and algebras over the Steenrod algebra.

spectral sequence of a space with coeffi-

tere for $(\mathbb{Z}/2)^k$. Topology 22 (1983), 83-


ds de la déstabilisation. Math. Z. 194


classifying spaces. Ann. of Math. 120

point of view. Lecture Notes in Math.

logy algebras of symmetric groups. J.

omology ring. I. II. Ann. of Math. 94


is over the Steenrod algebra. J. Pure

over the Steenrod algebra, variation on


logiques. Publications Mathématiques


Looping Massey-Peterson towers

J. R. Harper and H. R. Miller

Introduction

In this paper we begin a study of the mod $p$ cohomology of function spaces by means of an Adams resolution of the target space. For simplicity and concreteness we will require the source space to be a sphere, so we are dealing with iterated loop spaces. We will use the “classical” unstable Adams towers constructed by Massey and Peterson [4] and Barcus [1]. This technique restricts us to consideration of target spaces $X$ with “very nice” cohomology: $H^*(X) = U(M)$ for some unstable module $M$ over the Steenrod algebra $A$, where $U$ is the Steenrod-Epstein enveloping unstable $A$-algebra functor. The class of such spaces includes Stiefel manifolds over $\mathbb{C}$ and $\mathbb{H}$ (and over $\mathbb{R}$ too if $p = 2$), but excludes nontrivial wedges and suspensions.

Our method is simply to compute, to the extent possible, the cohomology of the spaces in the $k$-fold loop space of a Massey-Peterson tower. By construction, the cohomology of each space $E_i$ in such a tower for $X$ surjects to $H^*(X)$. This fails for the looped tower, and there results a filtration of $H^*(\Omega^k X)$ in the category of $A$-Hopf algebras. Our main result determines the associated quotient of $A$-Hopf algebras, under rather restrictive connectivity assumptions, in terms of certain homological functors on the category $U$ of unstable $A$-modules. To describe these, let $\Omega^k M$ denote the maximal unstable quotient of the $A$-module $\Sigma^{-k} M$. This is a right-exact functor from $U$ to $\mathcal{U}$, and has left-derived functors $\Omega^k_\ast$.

Theorem. Let $X$ be a simply connected finite complex such that $H^*(X) = U(M)$ for some $M \in U$. Let $b$ and $t$ be integers such that $M^t = 0$ unless $b < i < t$, and assume that $k - 1 \leq b - p^{-2} t$. Then there is a natural filtration of $H^*(\Omega^k X)$ by $A$-Hopf algebras $A_\ast$ such that $A_{-1} = F_p$, $A_0 = H^*(\Omega^k X)$, and

$$A_{\ast + 1}/A_\ast \cong U(\Omega^{k+1}_{\ast + 1} M).$$

A more precise version of this theorem is stated in Section 3 as Theorem 3.9.

We remark that, while examples (like 4.10 below) show that some restrictions on $M$ are necessary, the filtration by images of $H^*(\Omega^k E_\ast)$ seems to satisfy 3.9 in much
more general circumstances than we have been able to prove here. For example, computations of W.M. Singer [5] show that $H^*(\Omega^k S^{n+k})$ has this form for all $k \geq 0$ and $n \geq 1$, at least for $p = 2$. Indeed, the derived functors he computes are precisely the ones entering into 3.9 if $M$ is a $k$-fold suspension in $U$.

We begin by reviewing certain elementary facts about derived functors of $\Omega^k$. In Sections 2 and 3 we give a self-contained account of the Massey-Peterson-Barcus theory, using the Eilenberg-Moore spectral sequence. Finally, in Section 4, we prove the theorem.

We adopt the following convention on primes. When the case $p = 2$ can be handled by the substitution $\beta = Sq^1$, $P^1 = Sq^{2n}$, we make no further comment; only when something special happens for $p = 2$ do we take notice.

Most of the work on this paper was done before 1980. The intervening years have seen the exploitation by Jean Lannes and others of more sophisticated variants of this approach, but using source spaces with more convenient cohomology. We offer our apologies for the long delay in publication, and express our gratitude for the opportunity to submit it to these proceedings in honor of Ioan James.

1 Some algebra

We begin with the algebraic loop functor $\Omega$ and its derived functors [4,5]. For any unstable $\mathcal{A}$-module $M$, define a vector space $\Phi M$ by

\[
\begin{align*}
(\Phi M)^{2p} &= M^{2n}, \\
(\Phi M)^{2p+2} &= M^{2n+1}, \\
(\Phi M)^{1} &= 0 \quad \text{otherwise}.
\end{align*}
\]

Write $\bar{x}$ for the element of $\Phi M$ corresponding to $x \in M$. It is direct to check that

1.1 $\Phi M$ becomes an unstable $\mathcal{A}$-module if we declare

\[
\begin{align*}
p^{n} \bar{x} &= \bar{P}^{n}x, \\
p^{n+1} \bar{x} &= \beta \bar{P}^{n}x \quad \text{if } |x| \text{ is odd}, \\
\beta \bar{P}^{n}x &= 0 \quad \text{otherwise}.
\end{align*}
\]

1.2 The map $\lambda: \Phi M \to M$ defined by

\[
\lambda \bar{x} = \begin{cases} 
    P^{n}x & \text{if } |x| = 2n, \\
    \beta P^{n}x & \text{if } |x| = 2n + 1,
\end{cases}
\]

is $\mathcal{A}$-linear, and its kernel and cokernel are suspensions in $U$.

Define functors $\Omega_1$ and $\Omega_2$ by means of the resulting natural exact sequence

\[
0 \to \Sigma \Omega_1 M \to \Phi M \xrightarrow{\lambda} M \to \Sigma \Omega M \to 0. \tag{1.3}
\]
to prove here. For example, 

\[ \Phi^k \] has this form for all \( k \geq 0 \) torsors he computes are precisely in \( \mathcal{U} \).

ut derived functors of \( \Omega^k \). In : Massey-Peterson-Barcus the-

lly, in Section 4, we prove the case \( p = 2 \) can be handled further comment; only when

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or \( \Omega \) and its derived functors space \( \Phi M \) by

\[ \mathcal{M} \]. It is direct to check that we declare

\[ \to 0. \] (1.3)

Since \( \Phi \) is exact and \( \lambda \) is monic on projectives, standard homological methods imply that \( \Omega_1 \) is the first left derived functor of \( \Omega \) and that \( \Omega \) has no higher derived functors.

