# Comodules, sheaves, and the exact functor theorem

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Abstract. I sketch Mike Hopkins's proof of the Landweber exact functor theorem, and offer some related perspectives and notations.

In 1966 Pierre Conner and Ed Floyd [**3**] proved the following remarkable theorem: Complex $K$-theory is algebraically determined by complex bordism. To be precise, there is a ring homomorphism $Td : MU_* \to K_*$ (the "Todd genus") and a natural isomorphism from the resulting tensor product:

$$MU_*(X) \otimes_{MU_*} K_* \to K_*(X) \,.$$

Their proof required prior knowledge of the existence of the homology theory given by Bott periodicity. But it opened the possibility that other interesting homology theories might lie hidden inside complex bordism. Ten years later, Peter Landweber [**6**] provided a simple condition on an $MU_*$-module $M$ guaranteeing that

$$X \mapsto MU_*(X) \otimes_{MU_*} M$$

is a homology theory—and hence (according to George Whitehead and Ed Brown) is representable by a spectrum. This gives a construction of the Bott spectrum independent of Bott periodicity. It doesn't obviate the relevance of Bott periodicity, since it does not explain the relationship between the theory thus constructed and complex vector bundles. But Landweber's theorem opened the way to the wholesale construction of spectra, many of which have played dominant roles in algebraic topology throughout the intervening forty years.

This note has several loosely connected aims, all modest and none very novel. The theory of formal groups and their occurence in topology has been expressed many times in quite sophisticated terms ([**11, 5, 4, 8, 10**] are some examples). We hope here to achieve the level of generality and naturality one finds in those sources, without requiring as much infrastructure. So for example we avoid the language of stacks. On the other hand, we offer a coordinate-free definition of "formal group" directly suggested by the topology. We choose to work in an ungraded setting, which is to say that we restrict attention to even ("weakly") periodic theories. We obtain a grading at the appropriate moment by forming eigenspaces with respect to an action of the multiplicative group. This is quite standard in the case of even gradings, but the sign rules of topology—that is to say, the description in these terms of the symmetric monoidal category of all cobordism comodules (rather than just the evenly graded ones)—deserve further attention, as was pointed out to us by Jack Morava. We describe an extension of the Hopf algebroid underlying the theory of formal groups for which the category of comodules is equivalent to the category of $MU_*MU$-comodules as symmetric monoidal categories. The objects in the stack represented by this Hopf algebroid might be called spin formal groups, since they consist of a formal group with the added datum of a square root of the canonical line bundle.

Then comes an exposition of a proof of Landweber's theorem due to Mike Hopkins. It places the fact that height gives a complete classification of formal groups, up to faithfully flat extension, at the center of the proof, where it belongs. This proof doesn't quite give the full result, but it does cover all cases of interest.

This is a revision of notes that have been around for fifteen years; I spoke on this topic at meetings in Baltimore and Kinosaki in 2003, for example. I apologize for the long delay in publication. It's been long enough that I will probably forget to mention some of my creditors, but they certainly include Paul Goerss, Mike Hopkins, Mark Hovey, Gerd Laures, Jack Morava, Amnon Neeman, Charles Rezk, and Neil Strickland. And I thank the referee and editor, who have more recently rescued me from some of my more egregious errors.

I'm glad to have this opportunity to congratulate Paul Goerss and thank him for his many contributions to our subject!

## 1. Even periodic ring spectra and formal groups

DEFINITION 1.1. An *even periodic ring spectrum* is a commutative monoid $\mathbf{R}$ in the stable homotopy category such that (1) $\pi_1(\mathbf{R}) = 0$, and (2) for every integer $n$ the multiplication map

$$\pi_2(\mathbf{R}) \otimes_{\pi_0(\mathbf{R})} \pi_n(\mathbf{R}) \to \pi_{2+n}(\mathbf{R})$$

is an isomorphism.

As usual we will write $R_n = R^{-n} = \pi_n(\mathbf{R})$, and $R = R_0$. Since $R_2 \otimes_R R_{-2} \to R$ is an isomorphism, $R_2$ is an invertible $R$-module. We recall some facts about such modules.

PROPOSITION 1.2. Let $P$ be a module over the ring $R$. The following three conditions are equivalent.
1. There exists a module $Q$ such that $P \otimes_R Q \cong R$ as $R$-modules: $P$ is "invertible."

