THE BURNSIDE BICATEGORY OF GROUPOIDS

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Dedicated to the memory of Gaunce Lewis, 1950–2006,
and Sam Gitler, 1933–2014

Sometime around 1980 Gaunce Lewis described to me a "Burnside category" of groupoids. It provides an additive completion of the Burnside category of groups. In these notes we describe this construction in the more structured setting of bicategories [7].

Morphisms from H to G in the Burnside category of groups are provided by sets with commuting right and left actions of H and G, which are free and finite over G. We extend this construction to groupoids by considering a G-action on a set X as a homomorphism $G \to \text{Aut}(X)$. This has an alternative expression in terms of "bi-actions," a variant of the discrete version of a topological construction which provides a convenient account of the theory of orbifolds [6]. We obtain a bicategory $\mathbf{B}$, the "Burnside bicategory of groupoids." It comes equipped with a morphism [7] from the bicategory $\mathbf{G}$ of groupoids, functors, and natural transformations, and a compatible functor from the opposite of the sub-bicategory $\mathbf{G}_{cf}$ generated by "finite covers" of groupoids. We think of the first as "stabilization" and the second as providing "transfers."

The morphism categories are symmetric monoidal, and the morphism groups in the Burnside category (of groupoids, as of groups) are obtained by adjoining inverses to the commutative monoids of isomorphism classes.

Moreover, the composition functors are "bilinear" and presumably satisfy the axioms described by Bert Guillou [4]. Guillou shows that such pairings can be rigidified to provide the input required by Tony Elmendorf and Mike Mandell [3] to produce a spectral category. The functor $G \mapsto \Sigma^\infty_+ BG$ will then extend to a spectral functor to spectra. On the subcategory of finite groupoids, the maps on morphism spectra are appropriate completions; this is the content of the "Segal conjecture."

In fact the disjoint union of groupoids plays the role of both coproduct and product in these bicategories. In this sense we have a "semi-additive bicategory." We have not attempted to write down a proper definition of this, but presumably: there is such a notion (see [10] for some relevant work); it implies Guillou's axioms (for a "pre-additive bicategory"); and examples such as the ones presented here are additive bicategories. In any case, passing to the underlying Burnside category of groupoids we obtain an

additive category with compatible functors from the category of groupoids and functors and from the opposite of the subcategory in which morphisms are finite covering maps.

This construction can be copied easily to give a "Morita category" of rings, in which the morphism symmetric monoidal categories are the groupoids of bimodules which are finitely generated and projective over the source ring, and bimodule isomorphisms. This leads to an extension of the construction of the algebraic K-theory of a ring to a spectral functor from the "spectral Morita category."

An $H \times G$ bi-set is a functor $H^{\mathrm{op}} \times G \to \mathbf{Set}$. Such functors go by various names in the literature: "modules," "bimodules," or "profunctors." They are also sometimes called "correspondences," but it seems better to reserve that term for a "span," a diagram of the form $H \leftarrow K \to G$, with suitable conditions of the functor $K \to H$. In the second part of this paper, we investigate a bicategory $\mathbf{C}$ of correspondences of groupoids. The appropriate condition on $K \to H$ seems to be that it should be a "weak finite cover" (Definition 4.1). We set up this bicategory and show how it is related to the bicategory $\mathbf{B}$, by means of appropriate co-end and translation category constructions. The stabilization and transfer functors are tautologous in the setting of corespondences, and display it as a bicategorical form of Quillen's Q-construction.

My interest in making these constructions explicit was re-ignited by a lecture by Clark Barwick in April, 2009, in which he carried out the analogue for schemes, and showed that algebraic K-theory extends to an enriched functor on the resulting spectral category. I claimed at the time that the construction for groupoids was standard and well known, but was not able to back up that claim when Dustin Clausen asked what I meant. These notes are my attempt to make good on that claim. My objective is to be self-contained and explicit. Much deeper and far reaching work is being done in this area by many people, among them Clark Barwick, Anna Marie Bohmann, David Gepner, Bert Guillou, Rune Haugseng, Peter May, and Stefan Schwede on the topological side, and most recently Emily Riehl and Dominic Verity [8] on the category theoretic side.

The first section below describes various notions connected with actions of groupoids on sets. This is extended in Section 2 to bi-sets, and is used set up the "Burnside bicategory" of groupoids. We recall some standard constructions in Section 3, and then describe the connection with topology as well as the two connections with the usual bicategory of groupoids. Section 4 is devoted to the correspondence bicategory, and Section 5 to the relation between it and the bi-set bicategory.

I am grateful to Chris Schommer-Pries for pointing out the bi-action model (see [6, 9]). Conversations with Jacob Lurie, Clark Barwick, Matthew Gelvin, Angélica Osorno, and Martin Frankland have also been helpful, as have comments and corrections by the referee. My longtime friendships with

Sam Gitler and with Gaunce Lewis enriched my mathematical life over many years, and I am forever grateful.

1. G-SETS, G-ACTIONS, AND COVERS

We begin by establishing several equivalent ways to view an "action" of a groupoid on a set, and note how the conditions of being finite or free appear from these various perspectives. To start with, we set up some basic categorical features of the category of G-sets.

1.1. **G-sets.** Write G_0 for the set of objects of the groupoid G, and G_1 for the set of morphisms. Call a functor $X : G \to \mathbf{Set}$ from a groupoid G to the category of sets a "left G-set" (or just a "G-set") and a functor $Y : G^{\mathrm{op}} \to \mathbf{Set}$ a "right G-set." Write X_γ and Y^γ for the values at $\gamma \in G_0$. If $g : \gamma' \to \gamma$ is a morphism in G, write $gx \in X_\gamma$ for the image of $x \in X_{\gamma'}$ under γ, and $yg \in Y^{\gamma'}$ for the image of $y \in Y^\gamma$. Left G-sets form a category G-$\mathbf{Set}$ in which morphisms are natural transformations.

The forgetful functor from G-sets to G_0-sets (where we regard G_0 as the discrete subgroupoid of G consisting of the identity morphisms) preserves limits and colimits. Moreover:

Lemma 1.1. *The forgetful functor G-$\mathbf{Set} \to G_0$-$\mathbf{Set}$ creates coequalizers.*

Proof. Let $u, v : X \to Y$, and for each $\gamma \in G_0$ let Z_γ be the coequalizer of $u_\gamma, v_\gamma : X_\gamma \to Y_\gamma$. We need to see that Z extends uniquely to a functor from G, so let $g : \gamma' \to \gamma$. Since u and v are natural transformations, there is a unique map $g : Z_{\gamma'} \to Z_\gamma$ compatible with $g : Y_{\gamma'} \to Y_\gamma$ under the projection map $Y \to Z$. Uniqueness implies that with this structure Z is a G-set. To check that it is the coequalizer, let $w : Y \to W$ be a map of G-sets such that $wu = wv$. For each γ, there is a unique map $Z_\gamma \to W_\gamma$ factorizing f_γ. These maps are compatible with g because they are after composing with the surjection $Y_\gamma \to Z_\gamma$. $\square$

This lemma lets us recognize effective epimorphisms in G-$\mathbf{Set}$. Recall that a map $X \to Y$ is an effective epimorphism if it is the coequalizer of the canonical fork $X \times_Y X \rightrightarrows X$. This is the categorical notion of dividing by an equivalence relation. The effective epimorphisms in $\mathbf{Set}$ are the surjections.