The functor \( \Omega: \mathcal{U} \to \mathcal{U} \) is right adjoint to \( \Sigma \). Since the \( \Sigma \) is exact, \( \Sigma \) carries projectives to projectives. Consequently there is a Grothendieck spectral sequence associated to the composite \( \Omega^0 \Omega^k = \Omega^{j+k} \):

\[ \Omega^0 \Omega^k M \xrightarrow{g} \Omega^{j+k} M. \] (1.4)

In particular, with \( j = 1 \), we obtain the Singer exact sequence [5]

\[ 0 \to \Omega^0 \Omega^k M \to \Omega^{j+k+1} M \to \Omega_1 \Omega^{j-1} M \to 0. \] (1.5)

One sees also by induction that

\[ \Omega^s M = 0 \quad \text{if} \quad s > k, \quad \text{and} \]

\[ \Omega^s M = (\Omega_1)^s M. \] (1.6)

\[ \Omega^s M = (\Omega_1)^s M. \] (1.7)

We will need estimates on the connectivity and coconnectivity of \( \Omega^{s+k} M \). If \( N \) is a graded vector space let

\[ \text{conn} \, N = \min \{ i : N_i \neq 0 \} - 1 \]

\[ \text{coconn} \, N = \max \{ i : N_i \neq 0 \}. \]

1.8 Lemma. If \( M \in \mathcal{U} \), then

\[ \text{conn} \, \Omega^{s+k} M \geq p^t(\text{conn} \, M - k), \]

\[ \text{coconn} \, \Omega^{s+k} M \leq p^t \text{coconn} \, M - k. \]

Proof. From (1.3) we have

\[ \text{conn} \, \Omega^s M \geq p \text{ conn} \, M, \]

\[ \text{coconn} \, \Omega^s M \leq p \text{ coconn} \, M. \]

(If \( p \) is odd, both inequalities can be improved by dividing according to parity, and in any case the second can be improved by at least 1. But these suffice for our purposes.)

(1.7) then gives the result if \( k = 0 \). If \( s = 0 \) the result is clear, and the rest follows by induction using the Singer sequence (1.5). Notice that the cokernel determines the connectivity and the kernel determines the coconnectivity.

\[ \square \]

The above algebra is connected with the geometry by the contravariant functor \( K(\cdot) \) which associates to any projective unstable \( \mathcal{A} \)-module \( F \) of finite type a generalized Eilenberg-MacLane space \( K(F) \) with homotopy \( \pi_1 K(F) \cong \text{Hom}_A(F, \Sigma^i F_p) \). Then \( H^*(F) \cong U(F) \); and to a morphism \( f: F' \to F \) we associate the unique H-map.
The cohomology theory of induced fibrations

The basic homological tool in the study of pull backs of fibrations is the Eilenberg-Moore spectral sequence. Suppose we have a pull-back diagram

\[
\begin{array}{ccc}
E & \xrightarrow{h} & PB_0 \\
\downarrow p & & \downarrow p_0 \\
B & \xleftarrow{f} & B_0
\end{array}
\]  

(2.1)

with \( p_0 \) the path-loop fibration. Under certain restrictions, an analysis of (2.1) was carried out by Massey and Peterson [3] and by Barcus [1], using the Serre spectral sequence. In this section we offer a proof of a variant of their results, using the Eilenberg-Moore machinery. We state all our results and comment on them first, and then sketch proofs.

To express the result conveniently we need the following universal construction. Given an unstable \( A \)-algebra \( R \) and a map \( g: R \to G \) of unstable \( A \)-modules, we seek the initial unstable \( A \)-algebra \( B \) accepting an \( A \)-linear map from \( G \) such that the composite \( R \to B \) is an \( A \)-algebra homomorphism. Write \( \overline{U}_R(G) \) for this \( B \); it is a reduced version of a construction denoted \( U_R \) by Massey and Peterson. It is easily seen that if \( U(R) \to R \) is the unique \( A \)-algebra map extending the identity map \( R \to R \), then

\[ R \otimes_{U(R)} U(G) \cong \overline{U}_R(G). \]

From this we see:

2.2 Lemma. If \( g \) is monic then so is \( R \to \overline{U}_R(G) \); and the extension

\[ 0 \to R \to G \to Q \to 0 \]

in \( \mathcal{U} \) induces an extension

\[ F_p \to R \to \overline{U}_R(G) \to U(Q) \to F_p \]

of \( A \)-algebras: \( \overline{U}_R(G) \) is free over \( R \), and \( U(Q) = \overline{U}_R(G) \otimes_R F_p = \overline{U}_R(G)/R. \)
The functor $K$ is in a certain
jection

\[ \Omega^k K(F), \text{ so } H^*(\Omega^k K(F)) \cong M \], $H^*(\Omega^k X)$ is much larger shall see, a reflection of the
brations pull backs of fibrations is the pull-back diagram

\[ (2.1) \]

ons, an analysis of (2.1) was [1], using the Serre spectral t of their results, using the comment on them first, and
wing universal construction. of unstable $A$-modules, we near map from $G$ such that . Write $\bar{U}_R(G)$ for this $B$; Massey and Peterson. It is
\[ (G); \text{ and the extension} \]

\[ \otimes_R F_p = \bar{U}_R(G) \backslash R. \]

2.3 Theorem. Consider the fiber square

\[ E \quad \longrightarrow \quad PB_0 \]

\[ P \downarrow \quad \quad \downarrow \]

\[ B \quad \overset{f}{\longrightarrow} \quad B_0. \]

Assume that $B$ and $B_0$ are simply connected, that $H^*(B)$ and $H^*(B_0)$ are of finite type, and:

(i) $H^*(B)$ is a free module over $\text{Im} f^*$.
Assume further that there is given a sub $A$-module $Q$ of $H^*(B_0)$ such that

(ii) $\Omega_1 Q = 0$;

(iii) $H^*(B_0)$ is a free module over $U(Q)$; and

(iv) $\ker f^*$ is the ideal generated by $Q$.

