \quad \text{for} \quad i < n\,.$$

Theorem. (1) Every finite spectrum is of some type; only the contractible spectrum is of type n for every n; and for any n there are finite spectra of type n and not of type $n+1$.

(2) Let X be of type n. For some $k \geq 0$ there is a map

$$\phi : \Sigma^? X \to X$$

that induces multiplication by $v_n^{p^k}$ on $K(n)_*(X)$ and zero on $K(m)*(X)$ for $m \neq n$: a "v_n-self-map."

(3) v_n-self-maps are essentially canonical: Let $f : X \to Y$ be any map of type n spectra, let ϕ_X be any v_n-self-map of X and ϕ_Y any v_n-self-map of Y.

Then for some l and m, the following diagram commutes (in which I omit suspensions):

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \phi_X^{p^l}} & & \downarrow{\scriptstyle \phi_Y^{p^m}} \\ X & \xrightarrow{f} & Y \end{array}$$

So there's nothing special about these "generalized Moore spectra"! There's a *universal* "almost symmetry" on finite complexes of type n. The verification of these conjectures produced a revolution in our understanding of stable homotopy theory.

Among other things, this work shows that the telescopes described above are canonically associated to the finite spectrum you started with. In fact they are cases of a general construction. There's an exact triangle

$$\Sigma^{-1}X/\phi \longrightarrow X \xrightarrow{\phi} X$$

and since ϕ is a $K(n)$-isomorphism, and certainly a $K(i)$-isomorphism for $i < n$, the spectrum X/ϕ is a finite complex of type $(n+1)$. In general you can start with any spectrum X; cone off all maps from finite type n_1-spectra to get $X \to X_1$; and continue. In the direct limit you get the "finite localization"

$$X \to L_n^f X$$

with respect to $K(n)$. This is a canonical construction, functorial in the spectrum X, that may be arbitrary. But if X is finite of type n and ϕ is a v_n-self map, then

$$\phi^{-1}X = L_n^f X\,.$$

The finite localization is a "geometric" construction, reflecting properties of finite complexes.

There's another localization at play here, namely Bousfield localization. Bousfield showed this: For any homology theory E_*, there is a functor $X \mapsto L_{E_*}X$ and a natural transformation $X \to L_{E_*}X$ that is, in the homotopy category, terminal among E_*-isomorphisms out of X. This is obtained also by coning off maps from E_*-acyclics, but not necessarily finite ones. The construction is more elaborate, but the result is more computable. We'll write L_n for the localization functor analogus to L_n^f, that is,

$$L_n = L_{K(0)\vee K(1)\vee\cdots K(n)}\,.$$

For example, L_0 is rationalization.

Since $X \to L_n^f X$ is an isomorphism in $K(i)_*$ for $i \leq n$, we have the factorization

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ L_n^f X & \xrightarrow{\theta} & L_n X \end{array}$$

The right hand side here is amenable to attack by high powered machinery. In the case $n = 1$ for example it's explictly constructible by means of Adams operations. The Computation above (and an analogue at $p = 2$ due to Mark Mahowald) leads to the

Telescope Theorem. $L_1^f X \xrightarrow{\cong} L_1 X$.

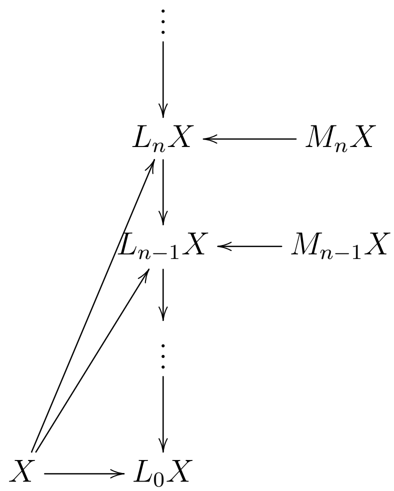
This is an analogue of Serre's theorem.

For $n > 1$ the whole of the theory of formal groups can be brought to bear on the homotopy of $L_n X$. For example, we know that (for $p \geq 5$) $\pi_*(L_2 S^0/p, v_1)$ is known, and quite interesting – computations using p-adic analytic groups show that it is of dimension 12 over $\mathbb{F}_p[v_2^{\pm 1}]$. But we don't know how it's related to $v_2^{-1}\pi_*(S^0/p, v_1)$. So the one conjecture remaining in Doug Ravenel's list is very important!

Telescope Question: Is the map θ an equivalence?

In my view this is the major outstanding question in chromatic homotopy theory. A single finite complex of type n but not $n+1$ for which the conjecture is true would verify it for all such. But Ravenel himself, and others, have spent a great deal of effort on it, and have presented evidence that it is probably false for $n > 1$. So perhaps it's like Kepler's zeroth law, the one about Platonic solids.

Because they localize at larger and larger spectra, the L_n's fit into a tower, the "chromatic tower":



This is like a prism, breaking the white light of X into its constituent wavelengths.