Corollary 1.2. *A morphism in G-$\mathbf{Set}$ is an effective epimorphism if and only if it is surjective on objects.*

Proof. Effective epimorphisms are preserved by the forgetful functor since it preserves limits and colimits. Conversely, suppose that $f : X \to Y$ is such that f_γ is surjective for each $\gamma \in G_0$. Surjections of sets are effective epimorphisms; that is,

$$X_\gamma \times_{Y_\gamma} X_\gamma \rightrightarrows X_\gamma \to Y_\gamma$$

is a coequalizer diagram. But the forgetful functor preserves pullbacks, so that $X_\gamma \times_{Y_\gamma} X_\gamma = (X \times_Y X)_\gamma$, and by the lemma the diagram extends to a coequalizer diagram in G-**Set**. $\square$

Definition 1.3. *A G-set X is* free *if for all $\gamma, \gamma' \in G_0$ and all $x \in X_{\gamma'}$ the map $G(\gamma', \gamma) \to X_\gamma$ sending g to gx is injective.*

For example, the *empty groupoid* $\varnothing$ has no objects (and hence no morphisms). It is initial in the category of categories. So there is a single $\varnothing$-set (namely the empty one). Its colimit is empty, and the freeness condition is satisfied vacuously. More interesting examples are provided by the following lemma.

Lemma 1.4. *The following are equivalent conditions on a G-set X.*
(1) X is a coproduct of co-representable G-sets.
(2) X is free.
*(3) X is projective in the category G-**Set**.*

Proof. To begin with, notice that any G-set X is canonically the target of an effective epimorphism from a coproduct of co-representable G-sets, given by the evident natural map

$$\coprod_{\gamma' \in G_0} \coprod_{x \in X_{\gamma'}} G(\gamma', -) \to X \,.$$

The source of this map can be made smaller, however: the map

$$\coprod_{\gamma' \in G_0} X_{\gamma'} \to G\backslash X$$

is surjective, so by the axiom of choice it admits a section. This means that there are subsets $X'_{\gamma'} \subseteq X'_\gamma$ such that the composite map

$$\coprod_{\gamma' \in G_0} X'_{\gamma'} \to G\backslash X$$

is bijective. The evident natural map

$$(1.1) \qquad\qquad i : \coprod_{\gamma' \in G_0} \coprod_{x \in X'_{\gamma'}} G(\gamma', -) \to X$$

is then still an effective epimorphism.

We aim to show that this map is an isomorphism if X is free. First, let $\gamma', \gamma'' \in G_0$, $x \in X_{\gamma'}$, $y \in X_{\gamma''}$, $f : \gamma' \to \gamma$ and $g : \gamma'' \to \gamma$, ans assume that $fx = gy$. Then $g^{-1}f : \gamma' \to \gamma''$ sends x to y, and so $x \in X'_\gamma$ and $y \in X_{\gamma''}$ have the same image in $G\backslash X$. Therefore $\gamma' = \gamma''$ and $x = y$.

If we now assume that X is free, this implies that $f = g$, and the map i is injective as well as surjective on objects, and hence an isomorphism: so any free G-set is a coproduct of co-representables.

We will also need the converse: For any $\gamma_0 \in G_0$, the co-representable functor $G(\gamma_0, -)$ is free: Given $\gamma, \gamma' \in G_0$ and $x \in G(\gamma_0, \gamma')$, $G(\gamma', \gamma) \to G(\gamma_0, \gamma)$ by $f \mapsto f \circ x$ is not just monic but bijective. So any coproduct of co-representable G-sets is free.

For any $\gamma_0 \in G_0$, the corepresentable functor $G(\gamma_0, -)$ is projective, and coproducts of corepresentables are too.

Finally, if X is projective then the map in (1.1) is split-epi, so X is a subobject of a coproduct of co-representables and hence of a free object. But clearly any subobject of a free G-set is free. $\square$

Two distinct finiteness conditions are relevant:

Definition 1.5. *A G-set X is* finitely generated *if its colimit is finite. It is* finite *if X_γ is finite for all $\gamma \in G_0$.*

A G-set is finitely generated if and only if it is a quotient of a coproduct of finitely many co-representable G-sets.

1.2. Some constructions. We recall a few standard constructions on G-sets.

The *co-end* of functors $X : G^{\mathrm{op}} \to \mathbf{Set}$ and $Y : G \to \mathbf{Set}$ is the set

$$X \otimes_G Y = \coprod_{\gamma \in G_0} X^\gamma \times Y_\gamma / \sim$$

where the equivalence relation is given by $(xg, y) \sim (x, gy)$ with $g : \gamma' \to \gamma$ in G, $x \in X^\gamma$, $y \in Y_{\gamma'}$.

Note that $* \otimes_G Y \xrightarrow{\cong} G \backslash Y$ (where $*$ denotes the constant functor with singleton values).

A functor $p : H \to G$ induces a pair of adjoint functors between the categories of G-sets and H-sets. The right adjoint sends a G-set $X : G \to \mathbf{Set}$ to the composite Xp. The left adjoint may be described using the co-end: Given a right H-set X or left H-set Y,

$$(p_* Y)_\gamma = G(p, \gamma) \otimes_H Y, \quad (p_* X)_\gamma = X \otimes_H G(\gamma, p).$$

In particular,

$$(1.2) \qquad G(-, \gamma) \otimes_G Y = Y_\gamma, \quad X \otimes_G G(\gamma, -) = X^\gamma.$$

It's easy to check that

$$p_* H(\eta, -) = G(p\eta, -), \quad p_* H(-, \eta) = G(-, p\eta).$$

Combining these we have

$$(1.3) \qquad G(p, \gamma) \otimes_H H(\eta, -) = G(p\eta, \gamma),$$

$$(1.4) \qquad H(-, \eta) \otimes_H G(\gamma, p) = G(\gamma, p\eta).$$

1.3. G-**actions.** The notion of a G-set follows the image of a G-action on X as a homomorphism $G \to \mathrm{Aut}(X)$. Normally however one thinks of an action of a group G on a set X as a map $\alpha : G \times X \to X$ satisfying certain properties. The groupoid story can be developed following this model as well.

A *left action* of a groupoid $G = (G_0, G_1)$ is a set P together with a map $p : P \to G_0$ and an "action map"

$$\alpha : G_1{}^s\times_{G_0} P \to P$$

over G_0 that is unital and associative. Here the prescript indicates that G_1 is to be regarded as a set over G_0 via the source map s. The fiber product is regarded as a set over G_0 via the target map t. In formulas, if we write $\alpha(g,x) = gx$, we are requiring $p(gx) = tg, (g'g)x = g'(gx), 1x = x$. A morphism of left G-actions is defined in the evident way.

A G-set $X : G \to \mathbf{Set}$ determines an action of G on the set

$$P_X = \coprod_{\gamma \in G_0} X_\gamma$$

by defining gx to be $X_g(x)$. Conversely, an action of G on P determines a G-set X_P by $(X_P)_\gamma = p^{-1}(\gamma)$, with functoriality determined in the evident way by the action map. These constructions provide an equivalence of categories between G-sets and G-actions.

Under this correspondence, a G-set is finitely generated precisely when the orbit set $P/(x \sim gx)$ of the corresponding action is finite. The equivalence relation on P is the one given by $x \sim gx$. P is finite when the fibers of $p : P \to G_0$ are finite. A G-set is free precisely when the corresponding G-action satisfies the following property: the shear map

$$\sigma : G_1{}^s\times_{G_0} P \to P \times P \quad , \quad (g,x) \mapsto (x,gx)$$

is injective. (Orbifold morphisms are modeled [6] by bi-actions in which the shear map is required to be an isomorphism.)

1.4. **Covering maps of groupoids.**

Definition 1.6. [2, 5] *A map of groupoids $p : H \to G$ is a* fibration *provided that for any $\eta' \in H_0$ and any $g : p\eta' \to \gamma$ there exists $h : \eta' \to \eta$ in H such that $ph = g$. It is a* cover *provided there's only one such pair (η, h), and a* finite cover *provided that in addition the set $p^{-1}\gamma$ is finite for all $\gamma \in G_0$.*

All three classes of maps are closed under composition and strict base-change.

