Nonabelian cohomology and obstructions, following Wojtkowiak [2]

Haynes Miller
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Let D be a small category. We define a variety of cohomology objects. Each starts with a functor to a category of "coefficents," and produces a different sort of object as output. The coefficient categories are: sets **Set**, groups **Gp**, abelian groups **Ab**, and "bands" **HGp**, that is, the category whose objects are groups and whose morphisms are conjugacy classes of homomorphisms.

Let $S : D \to \mathbf{Set}$ be a contravariant functor. A 0-*cocycle* is a system $w_a \in S(a)$ of elements such that for all $a \xrightarrow{\alpha} b$, $\alpha^* w_b = w_a$. The set of 0-cocycles coincides with the zero-dimensional cohomology set $H^0(D; S)$, and is just the inverse limit of the functor S.

If D is a group, S is a D-set and $H^0(D; S)$ is the subset of fixed points.

Let $G : D \to \mathbf{Gp}$ be a contravariant functor. We define the groupoid of 1-*cocycles*, $Z^1(D; G)$, as follows. An object is a choice of $f(\alpha) \in G(a)$ for each $a \xrightarrow{\alpha} b$, such that for all $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$,

$$f(\beta\alpha) = \alpha^* f(\beta) \cdot f(\alpha).$$

(Note that in particular $f(1_a) = f(1_a) \cdot f(1_a)$, which implies that $f(1_a) = 1 \in G(a)$ for all a.) A morphism $f \to f'$ is a choice of $h(a) \in G(a)$ for each a, such that for all $a \xrightarrow{\alpha} b$

$$h(a) \cdot f(\alpha) = f'(\alpha) \cdot \alpha^* h(b).$$

Composition is given by $(hk)(a) = h(a) \cdot k(a)$.

As an example, the *trivial cocycle* is f_0 given by $f_0(\alpha) = 1 \in G(a)$ for all $a \xrightarrow{\alpha} b$.

$H^1(D; G)$ is the set of components of $Z^1(D; G)$. It is a pointed set, with distinguished point given by the class of the trivial cocycle.

If D is a group, an object of $Z^1(D; G)$ is a crossed homomorphism from D to G. Isomorphism in $Z^1(D; G)$ is the usual equivalence relation, and those equivalent to f_0 are "principal."

Let $\Phi : D \to \mathbf{HGp}$ be a contravariant functor. We define the groupoid of 2-*cocycles*, $Z^2(D; \Phi)$, as follows. An object is a pair (F, f), where F is a choice of $F(\alpha) \in \mathrm{Hom}(\Phi(b), \Phi(a))$ for each $a \xrightarrow{\alpha} b$, and f is a choice of $f(\alpha, \beta) \in \Phi(a)$ for each $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$, which satifies the following conditions.

(0) For all a, $F(1_a) = 1_{\Phi(a)}$ and $f(1_a, 1_a) = 1 \in \Phi(a)$.

(1) For all $a \xrightarrow{\alpha} b$, $F(\alpha)$ is a representative of $\Phi(\alpha)$.

(2) For all $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$,

$$F(\alpha) \circ F(\beta) = c_{f(\alpha,\beta)} \circ F(\beta\alpha) \in \mathrm{Hom}(\Phi(c), \Phi(a)),$$

where for an element g of a group c_g denotes conjugation by that element.

(3) For all $a \xrightarrow{\alpha} b \xrightarrow{\beta} c \xrightarrow{\gamma} d$,

$$F(\alpha)(f(\beta, \gamma)) \cdot f(\alpha, \gamma\beta) = f(\alpha, \beta) \cdot f(\beta\alpha, \gamma) \in \Phi(a).$$

These conditions imply that for all $a \xrightarrow{\alpha} b$, $f(\alpha, 1_b) = 1 \in \Phi(a)$, and for all $b \xrightarrow{\beta} c$, $f(1_b, \beta) = 1 \in \Phi(b)$.

A morphism $(F, f) \to (F', f')$ of 2-cocycles is a choice of $h(\alpha) \in \Phi(a)$ for each $a \xrightarrow{\alpha} b$ such that

(0) For all a, $h(1_a) = 1 \in \Phi(a)$.