Let $R = H^*(B) \otimes H^*(B_0)_F p$. There is then a short exact sequence in $\mathcal{U}$, the "fundamental sequence"

\[ 0 \longrightarrow R \longrightarrow G \longrightarrow \Omega Q \longrightarrow 0, \]

and an extension $i: G \rightarrow H^*(E)$ of the natural $A$-algebra map $R \rightarrow H^*(E)$, such that the induced map

\[ \bar{U}_R(G) \longrightarrow H^*(E) \]

is an isomorphism, and

\[ G \quad \longrightarrow \quad \Omega Q \]

\[ \quad \downarrow \quad \quad \downarrow \omega \]

\[ H^*(E) \quad \overset{\partial^*}{\longrightarrow} \quad H^*(\Omega B_0) \]

commutes. Here $\partial$ is the Barratt-Puppe boundary map and $\omega$ is the natural "suspension" map.

Thus $R$ embeds in $H^*(E)$, so $R \cong \text{Im} f^*$.

As a special case, take $B = *$, so $E = \Omega B_0$. The theorem yields the

2.6 Corollary. Let $X$ be a simply connected space such that $H^*(X)$ is of finite type and $H^*(X) = U(Q)$, $\Omega_1 Q = 0$. Then the suspension map induces an isomorphism

\[ U(\Omega Q) \xrightarrow{\cong} H^*(\Omega^k X). \]

If $X$ is $k$-connected and $\Omega_i \Omega^j Q = 0$ for all $i$ with $0 \leq i < k$, then by induction we have

\[ U(\Omega^k Q) \xrightarrow{\cong} H^*(\Omega^k X). \]
The Singer sequence (1.5) shows that this condition is in fact equivalent to requiring \( \Omega_{s+1}^* Q = 0 \) for all \( s \geq 0 \); so our main theorem holds in this case without dimension assumptions.

\subsection{Example}

Typically one obtains the conditions of Theorem 2.3 by starting with \( B_0 \) such that \( H^*(B_0) = U(N) \) for some \( N \in \mathcal{U} \) for which \( \Omega_1 \mathcal{N} = 0 \); for instance, \( B_0 = K(N) \) for \( N \) projective in \( \mathcal{U} \). Any \( Q \subseteq N \) then has \( \Omega_1 Q = 0 \) as well, since \( \Omega_1 \) is left exact. Note that \( U(N) \) is an \( A \)-Hopf algebra, with \( PU(N) = N \). Now suppose that we can give \( H^*(B) \) a Hopf algebra structure such that \( f^* \) is a Hopf algebra map. Then condition (i) of Theorem 2.3 is automatic. Take

\[
Q = N \cap \ker f^* = \ker Pf^*.
\]  

Then we obtain (ii) of Theorem 2.3 as noted; (iii) holds because \( U(Q) \to H^*(B_0) \) is monic, being a Hopf algebra map which is monic on primitives; and (iv) holds since \( U(Q) \) is the Hopf algebra kernel \( \text{Ker} f^* = H^*(B_0) \cap U(B)F_p \).

The Hopf condition on \( f^* \) usually occurs in one of two ways:

(a) \( H^*(B) = U(M) \) for some \( M \in \mathcal{U} \), and the map \( f^* \) induced by \( f: B \to B_0 \) fits into a diagram

\[
\begin{array}{ccc}
U(M) & \xrightarrow{U(\phi)} & U(N) \\
\cong & & \cong \\
\downarrow & & \downarrow \\
H^*(B) & \xrightarrow{f^*} & H^*(B_0)
\end{array}
\]

for an \( A \)-module map \( \phi \).

(b) \( B \) and \( B_0 \) are homotopy-associative \( H \)-spaces such that the Hopf algebra structure on \( H^*(B_0) = \bar{U}(N) \) has \( N \) for primitives, and \( f \) is an \( H \)-map. If \( B_0 = K(N) \), the standard \( H \)-structure on \( B_0 \) will do, and \( f \) corresponds to a primitive cohomology class in \( B \).

\subsection{Remark}

In case (b) above,

\[
\begin{array}{ccc}
E & \rightarrow & PB_0 \\
p \downarrow & & \downarrow \\
B & \xrightarrow{f} & B_0
\end{array}
\]

is a Hopf fiber square. We have seen that \( R \cong \text{Im} p^* \). We can identify other elements of the situation as follows:

(i) The Hopf algebra kernel \( \text{Ker} f^* = H^*(B_0) \cap H^*(B)F_p = U(Q) \).

(ii) \( \text{Im} f^* = \text{Ker} p^* \).

(iii) \( \text{Ker} \partial^* \) has a description in terms of a fundamental sequence. The diagram (2.5) shows that \( \partial^* \) factors as
is in fact equivalent to requiring
in this case without dimension

conditions of Theorem 2.3 by
$e N \in U$ for which $\Omega_1 N = 0$;
$Q \subseteq N$ then has $\Omega_1 Q = 0$ as
opf algebra, with $PU(N) = N$.
structure such that $f^*$ is a Hopf
omonic. Take

\begin{align}
H^*(E) \xrightarrow{\partial^*} H^*(\Omega B_0) \\
\cong \xrightarrow{\cong} \Omega \Omega(N)
\end{align}

where $k: \ker P_f^* \to N$ is the inclusion. By Lemma 2.2 the map $q$ is epic with Hopf
algebra kernel $R$, but $U(\Omega k)$ is not generally monic; indeed, there is an exact sequence

$$0 \to \Omega_1 \text{Im } P_f^* \xrightarrow{k} \Omega \ker P_f^* \xrightarrow{\Omega k} \Omega N \to \Omega \text{Im } P_f^* \to 0.$$ 

If we form the pull-back diagram

\begin{align*}
0 & \to R \to K \to \Omega_1 \text{Im } P_f^* \to 0 \\
\delta \downarrow & \downarrow \delta \\
0 & \to R \to G \to \Omega \ker P_f^* \to 0
\end{align*}

then we see that the top row is a fundamental sequence for

$$\text{Ker } \delta \cong U_R'(K).$$

It will be important for us to compare the fundamental sequence for $E$ against
that for $\Omega E$. For this purpose we will write $Q(f)$ for the choice of $Q$, and agree to
take $Q(\Omega f) = \ker P(\Omega f)^*$ as in (2.8). Then the suspension $\omega: \Omega H^*(B_0) \to H^*(\Omega B_0)$
automatically maps $\Omega Q f$ to $Q(\Omega f)$. We will also write $R(f) = H^*(B) \otimes H^*(B_0) F_p$, etc.