A G-set X determines a new groupoid, the *translation groupoid GX* with

$$(GX)_0 = \coprod_{\gamma \in G_0} X_\gamma$$

and $GX(x, y) = \{g \in G_1 : gx = y\}$. The map of G-sets $X \to *$ induces a functor $GX \to G$, which is a cover.

Conversely, a cover $p : H \to G$ determines a G-set X with $X_\gamma = p^{-1}(\gamma) \subseteq H_0$ and $g : X_{\gamma'} \to X_\gamma$ defined using the unique morphism lifting. This establishes an equivalence of categories between G-sets and covers of G.

Our conditions on G-sets translate to conditions on covers as follows. The orbit set or colimit $G \backslash X$ of a G-set is precisely the set of components of the translation groupoid GX, so a cover $p : H \to G$ corresponds to a finitely generated G-set when the set of components of H is finite. It corresponds to a finite G-set when it is a finite cover. It corresponds to a free G-set when, for every $\eta \in H_0$ and $\gamma \in G_0$, the map $G(p\eta, \gamma) \to H_0$ that sends g to the target of its unique lift with source η is injective.

Covers provide another source of free H-sets, as shown by the following easily checked lemma.

Lemma 1.7. *Let $p : H \to G$ be a cover of groupoids. Then $G(\gamma, p)$ is a free H-set for all $\gamma \in G_0$, and if p is a finite cover then $G(\gamma, p)$ is finitely generated as an H-set for all $\gamma \in G_0$.*

The second statement follows from the identity $H \backslash G(\gamma, p) = p^{-1}(\gamma)$. $\square$

2. Bi-sets and bi-actions

Here we discuss the bivariant form of G-sets and G-actions, and show how they form the objects of the morphism categories in a bicategory structure on groupoids.

2.1. **Bi-sets.** For groupoids G and H, call a functor $H^{\mathrm{op}} \times G \to \mathbf{Set}$ an "(H, G)-bi-set." An (H, G)-bi-set is *admissible* if X^η is free and finitely generated as a G-set for every $\eta \in H_0$.

Much of what we do would work just as well if we dropped the finiteness condition; but it arises in the connection with topology, so we retain it anyway.

The admissible (H, G)-bi-sets form the objects of a category $\mathbf{B}(H, G)$ in which morphisms are the natural *isomorphisms*. $\mathbf{B}(H, G)$ is a symmetric monoidal groupoid, with the tensor product given by disjoint union.

This definition provides us with the morphism categories for a bicategory [7] in which the objects are groupoids. Define the composition functor $\mathbf{B}(H, G) \times \mathbf{B}(K, H) \to \mathbf{B}(K, G)$ by sending X, Y to $X \otimes_H Y$ with

$$(X \otimes_H Y)_\gamma^\kappa = X_\gamma \otimes_H Y^\kappa.$$

We must check that $X \otimes_H Y^\kappa$ is free and finitely generated. First freeness: Since $X \otimes_H -$ commutes with coproducts, we may assume by Lemma 1.4 that $Y^\kappa = H(\eta, -)$ for some $\eta \in H_0$. Then by (1.2) $X \otimes_H Y^\kappa \cong X^\kappa$, which is free by assumption.

Now finite generation: Note that

$$G\backslash(X \otimes_H Y^\kappa) \cong * \otimes_G (X \otimes_H Y^\kappa) \cong (* \otimes_G X) \otimes_H Y^\kappa = (G\backslash X) \otimes_H Y^\kappa.$$

Then use the facts that $G\backslash X^\eta$ is finite for every η, that $H\backslash Y^\kappa$ is finite, and the following observation.

Let X be a left G-set, and let $\{x_i \in X_{\gamma_i}\}$ represent the elements of $G\backslash X$. Then for any right G set W the map

$$\coprod_i W^{\gamma_i} \to W \otimes_G X$$

sending $w \in W^{\gamma_i}$ to (w, γ_i, x_i) is surjective. To see this, let $(w, \gamma, x) \in W \times_G X$. There exists i and $g : \gamma_i \to \gamma$ such that $gx_i = x$, so $(w, \gamma, x) = (x, \gamma, gx_i) = (wg, \gamma_i, x_i)$.

The identity object in $\mathbf{B}(G, G)$ is given by

$$(1_G)_\gamma^{\gamma'} = G(\gamma', \gamma)$$

The isomorphisms (1.2) provide the unitors

$$\rho : X \otimes_G 1_G \to X \quad , \quad \lambda : 1_G \otimes_G Y \to Y$$

The associator

$$\alpha : X \otimes_G (Y \otimes_H Z) \to (X \otimes_G Y) \otimes_H Z$$

sends a list $(\varphi, x \in X_\varphi^\gamma, \gamma, y \in Y_\gamma^\eta, \eta, z \in Z_\eta^\kappa, \kappa)$ to the same list, differently bracketed. The triangle identity for the unitors and the pentagon identity for the associator are easily checked.

It is clear that

$$(X \amalg X') \otimes_G Y = (X \otimes_G Y) \amalg (X' \otimes_G Y) , \ X \otimes_G (Y \amalg Y') = (X \otimes_G Y) \amalg (X \otimes_G Y').$$

This is the main ingredient in the check that these pairings are bilinear in the sense of [4].

2.2. **Bi-actions.** We now indicate the "bi-action" analogue of the bi-set bicategory.

Given a right G-action on Q and a left G-action on P, the "balanced product" is defined as the set

$$Q \times_G P = Q \times_{G_0} P / \sim$$

where $(qg, p) \sim (q, gp)$ for $q \in Q^\gamma$, $p \in P_{\gamma'}$, $g : \gamma' \to \gamma$.

A *bi-action* of a pair of groupoids G, H is a set P with a left action of G (with projection $p : P \to G_0$) and a right action of H (with projection $q : P \to H_0$), such that $p(xh) = p(x)$, $q(gx) = q(x)$, and $(gx)h = g(xh)$.

This is the same thing as a left action by $H^{\mathrm{op}} \times G$, so bi-actions are equivalent to functors $H^{\mathrm{op}} \times G \to \mathbf{Set}$, i.e. to bi-sets. The shear map for a

bi-action is compatible with projections to H_0:

$$G_1{}^s\times_{G_0} P \xrightarrow{\ \ \sigma\ \ } P \times_{H_0} P$$

$$H_0$$

Requiring X^η to be free for every $\eta \in H_0$ is equivalent to requiring that σ be injective. Requiring X^η to be finitely generated for $\eta \in H_0$ is equivalent to requiring that in the corresponding bi-action P, the G-action on the fiber of P over η is finite. Write $\mathbf{A}(H,G)$ for the category of bi-actions of (H,G) satisfying these conditions. This is category is equivalent to $\mathbf{B}(H,G)$.

Given another groupoid K, there is "composition" functor $\mathbf{A}(H,G) \times \mathbf{A}(K,H) \to \mathbf{A}(K,G)$ given by sending (P,Q) to the balanced product $Q \times_H P$, with the evident actions of K and G. We leave it to the reader to complete the structure of a bicategory, and to verify that this bicategory is bi-equivalent [7] to the bicategory $\mathbf{B}$. We will prefer the bi-set model.

2.3. **Additivity.** One of the most pleasing aspects of the bi-set bicategory $\mathbf{B}$ is that it is *additive*: the coproduct groupoid serves as both coproduct and product in $\mathbf{B}$.

Let G and H be two groupoids. Define the admissible $(G, G \amalg H)$-bi-set $X(1)$ by

$$X(1)_\gamma = G(-,\gamma) \quad , \quad X(1)_\eta = \varnothing \quad , \quad \gamma \in G_0, \eta \in H_0 .$$

The bi-set $X(2)$ has a similar description. These morphisms induce an equivalence of categories

$$\mathbf{B}(G \amalg H, K) \to \mathbf{B}(G,K) \times \mathbf{B}(H,K)$$

which establishes $G \amalg H$ as the coproduct in the bicategory $\mathbf{B}$.