2. $P$ is finitely generated, projective, and $\dim_{k(\mathfrak{p})}(P/\mathfrak{p}P)_{\mathfrak{p}} = 1$ for all $\mathfrak{p} \in \mathrm{Spec} R$ (where $k(\mathfrak{p}) = (R/\mathfrak{p})_{\mathfrak{p}}$): $P$ is "projective of rank one."
3. There exists a faithfully flat map $R \to S$ such that $P \otimes_R S \cong S$ as $S$-modules.

Proof. This is standard. See for example [**1**], II.§5 Ex. 8, pp 147 f. □

Write $\omega$ for the invertible $R$-module $\pi_2(\mathbf{R})$. The homotopy groups of $\mathbf{R}$ are given by

$$R_{2n} = \omega^n\,, \quad n \in \mathbb{Z}\,.$$

We will write $R(\omega)$ for the evenly graded "Laurent series ring" with $\omega^n$ in degree $2n$ and the evident product structure. It is commutative, as you can see by tensoring with a faithfully flat extension $R \to S$ such that $\omega \otimes_R S = S$. This is the coefficient ring of $\mathbf{R}$: $\pi_*(\mathbf{R}) = R(\omega)$.

For any spectrum $X$ and any integer $n$, the product map

$$R_n(X) \otimes_R \omega \to R_{n+2}(X)$$

is an isomorphism.

When applied to a pointed space, $R_*(-)$ will always mean the reduced theory. Complex projective space with a disjoint basepoint adjoined will be denoted by $\mathbb{C}P_0^\infty$.

Proposition 1.3. Let $\mathbf{R}$ be an even periodic ring spectrum, and let

$$I = \ker\left(R^0(\mathbb{C}P_0^\infty) \to R^0(S^0)\right).$$

Then $R^0(\mathbb{C}P_0^\infty)$ is $I$-adically complete, and the associated graded ring of the $I$-adic filtration is the evenly graded "polynomial ring" $R[\omega]$:

$$\mathrm{gr}^{2n} R^0(\mathbb{C}P_0^\infty) = \omega^n\,, \quad n \geq 0\,.$$

Proof. In fact, the $I$-adic filtration coincides with the filtration associated with the Atiyah-Hirzebruch spectral sequence –

$$I^k = \ker\left(R^0(\mathbb{C}P_0^\infty) \to R^0(\mathbb{C}P_0^{k-1})\right)$$

– and this spectral sequence collapses at $E_2$. We omit the details. □

This suggests the following definition.

Definition 1.4. A (one-dimensional smooth) *formal algebra* over a ring $R$ is a commutative $R$-algebra $\mathcal{O}$ equipped with an ideal $I$ (the "augmentation ideal") such that
(1) $\mathcal{O}$ is $I$-adically complete,
(2) the $R$-module $\omega = I/I^2$ (the "cotangent space") is invertible, and
(3) the natural map $R[\omega] \to \mathrm{gr}^*\mathcal{O}$ is an isomorphism (so in particular $\mathcal{O}/I = R$).
A morphism of formal $R$-algebras is an $R$-algebra homomorphism sending one augmentation ideal into the other.

So $R^0(\mathbb{C}P_0^\infty)$ is a formal $R$-algebra, with cotangent space $\omega = R^0(S^2)$, and the rest of the graded coefficient ring is a functor of this formal algebra.

Similarly, the skeleton filtration of

$$R^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = R^0(\mathbb{C}P^\infty)\widehat{\otimes}_R R^0(\mathbb{C}P^\infty)$$

has associated graded ring given by the polynomial algebra generated by two copies of $\omega$. The map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ classifying the tensor product of line bundles induces a completed Hopf algebra structure on $R^0(\mathbb{C}P_0^\infty)$ which, on the associated graded level, is the canonical diagonal defined in grading $2n$ by sending $\omega^n$ to each factor $\omega^i \otimes_R \omega^j$ (with $i + j = n$) by the inverse of the natural isomorphism.

The algebraic structure of $R^0(\mathbb{C}P_0^\infty)$ is captured by the following definitions.

Definition 1.5. A *formal Hopf algebra* over a ring $R$ is a commutative filtration preserving completed $R$-Hopf algebra structure on a formal $R$-algebra over $R$ inducing on the associated graded ring the canonical Hopf algebra structure. A morphism of formal $R$-Hopf algebras is a map of formal $R$-algebras compatible with the Hopf algebra structures.