Theorem 2.3 has the

2.10 Addendum. Suppose $f: B \to B_0$ and $Q(f)$ satisfy the conditions of
Theorem 2.3, that $B$ and $B_0$ are 2-connected, and that $\Omega_1 Q(\Omega f) = 0$. Then there is
$\omega: \Omega G(f) \to G(\Omega f)$ such that

$$\begin{array}{cccc}
\Omega R(f) & \to & \Omega G(f) & \to & \Omega^2 Q(\Omega f) & \to & 0 \\
\omega & \downarrow & \omega & \downarrow & & \omega & \\
0 & \to & R(\Omega f) & \to & G(\Omega f) & \to & \Omega Q(\Omega f) & \to & 0
\end{array}$$

and

$$\begin{array}{cccc}
\Omega G(f) & \to & \Omega H^*(E) \\
\omega & \downarrow & \omega & \downarrow \\
G(\Omega f) & \to & H^*(\Omega E)
\end{array}$$

The diagram commute.
We turn to a sketch of proofs. Write $E_0$ for $PB_0$. Let $E_0T$ be the mapping cylinder of the projection map $p_0: E_0 \to B_0$. Then $E_0 \to B_0$ factors canonically as $E_0 \to E_0T \to B_0$, and $p_0$ is a homotopy equivalence. Now $E_0 \to E_0T$ is a cofibration over $B$ inducing a surjection in cohomology. (This is the general assumption made in [3] by Massey and Peterson; here it is trivial since $E_0 \simeq B_0$.) Moreover $H^*(E_0T) \cong H^*(B_0)$ is a projective $H^*(B_0)$-module. It follows that the following diagram is the beginning of an exact couple yielding the Eilenberg-Moore spectral sequence (cf. Smith [6]):

$$
\begin{array}{ccccccc}
H^*(B \times B_0, E_0) & \xrightarrow{\delta} & H^*(B \times B_0, (E_0T, E_0)) & \xrightarrow{\delta} & \cdots \\
& \downarrow & & \downarrow & \\
& H^*(B \times B_0, E_0T) & & H^*(B \times B_0, B_0 \times (E_0T, E_0)). & \\
\end{array}
$$

Rewriting this, we have

$$
\begin{array}{ccccccc}
H^*(E) & \xrightarrow{\delta} & H^*(E_T, E) & \xrightarrow{\delta} & \cdots \\
& \downarrow p^* & & \downarrow p_1^* & \\
H^*(B) & & H^*(B) \otimes H^*(E_0T, E_0) & & (2.11) \\
& \downarrow \| & & \downarrow \| & \\
E_0^q & & E_0^q & & d_1 & & E_0^{q-1}.
\end{array}
$$

Therefore, in the Eilenberg-Moore filtration of $H^*(E)$,

$$
F^0 = \ker \delta = \text{Im} \ p^*, \\
F^{-1} = \ker \delta^2 = \delta^{-1} \text{Im} \ p_1^*.
$$

If we identify $H^*(E_0T, E_0) \cong H^*(B_0)$, then $d_1(b \otimes b_0) = b \cdot f^*b_0$. Since $Q \subset \ker f^*$, the map $\Sigma^{-1}Q \to E_0^{-1}$ sending $b_0$ to $1 \otimes b_0$ carries $\Sigma^{-1}Q$ to $\ker(d_1|E_0^{-1})$. Since $E_0^{-1} \in U$, there is a factorization

$$
\begin{array}{ccc}
\Sigma^{-1}Q & \xrightarrow{\delta} & \Omega Q \\
\downarrow & & \downarrow \\
\ker(d_1|E_0^{-1}) & \xrightarrow{\delta} & E_0^{-1}.
\end{array}
$$

Thus we may form the pull-back

$$
\begin{array}{ccccccc}
0 & \to & R & \to & G & \to & \Omega Q & \to & 0 \\
& & \| & & \downarrow & & \| & & \| \\
0 & \to & F^0 & \to & F^{-1} & \to & E_0^{-1} & \to & 0.
\end{array}
$$
\( E_{0T} \) be the mapping cylinder
actors canonically as \( E_0 \to E_{0T} \) is a cofibration over
ral assumption made in [3] by
oreover \( H^*(E_{0T}) \cong H^*(E_0) \)
ing diagram is the beginning
sequence (cf. Smith [6]):

\[ \delta \to \ldots \]

\[ \times \]

\[ B_0 \times (E_{0T}, E_0). \]

\[ \delta \to \ldots \]

\[ \otimes H^*(E_{0T}, E_0) \]

\[ \| \]

\[ E_1^{-1}. \]

\[ \z \cdot f^* b_0. \] Since \( Q \subset \ker f^*, \)
\(-1)Q \to \ker(d_1[E_1^{-1}]). \) Since

\[ \begin{align*}
E_0 \\
\to E_0 \\
\to E_0 \\
\to E_0 \\
\to E_0 \\
\to E_0 \\
\end{align*} \]

The top sequence here is the fundamental sequence. The composite \( G \to F^{-1} \to H^*(E) \) is the map \( i \) in the statement of Theorem 2.3. The composite \( R \to G \to H^*(E) \)
is an A-algebra homomorphism, induced by \( p^* \); so we get a map \( \tilde{\iota}: \tilde{U}_R(G) \to H^*(E) \).
To show it is an isomorphism, we filter \( \tilde{U}_R(G) \). First filter \( U(Q) \) by setting \( F^0 = F_p, \)
\( F^{-1} = F_p \otimes Q, \) and \( F^{-n} = (F^{-1})^n; \) thus \( F^{-p} = U(Q) \). Then let \( F^{-n} \tilde{U}_R(G) \) be the
inverse image of \( F^{-n}U(Q) \) under the natural map. It is then clear that \( \tilde{\iota} \) is filtration
preserving when \( H^*(E) \) is given the Eilenberg-Moore filtration. We will show that
\( E_0 \tilde{\iota} \) is an isomorphism.