Novikov. There is a systematic way to use operations in a cohomology theory to get more information than merely evaluating it on a map: the Adams spectral sequence. Since MU detects non-nilpotence, the Adams spectral sequence based on MU is particularly adapted to picking out chromatic phenomena. It takes the form

$$E_2^s(X;MU) = \mathrm{Ext}^s_{MU_*MU}(MU_*, MU_*(X)) \Longrightarrow \pi_*(X)\,.$$

Long ago, Doug Ravenel, Steve Wilson and I used this spectral sequence to study chromatic phenomena.

While every non-nilpotent self-map of X (for X finite) is detected on the 0-line, there are lots of non-nilpotent elements in E_2. For example, when $p = 2$ the element detecting η is non-nilpotent! The localization $\eta^{-1}E_2(S^0;MU)$ is "non-chromatic," and not directly amenable to the standard methods.

The 0-line contains only $\mathbb{Z}$ in dimension zero. Along the 1-line, the groups are trivial for n odd, and in his initiating work on this spectral sequence Novikov proved that the other groups are cyclic and computed their order. Doug, Steve, and I showed that with the exception of the one in $E_2^{1,4}$ all the generators of these cyclic groups support η towers.

A new result:

Theorem (with Michael Andrews) These generate the η-free part of E_2.

Our method provides suggestive names for the generators positioned along the zero-line: $v_1^2, v_2, v_1^4, \cdots$, so

$$\eta^{-1}E_2(S^0) = \mathbb{F}_2[\eta^{\pm 1}, v_1^2, v_2]/v_2^2 \quad |v_1^2| = (0,4), |v_2| = (0,6)\,.$$

Then

$$d_3 v_1^2 = \eta^3$$

terminates the spectral sequence.

Motivic homotopy theory. We now know that very nearby there are worldsheets containing alternate realities, homotopy theories very much like the one we inhabit but different. They are selected by a choice of base field k. These "motivic" homotopy theories, denoted $\mathbf{S}(k)$, were defined by Morel and Voevodsky, and have been much studied in the past decade or so. I am a novice at this, but Voevodsky and others have done enough to let a simple homotopy theorist such as myself have some fun here.

The first thing to know is that homotopy groups are naturally *bi*-graded, and the Hopf map lies in $\pi_{1,1}(S^0)$. The second thing to realize is:

η is non-nilpotent motivically.

This explains why MU thought that η was non-nilpotent; MU knew about motivic homotopy theory long before we did.

In case $k = \mathbb{R}$ or $\mathbb{C}$, there are realization functors $\mathbf{S}(\mathbb{R}) \to \mathbf{S}$ and $\mathbf{S}(\mathbb{C}) \to \mathbf{S}$, called "taking real" or "complex points."

It turns out there is a motivic analogue of the Novikov spectral sequence, analyzed by Hu, Kriz, and Ormsby and by Dugger and Isaksen. Its E_2 term is the classical one extended by a polynomial generator τ; so the computation we did above gives a computation of the η localization of the motivic Novikov E_2 term. Again, there is a d_3, but now

$$d_3 v_1^2 = \tau\eta^3$$

and we end up with the following computation, verifying a conjecture of Guillou and Isaksen:

Theorem (with Andrews). $\eta^{-1}\pi_{*,*}(S^0) = \pi_{*,*}(\eta^{-1}S^0) = \mathbb{F}_2[\eta^{\pm 1}, v_1^4, v_2]/v_2^2$.

So there are non-chromatic periodic operators in motivic homotopy. A related study of thick subcategories of the motivic homotopy category has been made by Ruth Joachimi in her thesis.

The chromatic experience leads one to ask whether there are any other non-nilpotent self-maps of the motivic sphere (unknown), and to wonder whether now S^0/η has an interesting non-nilpotent self-map.

Theorem (Andrews). There is a non-nilpotent self-map

$$w_1 : S^{20,12}/\eta \to S^{0,0}/\eta\,.$$

A periodicity operator of this type was studied many years ago, by Harvey Margolis and Mark Mahowald. Apparently they also knew about the existence of motivic homotopy theory long before the rest of us did.

In fact Andrews has a sketch of a construction of a spectrum X_n in $\mathbf{S}(\mathbb{R})$ with a self-map whose real points give a v_n self map of a type n spectrum, and which base-changes to $\mathbf{S}(\mathbb{C})$ to an operator with properties extending those of w_1. The detection scheme he used for powers of w_1 doesn't work anymore, though, so we don't yet know whether w_n is non-nilpotent for $n > 1$.

My guess is that there is a second series of non-nilpotent operators in motivic homotopy over $\mathbb{C}$, reflecting the existence of the real subfield. v_n corresponds to Milnor's ξ_{n+1}. The element η could be written w_0. It corresponds to ξ_1^2, and the higher w_n's should correspond to the squares of the other ξ_n's. Since it is beyond the standard color range of chromatic homotopy theory, I propose to call it "technicolor." Technicolor was an MIT high tech spin-off, introduced one century ago, in 1916. The current MIT campus opened in Cambridge opened the same year. This cinematic technique did not use colored pigments. Instead, each frame had two pictures. A red beam was projected through one and a green beam through the other. Just so; we have the chromatic family, in green, but now also a parallel family, in red.