(1) For all $a \xrightarrow{\alpha} b$,

$$F(\alpha) = c_{h(\alpha)} \circ F'(\alpha) \in \mathrm{Hom}(\Phi(b), \Phi(a)).$$

(2) For all $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$,

$$h(\beta\alpha) \cdot f'(\alpha, \beta) = f(\alpha, \beta) \cdot F(\alpha)(h(\beta)) \cdot h(\alpha) \in \Phi(a).$$

Composition is given by $(h \circ k)(\alpha) = h(\alpha) \cdot k(\alpha)$ for all $a \xrightarrow{\alpha} b$.

$H^2(D; \Phi)$ is the set of components of $Z^2(D; \Phi)$.

A 2-cocycle (F, f) is *split* if $f(\alpha, \beta) = 1 \in \Phi(a)$ for all $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$. Note that if (F, f) is split cocycle then F gives us a functor $D \to \mathbf{Gp}$ lifting Φ. A cohomology class is *split* if it contains a split 2-cocycle. The set of split classes forms a distinguished (possibly empty) subset $H^2_s(D; \Phi) \subseteq H^2(D; \Phi)$, which thus naturally has the structure of a *pair* of sets.

If $\Phi(a)$ is abelian for all a, then F is unique, and the only nontrivial conditions are (3), which form the usual definition of a normalized 2-cocycle. $H^2(D; \Phi)$ is thus just the usual second cohomology group, as defined below. There is a split class and only one, namely 0.

The category $\mathbf{\Delta}$ is the full subcategory of $\mathbf{Cat}$ generated by the ordered sets $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$. It is generated by the morphisms $d^i : [n] \to [n-1]$ and $s^i : [n] \to [n+1]$, where d^i is the injection which omits the value i and $s^i : [n] \to [n-1]$ is the surjection which assumes the value i twice. A *cosimplicial object* in some category $\mathbf{C}$ is a functor $\mathbf{\Delta} \to \mathbf{C}$.

Let $W : D \to \mathbf{C}$ be a contravariant functor to a category with products. We define a cosimplicial object $C^\bullet(D; W)$ by setting

$$C^n(D; W) = \prod_{\sigma:[n] \to D} W(\sigma_0)$$

where the value of the functor σ at $j \in [n]$ is denoted by σ_j. An order-preserving map $\phi : [n] \to [m]$ induces a map $\phi_* : C^n(D; W) \to C^m(D; W)$ defined by declaring, for each $\tau : [m] \to D$, that

$$\mathrm{pr}_\tau \circ \phi_* = \alpha^* \circ \mathrm{pr}_{\tau \circ \phi},$$

where the morphism $\alpha : \tau_0 \to \tau_{\phi(0)} = (\tau \circ \phi)_0$ in D is induced from $0 \leq \phi(0)$ in $[m]$. This is the *cosimplicial replacement* of W.

The normalized cochain complex $N^\bullet$ associated to a cosimplicial abelian group $C^\bullet$ has

$$N^n = \bigcap_{i=0}^{n} \ker\,(s^i | C^n),$$

and differential the given by the restriction of $\sum(-1)^i d^i$. When $C^\bullet = C^\bullet(D;W)$ for a functor $W : D \to \mathbf{Ab}$, the homology groups of this cochain complex form the sequence of derived functors of inverse limit evaluated at $W : D \to \mathbf{Ab}$: $H^s(D;W) = \lim^s_D W$.

Now let $W : D \to \mathbf{Top}$ be a contravariant functor to the category of fibrant spaces. For example, one might have $W = \mathrm{Map}(X, Z)$, where Z is a fixed space and $X : D \to \mathbf{Top}$ is a covariant functor to fibrant spaces. We wish to study

$$T = \operatorname*{holim}_a W(a).$$

If $W = \mathrm{Map}(X, Z)$, then $T = \mathrm{Map}(\mathrm{hocolim}\, X, Z)$. By [1], this space is the inverse limit of the tot tower $T^\bullet$ of the cosimplicial space $Y^\bullet = C^\bullet(D;X)$ associated to the diagram. By [1], p. 303, Y is a fibrant cosimplicial space and hence the tot tower is a tower of fibrations.