Before we carry out this computation, we prove the naturality statements (2.5) and
Addendum 2.10. The boundary map \( \partial: \Omega B_0 \to E \) extends to a map of squares

\[ \begin{array}{ccc}
\Omega B_0 & \to & E_0 \\
\downarrow & & \downarrow \\
\ast & \to & B_0 \\
\downarrow & & \downarrow \\
B_0 & \to & B_0, \\
\end{array} \]

so compatibility with \( \partial \) follows by naturality of the Eilenberg-Moore spectral sequence.

As for 2.10, note that the cohomology suspension map \( \omega \) is induced from a natural
transformation \( \Sigma \Omega \to \text{id} \). The relevant portion of the exact couple for the square with
classifying map \( f \) is induced in \( H^* \) by

\[ \begin{align*}
E & \to (E_T, E) \\
\downarrow & \downarrow \\
E_T & \to B \times (E_{0T}, E_0). \\
\end{align*} \]

The corresponding diagram for \( \Omega f \) is

\[ \begin{align*}
\Omega E & \to (\Omega E_T, \Omega E) \\
\downarrow & \downarrow \\
\Omega E_T & \to \Omega B \times (\Omega E_{0T}, \Omega E_0). \\
\end{align*} \]

The suspension of (2.15) maps to (2.14), and this leads to the desired naturality.
We now compute

\[ E_2^2 = \text{Tor}_H^2(B_0)(H^*(B), F_p) \]
in the Eilenberg-Moore spectral sequence. Let \( A = \text{Im } f^* \). Since \( H^*(B) \) is a free
A-module, the extension spectral sequence

\[ \text{Tor}_A^2(H^*(B), \text{Tor}_H^1(B_0)(A, F_p)) \Rightarrow \text{Tor}_H^1(B_0)(H^*(B), F_p) \]
collapses. Since \( H^*(B_0) \) is free over \( U(Q) \) and \( A \cong H^*(B_0) \otimes U(Q) F_p \),

\[ \text{Tor}_H^1(B_0)(A, F_p) \cong \text{Tor}_{U(Q)}(F_p, F_p). \]
There is thus an $A$-algebra isomorphism

$$R \otimes \text{Tor}_{U(Q)}^*(F_p, F_p) \cong \text{Tor}_{U^*(B_0)}^*(H^*(B), F_p).$$

(2.16)

Suppose first $p = 2$. The assumption $\Omega_1 Q = 0$ implies that $U(Q)$ is polynomial with $\text{Tor}_{U(Q)}^{-1}(F_2, F_2) \cong \Sigma Q$ as $A$-modules, so $\text{Tor}_{U^*(B_0)}^{-1}(H^*(B), F_p) \cong R \otimes E[\omega^{-1} \Sigma Q]$. By multiplicativity, $E_2 = E_{\infty}$. The map $i$ is thus an isomorphism in this case.

Now take $p > 2$. By 1.1, the sub vector space $\Phi^+ Q$ of $\Phi Q$ of elements of degree divisible by $2p$ forms a sub $A$-module. Since $\Omega_1 Q = 0$ we have a commutative diagram with exact rows and columns

\[
\begin{array}{cccccc}
0 & \rightarrow & \Phi^+ Q & \rightarrow & \Phi Q & \rightarrow & \Phi^- Q & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \rightarrow & \Phi^- Q & \rightarrow & I Q & \rightarrow & \Phi^- Q & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]

which defines $IQ$ and $\Phi^- Q$ in $U$. Then as an algebra $U(Q)$ is free-commutative and $\text{Tor}_{U(Q)}^{-1}(F_2, F_2) = I(Q)$ as $A$-modules. Therefore $\text{Tor}_{U^*(Q)}^*(F_p, F_p)$ is the free algebra with divided powers generated by $I(Q)$. The $p$th divided power $\gamma_p: \text{Tor}_{U(Q)}^{-1}(F_p, F_p) \rightarrow \text{Tor}_{U(Q)}^{-1}(F_p, F_p)$ factors through an embedding

$$\Sigma^{p-1} \Phi^- Q \rightarrow \text{Tor}_{U(Q)}^{-1}(F_p, F_p)$$

sending $\bar{z}$ to $\gamma_p(x)$. Under this correspondence, the differential $d_{p-1}$ satisfies

$$d_{p-1} \tau_n[z] = \tau_{n-p}[x] \cdot [\lambda^{-1} \bar{z}]$$

(2.17)

up to a unit, for $n \geq p$. This follows from the fact that the divided powers in Tor are natural, so we can use them to compute the effect on Tor of the map of squares

\[
\begin{array}{cccccc}
E & \rightarrow & E_0 & \rightarrow & K_{2n} & \rightarrow & PK_{2n+1} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
B & \rightarrow & B_0 & \rightarrow & * & \rightarrow & K_{2n+1} \\
\end{array}
\]

to the appropriate universal example; here $|z| = 2n + 1$ and $K_m = K(Z/p, m)$. There, (2.17) is known ([6], Prop. 4.4, page 85).

The effect of $d_{p-1}$ is that $E_{p-1}^* \cong E_{p-1}^*$ contains no generators in homological degree less than $-1$, so $E_{p-1}^* = E_{p-1}^*$; and $E_{p-1}^* = E_{p-1}^* U_R(G)$, so as before we conclude that $U_R(G) \cong H^*(E)$. \[\square\]
that $U(Q)$ is polynomial with \( \tau^*(B; F_p) \cong R \otimes E[\sigma^{-1} \Omega Q] \).

\[ \Phi^*Q \rightarrow 0 \]

\[ \Phi^*Q \rightarrow 0 \]

\( \phi^*Q \) is free-commutative and \( \phi^*(Q)(F_p, F_p) \) is the free algebra generated power \( \gamma_p: \text{Tor}^1_{U(Q)}(F_p, F_p) \) by \( \phi^*Q \) of elements of degree \( e \) have a commutative diagram

\[ \begin{array}{c}
\gamma_{s+1} \\
\downarrow \gamma_s \\
X \\
\downarrow \\
K(P_0) = E_0 \\
\downarrow f_0 \\
K(P_1)
\end{array} \]

where \( p_s \) is the pull-back under the "k-invariant" \( f_s \) of the path-loop fibration; and if \( \partial_s: K(\Omega^s P_s) \to E_s \) is the Barratt-Puppe boundary map, then

\[ f_s \circ \partial_s \cong K(\Omega^s d_s). \]