On the other hand, define the $(G \amalg H, G)$-bi-set $Y(1)$ by

$$Y(1)^\gamma = G(\gamma,-) \quad , \quad Y(1)^\eta = \varnothing \quad , \quad \gamma \in G_0, \eta \in H_0 ,$$

and define $Y(2)$ similarly. Since $G(\gamma,-)$ is co-representable, $Y(1)$ is admissible. Thus we have morphisms in the category $\mathbf{B}$ that together induce a functor

$$\mathbf{B}(K, G \amalg H) \to \mathbf{B}(K,G) \times \mathbf{B}(K,H)$$

This functor is also an equivalence The groupoid $G \amalg H$, together with these structure morphisms, serves as a product in the bicategory $\mathbf{B}$.

2.4. **Burnside category.** By passing to the commutative monoids of isomorphism classes of objects in the morphism symmetric monoidal categories and then group-completing, we receive the *Burnside category* of groupoids. It is preadditive by construction: abelian group structures on the morphism

sets are given, and composition is bilinear. Since group-completion commutes with finite products, the constructions given above continue to provide products and coproducts in the Burnside category, which is therefore actually additive.

3. Topology

The classifying space construction provides a functor from $\mathbf{G}$ to the category of spaces. Under this functor, covers pass to covering spaces, and finite covers to finite covers.

The topological interest in the Burnside bicategory arises from prospect of using the bicategorical structure to form a spectrally enriched category of groupoids. Given a spectrally enriched category $\mathbf{C}$, an object F in $\mathbf{C}$ defines a spectral functor $\mathbf{C} \to \mathbf{Sp}$ to the category $\mathbf{Sp}$ of spectra sending G to $\mathbf{C}(F, G)$. Then we may define

$$\mathbf{C}(H, G) \to \mathbf{Sp}(\mathbf{C}(K, G), \mathbf{C}(K, H))$$

as the adjoint of the composition morphism $\mathbf{C}(H, G) \wedge \mathbf{C}(F, H) \to \mathbf{C}(F, G)$. In our example, taking F to be the singleton groupoid yields a spectral functor $\mathrm{Sp}\mathbf{B} \to \mathbf{Sp}$ sending G to the spectrum associated to the category (with coproducts) consisting of free and finitely generated G-modules. Since a finitely generated free G-set splits canonically into a coproduct of corepresentables, and

$$\mathrm{Aut}(\coprod_i n_i G(\gamma_i, -)) = \prod_i \Sigma_{n_i} \wr \mathrm{Aut}(\gamma_i) \,,$$

we find

$$\mathrm{Sp}\mathbf{B}(*, G) = \Sigma_+^\infty BG$$

The resulting maps

$$\mathrm{Sp}\mathbf{B}(H, G) \to F(\Sigma_+^\infty BH, \Sigma_+^\infty BG)$$

are, by the Segal conjecture, appropriate completions when G and H are finite.

3.1. **Stabilization and transfer.** So you can think of $\mathbf{B}$ as the stabilization of the bicategory of groupoids. As such it should enjoy transfer maps as well. In this section we describe this construction.

Let $\mathbf{G}$ denote the bicategory of groupoids, functors, and natural transformations. (It is actually a 2-category.) Let $\mathbf{G}_{fc}$ denote the sub-bicategory with $\mathbf{G}_{fc}(H, G)$ given by the groupoid of finite covers $H \to G$ and natural transformations between them.

There are morphisms [7] (automatically "homomorphisms," in our gropoid context) or pseudo-functors of bicategories

$$S : \mathbf{G} \to \mathbf{B} \,, \quad T : \mathbf{G}_{fc}^{\mathrm{op}} \to \mathbf{B}$$

each of which is the identity on objects. The morphism S sends a functor $p : H \to G$ to the (H, G)-bi-set $S(p) \in \mathbf{B}(H, G)_0$ defined by

$$S(p)_\gamma^\eta = G(p\eta, \gamma),$$

which is admissible since corepresentable functors are free and have singleton orbit sets, while T sends a finite cover $p : H \to G$ to the (G, H)-bi-set $T(p) \in \mathbf{B}(G, H)_0$ defined by

$$T(p)_\eta^\gamma = G(\gamma, p\eta),$$

which is admissible by Lemma 1.7.

On 2-cells, the morphism S sends a natural transformation $\alpha : p' \to p$ to the isomorphism of (H, G)-bi-sets given by $(\alpha_\eta^{-1})^* : G(p'\eta, \gamma) \to G(p\eta, \gamma)$, while T sends α to $(\alpha_\eta)_* : G(\gamma, p'\eta) \to G(\gamma, p\eta)$.

Compatibility of S with composition $K \xrightarrow{q} H \xrightarrow{p} G$ is given by the isomorphisms

$$S(q)_\gamma \otimes_H S(p)^\kappa \xrightarrow{\cong} S(q \circ p)_\gamma^\kappa$$

provided by (1.3). The required compatibilities with the associators (a hexagon) and unitors are easily verified. Similarly, compatibility of T is given by the isomorphisms

$$T(q)_\kappa \otimes_H T(p)^\gamma \xrightarrow{\cong} T(p \circ q)_\kappa^\gamma$$

provided by (1.4).

These two morphisms are linked by a "double coset formula" of the following form. Let $p : H \to G$ and $q : K \to G$. Assume that q is a finite cover, and form the strict pullback diagram

$$
\begin{array}{ccc}
H \times_G K & \xrightarrow{\bar{p}} & K \\
\downarrow{\scriptstyle \bar{q}} & & \downarrow{\scriptstyle q} \\
H & \xrightarrow{p} & G
\end{array}
$$

Then $\bar{q}$ is again a finite cover, and the pair (q, p) defines a natural isomorphism of (H, K)-bisets

$$\alpha : S(\bar{p}) \otimes_{H \times_G K} T(\bar{q}) \xrightarrow{\cong} T(q) \otimes_G S(p).$$

For given $\kappa \in K_0$ and $\eta \in H_0$, this is an isomorphism

$$K(\bar{p}, \kappa) \otimes_{H \times_G K} H(\eta, \bar{q}) \xrightarrow{\cong} G(-, q\kappa) \otimes_G G(p\eta, -).$$

Such a morphism is given by sending (k, h) to (qk, ph). Surjectivity follows from the assumption that q is a cover, and injectivity follows from a careful inspection of the equivalence relations defining the co-ends.

These functors and natural transformations enjoy various further coherence properties. To describe some of them, let's write $p_* = S(p)$ and

$q^* = T(q)$. Suppose we have a diagram

$$
\begin{array}{ccccc}
 & & \cdot & \xrightarrow{\hat{p}} & \cdot \\
 & & \Big\downarrow{\bar{q}'} & \bar{p} & \Big\downarrow{q'} \\
\cdot & \xrightarrow{\bar{p}'} & \cdot & \longrightarrow & \cdot \\
\Big\downarrow{\hat{q}} & & \Big\downarrow{\bar{q}} & & \Big\downarrow{q} \\
\cdot & \xrightarrow{p'} & \cdot & \xrightarrow{p} & \cdot
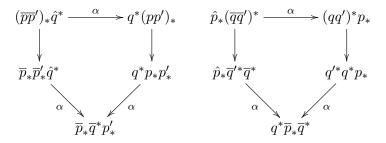\end{array}
$$

of pullback squares, and suppose that q and q' are finite covers. Then $\bar{q}$, $\bar{q}'$, and $\hat{q}$ are as well, and the following pentagons of natural transformation are commutative. We leave the composition compatibility morphisms, $(\overline{pp}')_* \to \bar{p}_* \bar{p}'_*$ and so on, undenoted.