The nth space in the tot tower of a cosimplicial space $Y^\bullet$ is defined by

$$T^n = \mathrm{Map}(\mathrm{Sk}_n\Delta^\bullet, Y^\bullet)$$

where $\Delta^\bullet$ is the standard cosimplicial space (the identity functor), and Sk_n is the n-skeleton functor.

For a start let's study $\pi_0(T)$. We will focus on vertices and remain silent about higher simplices.

Starting at the bottom, $T^0 = \prod_a W(a)$. Thus T is empty whenever some $W(a)$ is empty. Assume henceforth that they are all nonempty.

Since the tot tower is a tower of fibrations, all elements of a component of T^n lift to T^{n+s} whenever any one of them does. Let $F^s\pi_0(T^n)$ be the set of components of T^n which lift to T^{n+s}.

For each a pick $w_a \in W(a)$. This choice defines an element $w \in T^0$, which lifts to T^1 exactly when it defines an element $[w]$ of

$$H^0(D;\Phi_0) = \lim_a \pi_0(W(a)) \subseteq \prod_a \pi_0(W(a))\,,$$

where $\Phi_0(a) = \pi_0(W(a))$.

Giving a lift of $w \in T^0$ to $w' \in T^1$ is equivalent to giving, for each $a \xrightarrow{\alpha} b$, a path u_α in $W(a)$ from w_a to $\alpha^* w_b$ (i.e. $u_\alpha : \Delta^1 \to W(a)$ such that $u_\alpha \circ d^0 = \alpha^* w_b$ and $u_\alpha \circ d^1 = w_a$), with the proviso that $u_{1_a} = 1_{w_a}$ for each a. Denote by g_α the path class of u_α.

Next we wish to know whether some choice of lift to T^1 lifts further to T^2. For each $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$, w' determines a map $\dot\phi : \dot\Delta^2 \to W(a)$ characterized by

$$\dot\phi \circ d^0 = \alpha^* u_\beta, \quad \dot\phi \circ d^1 = u_{\beta\alpha}, \quad \dot\phi \circ d^2 = u_\alpha.$$

A lifting of w' to T^2 amounts to a choice of 2-simplex $\phi : \Delta^2 \to W(a)$ extending $\dot\phi$, with the proviso that $\phi = y_\beta \circ s^0$ if $\alpha = 1_a$ and $\phi = u_\alpha \circ s^1$ if $\beta = 1_b$.

The obstruction to the existence of such extensions can be measured using a loop class in $W(a)$ at w_a given by the composition of path classes:

$$f(\alpha, \beta) = g_{\beta\alpha}^{-1} \cdot \alpha^* g_\beta \cdot g_\alpha$$

where we use the "functional order" convention, starting with the rightmost path. The element $w' \in T^1$ lifts to T^2 if and only if each of these loop classes is trivial.

We can express this question in the following terms. For any a, let

$$\Phi_1(a) = \pi_1(W(a), w_a).$$

For any $a \xrightarrow{\alpha} b$, define $F(\alpha) \in \mathrm{Hom}(\Phi_1(b), \Phi_1(a))$ by

$$\pi_1(W(b), w_b) \xrightarrow{\alpha^*} \pi_1(W(a), \alpha^* w_a) \xrightarrow{g_{\alpha\#}^{-1}} \pi_1(W(a), w_a).$$

Up to conjugacy in $\pi_1(W(a), w_a)$ this is independent of choice of path g_α, and it extends Φ_1 to a functor $\Phi_1 : D \to \mathbf{HGp}$ which depends only on $[w] \in \lim_a \pi_0(W(a))$.