Begin by constructing \( q_0: X \to K(P_0) \) such that \( H^*(q_0) = U(\epsilon) \). Since \( \epsilon d_0 = 0 \), \( q_0 \) lifts to \( q_1 \). By Theorem 2.3 we have a diagram

\[ \begin{array}{c}
\Omega P_2 \\
\downarrow \Omega d_1 \\
H^*(X) \\
\downarrow i \\
H^*(E_1) \\
\downarrow \delta^* \\
H^*(K(\Omega P_1))
\end{array} \]

in which \( R(f_0) = H^*(K(P_0)) \otimes H^*(K(P_1)) \) \( F_p \cong U(M) \cong H^*(X) \) and \( \overline{U}(f_0)(G(f_0)) \) \( \cong H^*(E_1) \). Thus \( q^* i \) splits the fundamental sequence for \( E_1 \).

It follows that \( H^*(E_1) \cong U(M \oplus \Omega \ker d_0) \) as \( \Lambda \)-algebras, in such a way that the diagram...
Here the bottom horizontal arrows are the obvious ones.

Let \( \sigma: \Omega \ker d_0 \to M \oplus \Omega \ker d_0 \) include the summand, and let \( \Omega d'_1: \Omega P_2 \to \Omega \ker d_0 \) be induced from \( d_1 \). Then there exists a unique map \( f_1: E_1 \to K(\Omega P_2) \) such that \( U(\sigma \Omega d'_1) = H^*(f_1) \). It follows that \( f_1 \partial_1 = * \) and that \( f_1 \partial_1 = K(\Omega d_1) \).

To carry on we must know \( R(f_1) = H^*(E_1) \otimes_{H^*(K(\Omega P_2))} F \). For this we note the exact sequence

\[
\Omega^{s-1} P_{s+1} \xrightarrow{d'_s} \ker \Omega^{s-1} d_{s-1} \to \Omega^{s-1} M \to 0
\]

defining the derived functor, and remember that \( \Omega^{s-1} = 0 \). Since \( \Omega \) is right exact,

\[
\Omega d'_s: \Omega^s P_{s+1} \to \Omega \ker \Omega^{s-1} d_{s-1}
\]

is surjective. Thus (taking \( s = 1 \)) tensoring over \( H^*(K(\Omega P_2)) = U(\Omega P_2) \) is the same as tensoring over \( U(\Omega \ker d_0) \); and that gives us

3.7 \( q_1 \) induces an isomorphism \( R(f_1) \cong U(M) \).

The succeeding inductive steps are identical, and yield

3.8 Theorem [4]. There exists a tower (3.2) in which (3.3) holds and such that

\[
H^*(X) \xrightarrow{q_1} H^*(E_1) \xrightarrow{d'_1} H^*(K(\Omega P_2))
\]

commutes.

Here we have slipped in the identification

\[
\Omega^{s-1} \ker d_{s-1} \cong \ker \Omega^{s-1} d_{s-1},
\]

which follows from the exact sequence

\[
0 \to \Omega^{s-1} M \to \Omega^{s-1} \ker d_{s-1} \to \Omega^{s-1} P_s \xrightarrow{\Omega^{s-1} d_{s-1}} \Omega^{s-1} P_{s-1}
\]

(see e.g. [2, page 93]) together with the fact that \( \Omega^{s-1} M = 0 \). A generalization of this identification, Lemma 4.5 below, is a key element to our approach and explains the form of the associated quotients in our main theorem, which we now restate in more precise form.
3.9 Theorem. Let $M \in \mathcal{U}$ be of finite type, and let $X$ be a simply connected space with $H^*(X) = U(M)$. Choose a projective resolution $M \rightarrow P_*$ in $\mathcal{U}$ and construct the associated Massey-Peterson tower (3.2). For any

$$k \leq \text{conn } M - p^{-2} \text{conn } M + 1,$$

$$\text{Im}(H^*(\Omega^k E_*) \rightarrow H^*(\Omega^k X)) = \text{Im}(H^*(\Omega^k E_*) \rightarrow H^*(\Omega^k E_{*+1})),$$

and is independent of the resolution. If we call this Hopf algebra $A_*$, then there is a fundamental sequence giving rise to a Hopf algebra extension sequence over $A$

$$F_p \rightarrow A_1 \rightarrow A_{p+1} \rightarrow U(\Omega^p \Omega^* M) \rightarrow F_p.$$

4 Looping the tower: proof of the main theorem

Begin again as in Section 3, but assume $X$ is $(k+1)$-connected and $P_*$ is $(s+k)$-connected. We wish to compute $H^*(\Omega^k E_*)$ and the map $\Omega^k p_*^* : H^*(\Omega^k E_*) \rightarrow H^*(\Omega^k E_{*+1})$. Start with the fundamental sequence for $\Omega^k E_1$: by 2.10, we have a commutative diagram

$$
\begin{array}{cccc}
0 & \rightarrow & \Omega^k U(M) & \rightarrow & \Omega^k G(f_0) & \rightarrow & \Omega^{k+1} \text{ker } d_0 & \rightarrow & 0 \\
& & \downarrow{} & \downarrow{} & \downarrow{\omega} & & \downarrow{\Omega \tau} & & \\
0 & \rightarrow & U(\Omega^k M) & \rightarrow & G(\Omega^k f_0) & \rightarrow & \Omega \text{ker } \Omega^k d_0 & \rightarrow & 0.
\end{array}
$$

(4.1)

The bottom left term, $R(\Omega^k f_0)$, is as displayed because $\Omega^k$ is right exact. Notice that $R(\Omega^k f_0) \rightarrow H^*(\Omega^k E_1)$ is a Hopf algebra map and that $PR(\Omega^k f_0) = \Omega^k M$ is zero in degrees greater than $\text{conn } M - k$.