$$
\begin{array}{ccc}
(\overline{pp}')_* \hat{q}^* & \xrightarrow{\ \ \alpha\ \ } & q^*(pp')_* \\
\downarrow & & \downarrow \\
\bar{p}_* \bar{p}'_* \hat{q}^* & & q^* p_* p'_* \\
& \searrow^{\alpha} \quad \swarrow^{\alpha} & \\
& \bar{p}_* \bar{q}^* p'_* &
\end{array}
\qquad
\begin{array}{ccc}
\hat{p}_*(\overline{qq}')^* & \xrightarrow{\ \ \alpha\ \ } & (qq')^* p_* \\
\downarrow & & \downarrow \\
\hat{p}_* \bar{q}'^* \bar{q}^* & & q'^* q^* p_* \\
& \searrow^{\alpha} \quad \swarrow^{\alpha} & \\
& q^* \bar{p}_* \bar{q}^* &
\end{array}
$$

3.2. **Remark on the Mackey category.** I owe the following remark to Rune Haugseng. It is actually more natural to consider, instead of the category $\mathbf{B}(H, G)$, what one might call the *bivariant Mackey bicategory* $\mathbf{M}(H, G)$. The objects are the same as the objects of $\mathbf{B}(H, G)$, namely (H, G)-bi-sets X such that for each $\eta \in H_0$ the G-set X^η is free and finitely generated. For any pair X, Y, of objects, we define $\mathrm{Hom}(X, Y)$ to be the category of spans [1] over this pair. Thus an object in $\mathrm{Hom}(X, Y)$ is a diagram $X \leftarrow Z \to Y$ in $\mathbf{B}(H, G)$, and a morphism from $X \leftarrow Z' \to Y$ is an isomorphism $Z' \to Z$ that is compatible with the projections to X and Y.

By replacing $\mathrm{Hom}(H, G)$ with the set of isomorphism classes of objects in $\mathrm{Hom}(H, G)$ one obtains what one might call the "bivariant Mackey category" $\overline{\mathbf{M}}(H, G)$. We remark that if a map $f : X \to Y$ in $\mathbf{B}(H, G)$ is such that the span $X \xleftarrow{=} X \xrightarrow{f} Y$ is an isomorphism in $\overline{\mathbf{M}}(H, G)$, then f is itself an isomorphism. So $\mathbf{B}(H, G)$ is the subcategory of isomorphisms in $\overline{\mathbf{M}}(H, G)$.

When H is a finite group and G is the trivial groupoid 1, $\mathbf{M}(H, 1)$ usual Mackey category of finite right H-sets and spans. It has finite coproducts, and so a corresponding spectrum, which is the fixed point spectrum of the H-equivariant sphere spectrum. The category $\mathrm{Hom}(X, Y)$ also has coproducts, and hence a corresponding spectrum. These will fit together to give a spectral enrichment of the bicategory $\mathbf{M}(H, 1)$, and presheaves on that spectral category provide one model for H-equivariant stable homotopy theory.

This makes it seem likely that the various bicategories $\mathbf{M}(H, G)$ assemble to a "Burnside tricategory" that will provide a model for "global equivariant stable homotopy theory" in the sense of Bohmann or Schwede.

4. Correspondences of Groupoids

The construction of compatible suspension and transfer homomorphisms for groupoids suggests an alternative construction of the target bicategory in terms of spans or correspondences. In this section we will propose a construction of a bicategory $\mathbf{C}$ of correspondences and in the next section we will check that it is bi-equivalent to the bi-set bicategory $\mathbf{B}$.

We begin with some well-known considerations about the "slice" or "over-category" in the bicategorical context. This is quite general, but we will continue to speak of groupoids.

Suppose G is a groupoid. An object of the over-bicategory $\mathbf{G}/G$ is a map $p : L \to G$ in $\mathbf{G}$. A morphism from $p' : L' \to G$ to $p : L \to G$ is given by a functor $t : L' \to L$ together with a natural transformation $\theta : p' \to pt$. The composition $(t'', \theta'') \circ (t', \theta')$ is given by $(t'' \circ t', \theta''_{t'} \circ \theta')$. The identity morphism on $p : L \to G$ is $(1, 1)$.

The morphisms from p' to p form the objects of a category $\mathbf{G}/G(p', p)$, in which a morphism $(\bar{t}, \bar{\theta}) \to (t, \theta)$ consists in a natural transformation $\psi : \bar{t} \to t$ such that $p\psi \circ \bar{\theta} = \theta : p' \to pt$. Composition is given by composition of natural transformations.

The composition law $(t'', \theta'') \circ (t', \theta')$ extends to a functor $\mathbf{G}/G(p, p'') \times \mathbf{G}/G(p', p) \to \mathbf{G}/G(p', p'')$. Let $\psi' : (\bar{t}', \bar{\theta}') \to (t', \theta')$, $\psi'' : (\bar{t}'', \bar{\theta}'') \to (t'', \theta'')$ be morphisms. The composed morphism $\psi : (\bar{t}''\bar{t}', \bar{\theta}''_{\bar{t}'}\bar{\theta}') \to (t''t', \theta''_{t'}\theta')$ is given by the diagonal in the commutative diagram

$$
\begin{array}{ccc}
\bar{t}''\bar{t}' & \xrightarrow{\ \bar{t}''\psi'\ } & \bar{t}''t' \\
{\scriptstyle \psi''_{\bar{t}'}}\downarrow & {\scriptstyle \psi}\searrow & \downarrow{\scriptstyle \psi''_{t'}} \\
t''\bar{t}' & \xrightarrow{\ t''\psi'\ } & t''t'
\end{array}
$$

The required diagram commutes by virtue of the commutativity of

$$
\begin{array}{ccccc}
p' & \xrightarrow{\ \bar{\theta}'\ } & p\bar{t}' & \xrightarrow{\ \bar{\theta}''_{\bar{t}'}\ } & p''\bar{t}''\bar{t}' \\
& & & & \downarrow{\scriptstyle p''\bar{t}''\psi'} \\
{\scriptstyle \theta'}\downarrow & {\scriptstyle p\psi'}\nearrow & & & \\
& & {\scriptstyle \bar{\theta}''_{t'}}\nearrow & p''\bar{t}''t' & \\
& & & & \downarrow{\scriptstyle p''\psi''_{t'}} \\
pt' & \xrightarrow{\ \theta''_{t'}\ } & & & p''t''t'
\end{array}
$$

Definition 4.1. *A groupoid is* weakly discrete *if there is at most one mor-phism between any two objects—that is, each component is "unicursal." A weakly discrete groupoid is* weakly finite *if it has finitely many components. A map $p : H \to G$ is a* weak cover *if it is a fibration and for every $\gamma \in G_0$ the fiber $p^{-1}\gamma$ is a weakly discrete groupoid, and a* weak finite cover *if all the fibers are weakly finite as well.*

Notice that the condition of being weakly discrete or weakly discrete and weakly finite is invariant under equivalence of groupoids. Weak covers and weak finite covers pull back to the same. It's also easy to see that compo-sitions of weak finite covers are weak finite covers, and that if $K' \to G$ and $K'' \to G$ are both weak finite covers then so is $K' \amalg K'' \to G$.
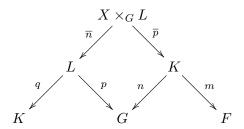
We are now in a position to define the groupoid of correspondences be-tween two groupoids. I am grateful to the referee for pointing out that my first attempt at this was not matching $\mathbf{B}(H, G)$ but was closer to the tricategorical speculation described in Section 3.2 above.

A "correspondence" from H to G is a diagram $H \xleftarrow{q} L \xrightarrow{p} G$ of groupoids in which q is a weak finite cover. A morphism from (q', L', p') to (q, L, p) is an equivalence class of triples (ϕ, t, θ), where $t : L' \to L$ is a functor of groupoids, $(t, \phi) : L' \to L$ in the groupoid $\mathbf{G}/H$, and $(t, \theta) : L' \to L$ in the groupoid $\mathbf{G}/G$. Two triples $(\overline{\phi}, \overline{t}, \overline{\theta})$ and (ϕ, t, θ) are equivalent if there is a natural transformation $\psi : \overline{t} \to t$ such that $q\psi \circ \overline{\phi} = \phi$ and $p\psi \circ \overline{\theta} = \theta$. We leave it to the reader to verify that the composition operation defined in $\mathbf{G}/G$ passes to equivalence classes and defines a category.