The pair (F, f) is then a 2-cocycle, $(F, f) \in Z^2(D; \Phi_1)$. It does depend upon the choice of path classes g_α's, of course, but any other such choice—say g'_α, giving rise to the 2-cocycle (F', f')—differs from g_α by premultiplication by some (uniquely defined) $h(\alpha) \in \pi_1(W(a), w_a)$: $g'_\alpha = g_\alpha \cdot h(\alpha)$. The association $\alpha \mapsto h(\alpha)$ constitutes a morphism from (F, f) to (F', f'):

$$(c_{h(\alpha)} \circ F'(\alpha))(x) = h(\alpha) \cdot {g'}_\alpha^{-1} \cdot \alpha^*(x) \cdot g'_\alpha \cdot h(\alpha)^{-1} = g_\alpha^{-1} \cdot \alpha^*(x) \cdot g_\alpha = F(\alpha)$$

and

$$h(\beta\alpha) \cdot f'(\alpha, \beta) = h(\beta\alpha) \cdot {g'}_{\beta\alpha}^{-1} \cdot \alpha^*(g'_\beta) \cdot g'_\alpha =$$
$$g_{\beta\alpha}^{-1} \cdot \alpha^*(g_\beta) \cdot \alpha^*(h(\beta)) \cdot g_\alpha \cdot h(\alpha) = f(\alpha, \beta) \cdot F(\alpha)(h(\beta)) \cdot h(\alpha).$$

The cohomology class $o_w = [F, f] \in H^2(D; \Phi_1)$ depends also only on $[w] \in \lim_a \pi_0(W(a))$. We thus have a configuration naturally associated to $[w]$ consisting of a set $H^2(D; \Phi_1)$, its subset of split classes, and an element o_w. The class $[w]$ lifts to $\pi_0(T^2)$ if and only if o_w lies in the subset of split classes.

Next, fix a choice of $u_\alpha : \Delta^1 \to W(a)$ for each $a \xrightarrow{\alpha} b$ which determines a split 2-cocycle (F, f_0). Thus we have $w' \in T^1$ which lifts $w \in T^0$ and which lifts to T^2. We ask when w' lifts further to $w^{(3)} \in T^3$. A lift of w' to $w'' \in T^2$ consists of a choice of 2-simplex $\phi_{\alpha,\beta} : \Delta^2 \to W(a)$ for each $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$, as described above. The four coface maps from codegree 2 to codegree 3 lift these 2-simplices to 2-simplices which, for each $a \xrightarrow{\alpha} b \xrightarrow{\beta} c \xrightarrow{\gamma} d$, fit together to give a map from the boundary of the 3-simplex into $W(a)$. A lift of w'' to T^3 consists in an extension to a map from the full 3-simplex, with certain restrictions if one of the maps α, β, or γ, is an identity map. As before, one considers a functor on D, which I will write Φ_2, assigning to a the group $\pi_2(W(a), w_a)$. These are abelian groups, but there is a potential basepoint issue nevertheless. One may worry that $\Phi_2(\beta\alpha)$ will differ from $\Phi_2(\alpha)\Phi_2(\beta)$ by some automorphism of $\pi_2(W(a), w_a)$ (determined by a class in $\pi_1(W(a), w_a)$, so presumably not even an inner automorphism now). However, the assumption that the 2-cocycle is split provides exactly what is needed to guarantee that this doesn't happen. For a start, F is a choice of lift of Φ to a functor $F : D \to \mathbf{Gp}$. The obstruction to this is given by conjugation by the elements $f(\alpha, \beta)$, and not only are these elements central in $\pi_1(W(a), w_a)$, they are actually trivial there. This triviality implies that we also have determined a natural structure of functor $\Phi_2 : D \to \mathbf{Ab}$. Moreover, the resulting cohomology group is just $H^3(D; \Phi_2)$. This is an abelian group, and it is determined by the choice of element $w' \in T^1$. This element lifts to T^3 if and only if the obstruction in $H^3(D; \Phi_2)$ is 0. This pattern continues; the element $w' \in T^1$ determines functors $\Phi_n : D \to \mathbf{Ab}$ for all $n > 1$, and a class $w^{(n-1)} \in T^{n-1}$ lifting w' determines an obstruction in $H^{n+1}(D; \Phi_n)$ such that $w^{(n-1)}$ lifts to T^{n+1} if and only if the obstruction class vanishes.

Now we address uniqueness of liftings. A necessary and sufficient condition for w and x to lie in the same component of T^0 is that their components w_a and x_a lie in the same component of $W(a)$ for every a.