We next state the inductive assumptions:

4.2 (i). The fundamental sequence for $H^*(\Omega^k E_*)$ is given by the bottom row of the commutative diagram

$$
\begin{array}{cccc}
0 & \rightarrow & \Omega^k R(f_{*+1}) & \rightarrow & \Omega^k G(f_{*+1}) & \rightarrow & \Omega^{*+k} \text{ker } d_{*+1} & \rightarrow & 0 \\
& & \downarrow{} & \downarrow{\omega} & & \downarrow{\Omega \tau} & & \\
0 & \rightarrow & R(\Omega^k f_{*+1}) & \rightarrow & G(\Omega^k f_{*+1}) & \rightarrow & \Omega \text{ker } \Omega^{*+k-1} d_{*+1} & \rightarrow & 0
\end{array}
$$

in which $\Omega \tau$ is the natural map.

(ii). There is a Hopf algebra fundamental sequence for $R(\Omega^k f_{*+1})$ of the form

$$0 \rightarrow R(\Omega^k f_{*+2}) \rightarrow G_{*+1} \rightarrow \Omega^p \Omega^{*+k-2} M \rightarrow 0.$$

(iii). $\text{conn } PR(\Omega^k f_{*+1}) \leq p^{-1} \text{conn } M - k.$
Let $\sigma: \Omega^k \ker d_{s-1} \to G(f_{s-1})$ be the splitting (as in Section 3). The $k$-invariant $f_s: E_s \to K(\Omega^k P_{s+1})$ is defined as induced by the composite

$$\Omega^k P_{s+1} \xrightarrow{\Omega^k d_{s+1}} \Omega^k \ker d_{s-1} \xrightarrow{\sigma} G(f_{s-1}) \xrightarrow{f} H^*(E_s).$$

Since $\omega$ is compatible with the cohomology suspension by 2.10, it follows that the $k$-invariant $\Omega^k f_s$ is induced by the composite

$$\Omega^{++k} P_{s+1} \xrightarrow{\Omega^{++k} d_{s+1}} \Omega^{++k} \ker d_{s-1} \xrightarrow{\Omega^{++k} \sigma} \Omega^k G(f_{s-1}) \xrightarrow{\omega} G(\Omega^k f_{s-1}) \xrightarrow{} H^*(\Omega^k E_s).$$

Write $\hat{\sigma} = \omega \circ \Omega^k \sigma$. We need to know $H^*(\Omega^k E_s)/\ker \Omega^k f_s^*$ and $\ker P(\Omega^k f_s)^*$, and for this it will suffice to find $\ker \hat{\sigma}$ and $\coker \hat{\sigma}$.

A diagram chase shows that $\hat{\sigma}$ is compatible with $\Omega^k$, and there results a commutative diagram with exact rows which defines an important map "fat theta".

$$0 \to \ker \Omega^k \tau \to \Omega^{++k} \ker d_{s-1} \to \Im \Omega^k \tau \to 0 \xrightarrow{\Theta} \Omega^k \tau \xrightarrow{\hat{\sigma}} \Im \Omega^k \tau \xrightarrow{\tau} 0.$$

We pause to record some information about $\tau$:

4.5 Lemma. There exists a natural commutative diagram in which each row and column is exact:

$$0 \to \ker \Omega^k \tau \to \Omega^{++k} \ker d_{s-1} \to \Omega^{++k} \ker d_{s-1} \to 0 \xrightarrow{\Theta} \Im \Omega^{++k} \tau \to 0.$$

Proof. The top row defines $\Omega^{++k} M$, and the second row follows from exactness of

$$P_{s+2} \xrightarrow{d_{s+1}} P_{s+1} \to \ker d_{s-1} \to 0.$$
and right-exactness of $\Omega^{*+k}$. The rows are thus exact. The map $\ker \Omega^{*+k}d_s \to \Omega^{*+k}P_{s+1}$ is the inclusion. The box to its left is clearly commutative, and this defines the vertical arrow to its right. We now construct the rest of the middle column and show it is exact; a diagram chase shows that this suffices.

The bottom three terms of the middle column are $\Omega$ applied to the exact sequence

$$\Omega^{*+k-1}P_{s+1} \overset{\alpha}{\longrightarrow} \ker \Omega^{*+k-1}d_{s-1} \longrightarrow \Omega^{*+k-1}M \longrightarrow 0.$$  

It remains to prove exactness at $\Omega^{*+k}P_{s+1}$. For this, consider the short exact sequence

$$\Omega^{*+k-1}P_{s+1} \begin{array}{c} \alpha \downarrow \end{array} \begin{array}{c} \Omega^{*+k-1}d_s \longrightarrow \Omega^{*+k-1}P_s \longrightarrow \Im \Omega^{*+k-1}d_{s-1} \longrightarrow 0.
\end{array}$$

Since $\Im \Omega^{*+k-1}d_{s-1}$ is a submodule of the projective $\Omega^{*+k-1}P_{s+1}$, and $\Omega$ is left exact, $\Omega$ is monic. Therefore $\ker \Omega \alpha$ and $\ker \Omega^{*+k}d_s$ coincide.

Now by 2.10, $\omega : \Omega^kG(f_{s-1}) \to G(\Omega^k f_{s-1})$ is compatible with the cohomology suspension $\Omega^k \to H^*(\Omega^k E_1)$, which lands in the coalgebra primitives. Since $R(\Omega^k f_{s-1}) = H^*(\Omega^k E_1) / \Im (\Omega^k f_{s-1}) \to H^*(\Omega^k E_1)$ is a Hopf algebra monomorphism, we conclude that $\Theta$ takes values in $PR(\Omega^k f_{s-1})$. By Lemmas 4.5 and 1.8, we know that

$$\text{conn } \ker \Omega r \geq p^{s+1}(\text{conn } M - k + 1).$$

By inductive assumption 4.2(ii)

$$\text{coconn } PR(\Omega^k f_{s-1}) \leq p^{s-1} \text{coconn } M - k.$$

The assumption

$$k - 1 \leq \text{conn } M - p^2 \text{coconn } M$$

of Theorem 3.9 lets us conclude that

$$\Theta = 0.$$

The serpentine lemma applied to (4.4), together with Lemma 4.5, then gives an isomorphism

$$\Omega^{*+k}M \overset{\psi}{\longrightarrow} \ker \delta$$

and a short exact sequence

$$0 \longrightarrow R(\Omega^k f_{s-1}) \longrightarrow \text{coker } \delta \longrightarrow \Omega^{*+k-1}M \longrightarrow 0.$$
We can now verify the next stage of the inductive assumption 4.2. The third term in the fundamental sequence for $H^*(\Omega^k E_{s+1})$ is the algebraic loops of the kernel of (4.3). To find this kernel, consider the following diagram, whose top half coincides with the diagram in Lemma 4.5. The map $f$ occurs in (4.3).