One should think of a formal $R$-algebra as the ring of functions or coordinate ring on a (smooth) "formal curve" over $R$. So the category of *formal curves* over $R$ is the opposite of the category of formal $R$-algebras. Because of this, we may write $\mathcal{O}_X$ for a formal $R$-algebra corresponding to a formal curve $X$, and $\varphi^* : \mathcal{O}_Y \to \mathcal{O}_X$ for the homomorphism corresponding to a map $\varphi : X \to Y$ of curves. Similarly, the category of *formal groups* over $R$ is the opposite of the category of formal $R$-Hopf algebras.

So an even periodic ring spectrum $\mathbf{R}$ determines more algebraic data than merely its graded coefficient ring $R_*$; one also acquires a formal group over $R = R_0$, with function ring given by $R^0(\mathbb{C}P_0^\infty)$ with its natural structure. We will denote this formal group by $G_{\mathbf{R}}$.

For a fixed ring $R$, denote by $\mathbf{FG}_R$ the category of formal groups over $R$ and isomorphisms between them. An isomorphism $\varphi$ induces an isomorphism on the cotangent spaces. A ring homomorphism $f : R \to S$ determines a functor $\mathbf{FG}_R \to \mathbf{FG}_S$ sending $\mathcal{O}_G$ to $\mathcal{O}_G\widehat{\otimes}_R S$. We will write the image of $G$ under $f$ as $fG$: so $\mathcal{O}_{fG} = \mathcal{O}_G\widehat{\otimes}_R S$.

These are subcategories of a larger category $\mathbf{FG}$, whose objects are pairs $G/R$ in which $G$ is a formal group over $R$. A morphism $G/R \to H/S$ consists of a homomorphism $f : R \to S$ together with an isomorphism of formal groups over $S$, $H \to fG$. Then $\omega_G \otimes_R S \xrightarrow{\cong} \omega_H$.

The assignment $\mathbf{R} \mapsto G_{\mathbf{R}}/R$ provides a functor from even periodic ring spectra to the category of formal groups.

The category $\mathbf{FG}$ should be compared with the "category of lines." For a ring $R$ let $\mathbf{Lines}_R$ be the category of invertible $R$-modules and their isomorphisms. A ring homomorphism $f : R \to S$ induces a functor $f : \mathbf{Lines}_R \to \mathbf{Lines}_S$, sending $\omega$ to $\omega \otimes_R S$. These categories fit into the larger category

**Lines**, whose objects are pairs $\omega/R$ and in which a morphism $\omega/R \to \alpha/S$ is a ring homomorphism $f : R \to S$ together with an isomorphism $\omega \otimes_R S \to \alpha$. Formation of the cotangent space provides a functor $\mathbf{FG} \to \mathbf{Lines}$.

Write $I_G$ for the augmentation ideal in the formal algebra $\mathcal{O}_G$, and $d : I_G \to \omega$ for the natural projection. In the topological situation, this corresponds to restriction to $\mathbb{C}P^1$:

$$d : R^0(\mathbb{C}P^\infty) \to R^0(S^2) = R_2 \,.$$

Definition 1.6. Let $\mathcal{O}$ be a formal $R$-algebra. A *parameter* is an element $t \in I$ such that $dt$ generates $\omega$ as a free $R$-module.

Equivalently, $t : R \to I$ is an $R$-module map such that the composite $dt : R \to \omega$ is an isomorphism; or such that the induced map

$$R[[t]] \to \mathcal{O}$$

is an isomorphism. Since $d : I \to \omega$ is surjective, a parameter exists exactly when $\omega$ is trivial as an invertible $R$-module. According to Proposition 1.2, any formal curve admits a parameter after passing to a suitable faithfully flat extension.