Definition 4.2. *The* correspondence groupoid $\mathbf{C}(H, G)$ *is the subcategory of isomorphisms in the category of correspondences from H to G.*

The equivalence class of a morphism $(\phi', t', \theta') : (q, L, p) \to (q, L, p)$ is an isomorphism exactly when (ϕ, t, θ) is an "internal equivalence" [7] in the bicategory $\mathbf{G}/(H \times G)$; that is to say, when there exists $(\phi', t', \theta') : (q, L, p) \to (q', L', p')$ and natural transformations $\psi : 1 \to tt' : L \to L$ and $\psi' : 1 \to t't : L' \to L'$ such that each of the following diagrams commutes.

(4.1)

$$
\begin{array}{ccc}
p' & \xrightarrow{\theta} & pt \\
{\scriptstyle p'\psi'} \searrow & & \downarrow {\scriptstyle \theta'_t} \\
& & p't't
\end{array}
\qquad
\begin{array}{ccc}
q' & \xrightarrow{\phi} & qt \\
{\scriptstyle q'\psi'} \searrow & & \downarrow {\scriptstyle \phi_t} \\
& & q't't
\end{array}
$$

$$
\begin{array}{ccc}
p & \xrightarrow{\theta'} & p't' \\
{\scriptstyle p\psi} \searrow & & \downarrow {\scriptstyle \theta_{t'}} \\
& & ptt'
\end{array}
\qquad
\begin{array}{ccc}
q & \xrightarrow{\phi'} & q't' \\
{\scriptstyle q\psi} \searrow & & \downarrow {\scriptstyle \phi_{t'}} \\
& & qtt'
\end{array}
$$

These groupoids form the morphism categories in a bicategory $\mathbf{C}$, the *correspondence bicategory* of groupoids. The composition functor $\mathbf{C}(G,F) \times \mathbf{C}(H,G) \to \mathbf{C}(H,F)$ is defined using the strict pullback:
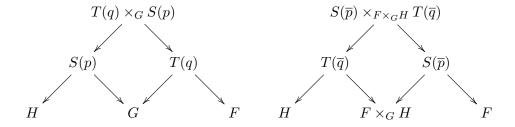
$$
\begin{array}{ccccccc}
& & & X \times_G L & & & \\
& & \swarrow^{\overline{n}} & & \searrow^{\overline{p}} & & \\
& & L & & K & & \\
& \swarrow^{q} & & \searrow^{p} \quad \swarrow^{n} & & \searrow^{m} & \\
K & & & G & & & F
\end{array}
$$

The pulled back functor $\overline{n}$ is a weak finite cover because n is, and so the composite $q \circ \overline{n}$ is again a weak finite cover.

We leave a discussion of the associator and unitors to the interested reader.

Stabilization and the transfer construction are even more tautologous in this setting than when expressed in terms of bimodules. The bicategory morphism $S : \mathbf{G} \to \mathbf{C}$ (for "stabilization") that sends a groupoid to itself, a functor $p : H \to G$ to the object $H \xleftarrow{1} H \xrightarrow{p} G$ of $\mathbf{C}(H,G)$, and a natural transformation $\theta : p' \to p$ to the class of $(1, 1, \theta)$. The transfer construction is a morphism $T : \mathbf{G}_{wf}^{\mathrm{op}} \to \mathbf{C}$ from the full sub bicategory of weak finite covers. It sends a groupoid to itself, a weak finite cover $q : H \to G$ to the object $G \xleftarrow{q} H \xrightarrow{1} H$ of $\mathbf{C}(G,H)$, and a natural transformation $\phi : q' \to q$ to the equivalence class of $(\phi, 1, 1)$.

These two functors are related by a "double coset formula": Let $p : H \to G$ be a functor and $q : F \to G$ a weak finite cover, and form the pullback $H \xleftarrow{\overline{q}} F \times_G H \xrightarrow{\overline{p}} F$. Write $S(p)$ and $T(q)$ for the apexes of the spans, so that we have two composed spans:

$$
\begin{array}{ccccc}
& T(q) \times_G S(p) & & & \\
& \swarrow \qquad \searrow & & & \\
S(p) & & T(q) & & \\
\swarrow \quad \searrow & & \swarrow \quad \searrow & & \\
H \qquad\quad G & & G \qquad\quad F & &
\end{array}
\qquad
\begin{array}{ccccc}
& S(\overline{p}) \times_{F \times_G H} T(\overline{q}) & & & \\
& \swarrow \qquad \searrow & & & \\
T(\overline{q}) & & S(\overline{p}) & & \\
\swarrow \quad \searrow & & \swarrow \quad \searrow & & \\
H \quad F \times_G H & & F \times_G H \quad F & &
\end{array}
$$

It is easy to construct a natural isomorphism

$$
T(q) \times_G S(p) \cong S(\overline{p}) \times_{F \times_G H} T(\overline{q})
$$

in the groupoid $\mathbf{C}(H,F)$. The pair of functors S, T, together with this natural isomorphism, is in a certain sense universal among such. The bicategory $\mathbf{C}$ is a bicategorical form of Quillen's Q-construction.

5. Comparison

Now we will see how the bicategories **B** and **C** are related.

It may help to look first at the case of groups. So let G be a group and K a subgroup. The inclusion $p : K \to G$ expresses K as a groupoid over G; but there is another groupoid over G that is expressed in terms of the transitive G set G/K, namely the translation groupoid $G(G/K)$. These two groupoids over G are in a suitable sense equivalent.

There is a natural map $\alpha : K \to G(G/K)$ in $\mathbf{G}/G$ that sends the unique object in K to $K \in (G(G/K))_0$. Note that in $G(G/K)$, $\mathrm{Aut}(K) = \{g \in G : gK = K\} = K$, and this isomorphism defines the functor on morphisms. This functor commutes strictly with the projection maps to G.

The map α has a non-natural quasi-inverse β obtained by picking coset representatives g_i, so that $G = \coprod_i g_i K$. In terms of these, define $\beta : G(G/K) \to K$ by sending $g_i K$ to the unique object of K, and $g : g_i K \to g_j K$ to $g_j^{-1} g g_i \in K$. This is functorial, but it does not commute with the projections to G. However, there is a natural transformation $\theta : q \to p\beta$ (where $q : G(G/K) \to G$ is the projection which forgets the objects). It is defined by $\theta_{g_i K} = g_i^{-1}$.

To simplify the formulae, we might as well chose the identity element of G as the representative of the identity coset K. With this choice, the composite $K \xrightarrow{\alpha} G(G/K) \xrightarrow{\beta} K$ is the identity morphism.

The composite $G(G/K) \xrightarrow{\beta} K \xrightarrow{\alpha} G(G/K)$ sends every object $g_i K$ to K, and $g : g_i K \to g_j K$ to $g_j^{-1} g g_i$. The natural transformation $q \to q\alpha\beta$ is given by $\theta_{g_i K} = g_i^{-1}$.

This morphism in $\mathbf{G}/G$ is far from an isomorphism, but there is an isomorphism from the identity to it in the category of endomorphisms of $G(G/K) \to G$ in the bicategory $\mathbf{G}/G$, given by $\psi_{g_i K} = g_i$.

Now we describe analogous constructions extending in two directions: to groupoids and to the bivariant situation.