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{Im} \Omega^{s+k}d_{s+1} & \rightarrow & \ker \Omega^{s+k}d_s & \rightarrow & \Omega^{s+k}_{s+1}M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \text{Im} \Omega^{s+k}d_{s+1} & \rightarrow & \Omega^{s+k}P_{s+1} & \rightarrow & \Omega^{s+k}\ker d_{s-1} & \rightarrow & 0 \\
\downarrow f & = & \downarrow & & \downarrow & & \downarrow & & \\
G(\Omega^k f_{s-1}) & \rightarrow & G(\Omega^k f_{s-1}) & \rightarrow & \text{coker} f & = & \text{coker} \hat{\sigma} & \rightarrow & 0.
\end{array}
\tag{4.3}
\]

Now the rows and the right column are known to be exact, so the middle column is too. We conclude that 4.2(i) holds.

To obtain (ii) we establish a fundamental sequence for

\[R(\Omega^k f_s) = H^*(\Omega^k E_s) \otimes_{U(\Omega^{s+k}\Pi_{s+1})} F_p.\]

By (4.3), the $k$-invariant $f_s^*$ factors through $U(\hat{\sigma})$ mapping to the second factor in

\[H^*(\Omega^k E_s) = R(\Omega^k f_{s-1}) \otimes_{U(\Omega^k f_{s-1})} UG(\Omega^k f_{s-1}),\]

so by associativity of tensor product and right exactness of $U$,

\[R(\Omega^k f_s) \cong R(\Omega^k f_{s-1}) \otimes_{U(\Omega^k f_{s-1})} U\text{coker} \hat{\sigma}.\]

Thus (4.7) is a fundamental sequence for $R(\Omega^k f_s)$, establishing (ii).

Assumption 4.2(ii) follows by applying $P$ to the Hopf algebra short exact sequence

\[
F_p \rightarrow R(\Omega^k f_{s-1}) \rightarrow R(\Omega^k f_s) \rightarrow U(\Omega^{s+k-1}M) \rightarrow F_p \tag{4.9}
\]

associated to this fundamental sequence, and using Lemma 1.8.

This completes the proof of Theorem 3.9. The unfortunate restrictions on the coconnectivity of $M$ are needed to kill fat theta in the above argument. A reduction in the estimate of the size of the module of primitives in $R(\Omega^k f_s)$ would lead to a better theorem; in homology one wants to guarantee the creation of squares. Examples of D. Kraines seem relevant here. However, some restrictions are necessary, as shown by the very simple
The third term of the algebraic loops of the kernel of $\xi$ diagram, whose top half coincides with $\xi$ in (4.3).

$$
\begin{align*}
0 & \quad \downarrow \\
\Omega^{+}M & \rightarrow 0 \\
\Omega^{+k} \ker d_{n-1} & \rightarrow 0 \\
\delta & \\
G(\Omega^{k}f_{s-1}) & \\
\text{coker } \delta & \\
0.
\end{align*}
\tag{4.8}
$$

Let $F'(n) = PH^*K(Z,n)$, so that $H^*K(Z,n) = UF'(n)$. Then the Massey-Peterson theory of Section 2 applies to compute $H^*(E)$, since $(\text{Sq}^2)^* = U(\phi)$ where $\phi: F'(4) \rightarrow F'(2)$ sends $t_4$ to $\text{Sq}^2t_2$. The fundamental sequence

$$
0 \rightarrow E[t_2] \rightarrow G \rightarrow \Omega \ker \phi \rightarrow 0
$$

splits for degree reasons, so $H^*(E) = U(M)$ with

$$
M = \langle t_2 \rangle \oplus \Omega \ker \phi.
$$

Consider $H^*(\Omega E)$. The modules $\Omega\Omega_1^s(\Omega \ker \phi)$ are involved in Theorem 3.9. To compute them we have the short exact sequence

$$
0 \rightarrow \ker \phi \rightarrow F'(4) \rightarrow \langle t_4, \text{Sq}^4t_4, \text{Sq}^8\text{Sq}^4t_4, \ldots \rangle \rightarrow 0.
$$

The right term has trivial $\Omega_1$, so

$$
0 \rightarrow \Omega \ker \phi \rightarrow F'(3) \rightarrow \langle t_3 \rangle \rightarrow 0
$$

is still exact. Since $\Omega_1$ is left exact and $\Omega_1 F'(n) = 0$, $\Omega_1 \Omega \ker \phi = 0$. Thus

$$
\Omega\Omega_1^s M = \langle t_2 \rangle.
$$

for any $s \geq 1$, while

$$
U(\Omega \ker \phi) = H^*(S^2(3)).
$$

So our estimate of $H^*(\Omega E)$ resembles $H^*(\Omega S^2 \times S^2(3))$.

But in fact $\Omega E = K(Z,1) \times K(Z,2)$, since $\Omega \text{Sq}^2 \simeq \epsilon$. In understanding this example it may help to notice the square of fibrations

$$
\begin{array}{ccc}
S^3(3) & = & S^3(3) \\
\downarrow & & \downarrow \\
S^1 & \rightarrow & S^3 \rightarrow S^2 \\
\| & & \downarrow \\
S^1 & \rightarrow & K(Z,3) \rightarrow E
\end{array}
$$
which when looped back gives

\[ \Omega S^3 \rightarrow \Omega S^2 \rightarrow S^1 \]
\[ \downarrow \quad \downarrow \]
\[ K(2,2) \rightarrow \Omega E \rightarrow S^1 \]
\[ \downarrow \quad \downarrow \]
\[ S^0(3) = S^0(3). \]

The horizontal sequence splits; we have picked up the vertical sequence, which does not.

References