Next suppose that w' and x' are two elements of T^1, and assume that w' lifts to T^2 and that their images w and x are in the same component of T^0. If follows that x' is liftable to T^2 as well. We ask for conditions guaranteeing that they are in the same component of T^1. For a start, w_a and x_a must lie in the same component of $W(a)$ for each a. Pick a path class k_a in $W(a)$ from w_a to x_a. The elements w' and x' determine, for each $a \xrightarrow{\alpha} b$, path classes g_α and h_α. Together these path classes determine a loop class at w_a:

$$f(\alpha) = g_\alpha^{-1} \cdot \alpha^*(k_b)^{-1} \cdot h_\alpha \cdot k_a.$$

The problem of lifting the path from from w to x to a path in T^1 from w' to x' amounts to the problem of finding a null-homotopy of this loop, with the proviso that the null-homotopy of $f(1_a)$ is given by composing k_a with the projection to one factor of the square. This proviso can be arranged by virtue of the agreement that $g_{1_a} = 1$ and $h_{1_a} = 1$.

Since w' lifts to T^2, for any $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$, $g_{\beta\alpha} = \alpha^*(g_\beta) \cdot g_\alpha$ and the rules $\Phi_1(a) = \pi_1(W(a); w_a)$ for all a and $\Phi_1(\alpha)(x) = g_\alpha^{-1} \cdot \alpha^*(x) \cdot g_\alpha$ for all $a \xrightarrow{\alpha} b$ determine a functor $\Phi_1 : D \to \mathbf{Gp}$. Since x' also lifts to T^2, $h_{\beta\alpha} = \alpha^*(h_\beta) \cdot h_\alpha$ as well, and this implies that $f \in Z^1(D; \Phi_1)$:

$$\begin{aligned}\Phi_1(\alpha)(f(\beta)) \cdot f(\alpha) &= g_\alpha^{-1} \cdot \alpha^*(g_\beta^{-1} \cdot \beta^*(k_c)^{-1} \cdot h_\beta \cdot k_b) \cdot g_\alpha^{-1} \cdot (g_\alpha \cdot \alpha^*(k_b)^{-1} \cdot h_\alpha \cdot k_a) \\ &= ((\alpha^*(g_\beta) \cdot g_\alpha)^{-1} \cdot \alpha^*\beta^*(k_c)^{-1} \cdot \alpha^*(h_\beta) \cdot \alpha^*(k_b) \cdot \alpha^*(k_b)^{-1} \cdot h_\alpha \cdot k_a \\ &= g_{\beta\alpha}^{-1} \cdot (\beta\alpha)^*(k_c)^{-1} \cdot h_{\beta\alpha} \cdot k_a = f(\beta\alpha).\end{aligned}$$

Suppose we choose different path classes, k'_a, joining w_a to x_a. Together they determine a different 1-cocycle, f'. The loop classes $l_a = k_a \cdot {k'_a}^{-1}$ constitute a morphism from f to f': $f'(\alpha) = F(\alpha)(l_a)^{-1} \cdot f(\alpha) \cdot l_a$, for all $a \xrightarrow{\alpha} b$. Thus the pair (w', x') (each liftable to T^2) determines a "difference" class

$$\delta_{w',x'} \in H^1(D; \Phi_1),$$

which is trivial if and only if they are in the same component of T^1.

This construction can be reversed, too: given a component $[w']$ of T^1 which is liftable to T^2, there is a bijective corresondance between elements $\delta \in H^1(D; \Phi_1)$ and components of T^1 which lift $[w] \subseteq T^0$.

This process continues. Given classes w'', x'' in T^2 such that w'' is liftable to T^3, and a path in T^0 from w to x which is liftable to a path in T^1 from w' to x', the obstruction to lifting that path in T^0 to a path from w'' to x'' in T^2 lies in $H^2(D; \Phi_2)$, where $\Phi_2(a) = \pi_2(W(a), w_a)$. This gives a bijection between the set of components of T^2 which contain lifts of w' and the set $H^2(D; \Phi_2)$.

References

[1] Pete Bousfield and Dan Kan, *Homotopy Limits, Completions, and Localizations*, Lecture Notes in Math. 304 (1972).

[2] Zdzisław Wojtkowiak, On maps from $\underrightarrow{\mathrm{holim}}\, F$ to Z, *Algebraic Topology, Barcelona 1986*, Lecture Notes in Math. 1298 (1987) 227–236.