Let G and H be groupoids and X an (H, G)-bi-set. The "double translation groupoid" GXH has objects

$$(GXH)_0 = \coprod_{(\gamma, \eta) \in G_0 \times H_0} X_\gamma^\eta$$

and morphisms $(\overline{\gamma}, \overline{x}, \overline{\eta}) \to (\gamma, x, \eta)$ given by pairs $g : \overline{\gamma} \to \gamma, h : \overline{\eta} \to \eta$ such that $g\overline{x} = xh \in X_\gamma^{\overline{\eta}}$. Composition is evident: Given also $(g', h') : (\gamma, x, \eta) \to (\gamma', x', \eta')$, the composite is given by $(g'g, h'h)$.

The double translation groupoid comes equipped with functors $p : GXH \to G$ and $q : GXH \to H$, both covariant.

To better understand the double translation category, it is useful to re-express it in terms of the Grothendieck construction. Let $F : H^{\mathrm{op}} \to \mathbf{Cat}$

be a functor. The *Grothendieck construction* of F is the category FH with objects

$$(FH)_0 = \coprod_{\eta \in H_0} F_0^\eta$$

and morphisms $(\eta', \varphi' \in F_0^{\eta'}) \to (\eta, \varphi \in F_0^\eta)$ given by $(h : \eta' \to \eta, f : \varphi' \to \varphi h)$. There is a covariant functor $FH \to H$ sending (η, φ) to η. If F^η has only identity morphisms, for each η, then the Grothendieck construction coincides with the translation category. The evident projection functor $FH \to H$ is a always fibration.

Similarly, there is a Grothendieck construction GF for a functor $F : G \to \mathbf{Cat}$. If X is an (H, G)-biset, then it is easy to construct natural isomorphisms

$$(GX)H \cong GXH \cong G(XH)$$

Since the fiber over η of $GXH \to H$ is the translation groupoid GX^η, we see:

Lemma 5.1. *If X^η is free and finitely generated as a G-set for all $\eta \in H_0$ then $q : GXH \to H$ is a weak finite cover.* $\square$

So far we have verified that the double translation groupoid construction gives a map on objects from $\mathbf{B}(H, G)$ to $\mathbf{C}(H, G)$. This construction extends to a functor: Let $f : X' \to X$ be an isomorphism of (H, G)-bi-sets. It induces a functor $t : GX'H \to GXH$ sending the object (γ, x', η) to (γ, fx', η) and the morphism $(g, h) : (\overline{\gamma}, \overline{x}', \overline{\eta}) \to (\gamma, x', \eta)$ to the morphism $(g, h) : (\overline{\gamma}, f\overline{x}', \overline{\eta}) \to (\gamma, fx', \eta)$: $g(f\overline{x}') = f(g\overline{x}') = f(x'h) = (fx')h$. The natural transformations needed to define a morphisms in $\mathbf{C}(H, G)$ from $GX'H$ to GXH can both be taken to be identity transformations.

We have now constructed a functor $\mathbf{B}(H, G) \to \mathbf{C}(H, G)$ for each pair of groupoids H, G. This is the start of the description of a morphism of bicategories [7], and the main result of this section is this:

Theorem 5.2. *These functors participate in a morphism of bicategories $\mathbf{B} \to \mathbf{C}$ that is in fact a bi-equivalence.*

The construction of this bi-equivalence occupies the rest of this section. By [7], it will suffice to construct a morphism that induces equivalences of morphism categories and is surjective up to equivalence on objects. The latter condition is obvious!

To specify a morphism of bicategories one needs to specify a transformation natural in the triple G, H, K

(5.1) $$[s] : (GXH) \times_H (HYK) \longrightarrow G(X \otimes_H Y)K$$

in $\mathbf{C}(H, K)$ (as well as a natural transformation comparing units which we leave to the reader). The map $[s]$ is represented by a functor s given on

objects by sending

$$((\gamma, x \in X_\gamma^\eta, \eta), (\eta, y \in Y_\eta^\kappa, \kappa))$$

to

$$(\gamma, x \in X_\gamma^\eta, y, \kappa) = (\gamma, x, y \in Y_\eta^\kappa, \kappa).$$

A morphism

$$((\overline{\gamma}, \overline{x}, \overline{\eta}), (\overline{\eta}, \overline{y}, \overline{\kappa})) \to ((\gamma, x, \eta), (\eta, y, \kappa))$$

in the pullback $(GXH) \times_H (HYK)$ consists of morphisms $g : \overline{\gamma} \to \gamma$, $h : \overline{\eta} \to \eta$, $k : \overline{\kappa} \to \kappa$ such that $g\overline{x} = xh$ and $h\overline{y} = yk$. This triple determines a morphism

$$(\overline{\gamma}, \overline{x}\overline{h}, \overline{y}, \overline{\kappa}) \to (\gamma, xh, y, \kappa)$$

in $G(X \otimes_H Y)K$ given by (g, k). The required equality $(g\overline{x}\overline{h}, \overline{y}) = (xh, yk)$ in $X \otimes_H Y$ is established by the morphism h.

This functor is strictly compatible with the projections to K and G. It is not bijective on objects, but it does give an isomorphism of correspondences and hence a map in the category $\mathbf{C}(K, G)$. To construct an inverse we begin by constructing a quasi-inverse s' for the functor s. Begin by choosing, for each $(\kappa, \gamma) \in K_0 \times G_0$, a splitting of the surjection

$$\coprod_{\eta \in H_0} X_\gamma^\eta \times Y_\eta^\kappa \to (X \otimes_H Y)_\gamma^\kappa$$

—so for each class $[x, y] \in (X \otimes_H Y)_\gamma^\kappa$, pick $\eta \in H_0$ and a representative $(x, y) \in X_\gamma^\eta \times Y_\eta^\kappa$. This defines the quasi-inverse s' on objects. We need to define the quasi-inverse on the morphism $g : \overline{\gamma} \to \gamma, k : \overline{\kappa} \to \kappa$ with $(g, k) : [\overline{x}, \overline{y}] \mapsto [x, y]$ (where $x \in X_\gamma^\eta$, $y \in Y_\eta^\kappa$, $\overline{x} \in X_{\overline{\gamma}}^{\overline{\eta}}$, $\overline{y} \in Y_{\overline{\eta}}^{\overline{\kappa}}$ are the chosen representatives). The fact that (g, k) is a morphism provides us with $h : \overline{\eta} \to \eta$ such that $g\overline{x} = xh$ and $h\overline{y} = yk$. Since X_γ is a free right H-set, the pair $(\overline{\eta}, h)$ is actually uniquely determined by the first equation. Declare $s'(g, k) = (g, h, k)$.

It is clear that $ss' = 1$; we can take $\psi = 1$ in the definition of equivalence of correspondences. The natural transformation $\psi' : 1 \to s's$ is given on the object $(x, y) \in X_\gamma^\eta \times Y_\eta^\kappa$ by $(1, h^{-1}, 1)$, where $h : \overline{\eta} \to \eta$ is the morphism described above relating the pair (x, y) to the chosen representative $s's(x, y) = (\overline{x}, \overline{y})$ of $[x, y] \in X_\gamma \otimes_H Y^\kappa$. The naturality diagram is assured again by the fact that X_γ is free as an H-set.

We leave to the reader the verification of the required compatibilities of the natural transformations $[s]$ and the unital analogue.

Up to this point we have been able to use functors between the summits of spans which strictly commute with the projection maps. The next step is to construct a quasi-inverse of the functor $\mathbf{B}(H, G) \to \mathbf{C}(H, G)$, and this will require the weaker notion of morphism. In the case of groups, with $G = 1$ you want to send a finite index subgroup K of H to the H-set H/K.

When $G \neq 1$ we get the (H, G)-set $H \times_K G$. The groupoid version sends $H \xleftarrow{q} K \xrightarrow{p} G$ to the (H, G)-set X with X_γ^η defined as the coend

$$(5.2) \qquad\qquad X_\gamma^\eta = G(p, \gamma) \otimes_K H(\eta, q).$$

Lemma 5.3. *If q is a weak finite cover then X^η is free and finitely generated for all $\eta \in H_0$.*

Proof. To see that X^η is free we must show that for all $x = (g : p\kappa \to \gamma, \kappa, h : \eta \to q\kappa) \in X_\gamma^\eta$, the map $G(\gamma, \gamma') \to X_{\gamma'}^\eta$ sending $\bar{g}$ to $\bar{g}x = (\bar{g}g, \kappa, h)$ is injective. Suppose that $\bar{g}'x = \bar{g}x \in X_{\gamma'}^\eta$. By the relation in the definition of the coend, there exists $k : \kappa \to \kappa$ such that $q(k)h = h$ and $\bar{g}'gp(k) = \bar{g}g$. The first identity implies that $k = 1$ since by hypothesis η/q is weakly discrete. Then the second relation implies that $\bar{g} = \bar{g}'$.

To see that X^η is finite note that $* \otimes_G G(\gamma_0, -) = \pi_0(GG(\gamma_0, -))$ is a singleton because the translation category is unicursal, so (using an evident notation)

$$* \otimes_{\gamma \in G} X_\gamma^\eta = (* \otimes_{\gamma \in G} (G(p, \gamma)) \otimes_K H(\eta, q) = * \otimes_K H(\eta, q).$$

Now $KH(\eta, q)$ is the homotopy fiber of q over η, and its set of components $* \otimes_K H(\eta, q)$ is finite since q is a weak finite cover. $\square$

This construction extends to a functor $\mathbf{C}(H, G) \to \mathbf{B}(H, G)$. Let $(t : K' \to K, \theta : p' \to pt, \phi : q' \to qt)$ be a morphism $(q', K', p') \to (q, K, p)$. To define the induced map $G(p', \gamma) \otimes_{K'} H(\eta, q') \to G(p, \gamma) \otimes_K H(\eta, q)$, pick $\kappa' \in K_0'$ and look at the natural map $\text{in}_{\kappa'} : G(p'\kappa', \gamma) \times H(\eta, q'\kappa') \to G(p', \gamma) \otimes_{K'} H(\eta, q')$. The map we are looking for is induced by $\text{in}_{t\kappa'} \circ ((\theta_{\kappa'}^{-1})^* \times (\phi_{\kappa'})_*) : G(p'\kappa', \gamma) \times H(\eta, q'\kappa') \to G(p, \gamma) \otimes_K H(\eta, q)$. It is straightforward to check that the equivalence relation defining morphisms in $\mathbf{C}(G, H)$ perfectly matches the equivalence relation defining elements in the coend (5.2).

Proposition 5.4. *These two constructions are the functors in an adjoint equivalence between $\mathbf{B}(H, G)$ and $\mathbf{C}(H, G)$.*

Proof. Given an (H, G)-bi-set X, there is a map

$$\beta : G(p, \gamma) \otimes_{GXH} H(\eta, q) \to X_\gamma^\eta$$

given as follows. An object of GXH is given as $(\bar{\gamma}, x \in X_{\bar{\gamma}}^{\bar{\eta}}, \bar{\eta})$. The map is

$$(g : \bar{\gamma} \to \gamma, (\bar{\gamma}, x, \bar{\eta}), h : \bar{\eta} \to \eta) \mapsto gxh \in X_\gamma^\eta$$

This map is an isomorphism. It is surjective since $(1, (\gamma, x, \eta), 1) \mapsto x \in X_\gamma^\eta$. To check that it is injective, let $(g' : \bar{\gamma}' \to \gamma, (\bar{\gamma}', x', \bar{\eta}'), h' : \eta \to \bar{\eta}')$ map to the same element of X_γ^η: $g'x'h' = gxh$. We seek a morphism in GXH— $(\hat{g} : \bar{\gamma}' \to \bar{\gamma}, \hat{h} : \bar{\eta}' \to \bar{\eta})$ with $\hat{g}x' = x\hat{h} \in X_{\bar{\gamma}}^{\bar{\eta}'}$—such that $g\hat{g} = g'$ and $\hat{h}h' = h$. There is a unique solution, $\hat{g} = g^{-1}g', \hat{h} = hh'^{-1}$. Then we calculate that $\hat{g}x' = g^{-1}g'x' = g^{-1}gxhh'^{-1} = x\hat{h}$.

Given a correspondence $H \xleftarrow{q} K \xrightarrow{p} G$, there is a functor

$$\alpha : K \to G(G(p,-) \otimes_K H(-,q))H$$

defined as follows. An object in the double translation category is determined by $(\gamma \in G_0, g : p\kappa \to \gamma, \kappa \in K_0, h : \eta \to q\kappa, \eta \in H_0)$. We send $\kappa \in K_0$ to $(p\kappa, 1, \kappa, 1, q\kappa)$. On morphisms, $k : \overline{\kappa} \to \kappa$ is sent to the pair (pk, qk); the coherence morphism is provided by k itself. This functor α is a morphism in the category of groupoids over G and over H; in fact the natural transformation allowed in a morphism is the identity map in both cases.

The required compatibilities are easily checked.

The functor α is not an isomorphism of groupoids, but it is an isomorphism in $\mathbf{C}(H, G)$. To see this, observe first that it factors as

$$K \xrightarrow{t} GG(p,-)K \times_K KH(-,q)H \xrightarrow{s} G(G(p,-) \otimes_K H(-,q))H$$

where s is the equivalence of (5.1) and t is the functor sending κ to $(1 : p\kappa \to p\kappa, \kappa, 1 : q\kappa \to q\kappa)$ and a morphism $k : \overline{\kappa} \to \kappa$ to (pk, k, qk). We construct a quasi-inverse t' for t that is compatible with the projection functors

$$GG(p,-)K \times_K KH(-,q)H \qquad .$$
$$\hat{q} \swarrow \qquad \qquad \searrow \hat{p}$$
$$H \qquad \qquad \qquad G$$

The functor t' sends $(\gamma, g : p\kappa \to \gamma, \kappa, h : \eta \to q\kappa, \eta)$ to κ, and a morphism

$(g', k', h') : (\overline{\gamma}, \overline{g} : p\overline{\kappa} \to \overline{\gamma}, \overline{\kappa}, \overline{h} : \overline{\eta} \to q\overline{\kappa}, \overline{\eta}) \to (\gamma, g : p\kappa \to \gamma, \kappa, h : \eta \to q\kappa, \eta)$

(where $g' : \overline{\gamma} \to \gamma$, $k' : \overline{\kappa} \to \kappa$, and $h' : \overline{\eta} \to \eta$ make

$$
\begin{array}{ccc}
p\overline{\kappa} & \xrightarrow{\overline{g}} & \overline{\gamma} \\
\downarrow{\scriptstyle pk'} & & \downarrow{\scriptstyle g'} \\
p k & \xrightarrow{g} & \gamma
\end{array}
\qquad\qquad
\begin{array}{ccc}
\overline{\eta} & \xrightarrow{\overline{h}} & q\overline{\kappa} \\
\downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle qk'} \\
\eta & \xrightarrow{h} & q\kappa
\end{array}
$$

commute) to k'. This functor does not satisfy $qt' = \hat{q}$ or $pt' = \hat{p}$, but there are natural transformations

$$\theta' : \hat{p} \to pt', \quad \phi' : \hat{q} \to qt'$$

given by

$$\theta' = g^{-1} : \gamma \to p\kappa, \quad \phi' = h : \eta \to q\kappa.$$

The required natural transformation $\psi : 1 \to t't$ can be taken to be the identity. The other one, $\psi' : 1 \to tt'$ is given on the object $(\gamma, g, \kappa, \eta, \eta)$ by the morphism $(g^{-1}, 1, h)$. Naturality is assured by virtue of the identities $g'\overline{g} = g \circ pk'$ and $hh' = qk' \circ \overline{h}$. The compatibility identities (4.1) are easily checked. □

This completes the verification that $\mathbf{B} \to \mathbf{C}$ is locally an equivalence, and hence the proof of Theorem 5.2.

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