A parameter $t$ on a formal group $G$ transports the group structure (i.e. the diagonal) from $\mathcal{O}_G$ to $R[[t]]$. The structure of a formal group on the formal curve $R[[t]]$ is determined by the image of $t$ under the diagonal. This is precisely a *formal group law*, that is, a power series

$$F(s,t) = \sum_{i,j} a_{i,j} s^i t^j \in R[[s,t]]$$

such that

$$F(s,0) = s\,, \quad F(0,t) = t\,,$$
$$F(s, F(t,u)) = F(F(s,t),u)\,, \quad F(s,t) = F(t,s)\,.$$

The dual $\omega^{-1}$ of $\omega$ is the *tangent space* to the curve. If $t$ is a parameter, there is a unique element

$$\frac{1}{dt} \in \omega^{-1}$$

dual to $dt$, and it generates $\omega^{-1}$ as a free $R$-module. The composite

$$\frac{d}{dt} : I \to R$$

sends $f(t)$ with $f(0) = 0$ to $f'(0) \in R$. In $R(\omega)$, $dt$ and $1/dt$ are inverse units. The "Euler operator"

$$x = t \otimes \frac{1}{dt} \in I \otimes_R \omega^{-1}$$

is an *Euler class*, meaning that it maps to 1 under the natural map $I \otimes_R \omega^{-1} \to \omega \otimes_R \omega^{-1} = R$.

Giving a parameter for a formal curve is equivalent to giving a unit and an Euler class. We have just obtained a unit $dt$ and an Euler class $x = t/dt$ from a parameter $t$. Conversely, suppose we are given an element $u \in \omega$

generating it as a free $R$ module and an element $x \in I \otimes \omega^{-1}$ mapping to $1 \in R$ under $d \otimes 1$. Then $t = ux$ is a parameter with $dt = u$ and $x = t/dt$.

In the topological context,

$$\mathcal{O} \otimes_R R(\omega) = R^*(\mathbb{C}P_0^\infty)\,,$$

and $x \in I \otimes_R \omega^{-1} = R^2(\mathbb{C}P^\infty)$ is a choice of Euler class for complex line bundles.

The units allow us to define a graded formal group law, lying in degree 2:

$$\widetilde{F}(x,y)\,dt = F(x\,dt, y\,dt)\,.$$

That is,

$$\widetilde{F}(x \otimes 1, 1 \otimes x) = \sum_{i,j} a_{i,j}(dt)^{i+j-1} x^i \otimes x^j\,.$$

Topologically, this is the law controlling the Euler class of a tensor product of line bundles.

## 2. Cobordism comodules

Let $F$ and $G$ be formal group laws over a ring $R$. A homomorphism $F \to G$ is determined by a formal power series $\varphi(t)$ such that $\varphi(0) = 0$ and $\varphi(F(u,v)) = G(\varphi(u), \varphi(v))$. The corresponding Hopf algebra map $\varphi^* : \mathcal{O}_G \to \mathcal{O}_F$ is determined by sending $t$ to $\varphi(t)$, and hence $(\varphi^* f)(t) = f(\varphi(t))$. The map induced on cotangent spaces is given by multiplication by $\varphi'(0)$, so $\varphi$ is an isomorphism exactly when $\varphi'(0)$ is a unit in $R$. It is *strict* if $\varphi'(0) = 1$.

The functor assigning to a ring $R$ the groupoid $\mathcal{F}_R$ of formal group laws and their isomorphisms over $R$ is representable. The representing object plays a fundamental role in the topological applications of formal groups.

The universal formal group law is easily constructed: form the symmetric algebra $\mathbb{Z}[a_{i,j} : i, j \geq 1]$ and divide it out by the ideal generated by the relations implied by requiring

$$G_u(s,t) = s + t + \sum_{i,j \geq 1} a_{i,j} s^i t^j$$

to be a formal group law. The quotient ring is the *Lazard ring* $L$. Given a formal group law $G/R$, the set of morphisms in $\mathcal{F}_R$ into it is precisely given by the set of invertible power series over $R$: an invertible power series $\varphi(t) \in tR[[t]]$ is an isomorphism to $G$ from a unique formal group law, namely $G^\varphi$ given by

$$G^\varphi(s,t) = \varphi^{-1}(G(\varphi(s), \varphi(t))).$$

Thus the functor sending a ring to the set of all morphisms in $\mathcal{F}_R$ is representable by the ring

$$W = L[b_0^{\pm 1}, b_1, \ldots]$$

with universal morphism given by the invertible power series

$$\varphi_u(t) = \sum_{i=1}^{\infty} b_{i-1} t^i$$

regarded as an isomorphism to $G_u$.

The structure maps for the groupoid $\mathcal{F}$ are represented by ring homomorphisms involving $L$ and $W$:

$$\eta_R : L \to W \quad \text{represents the source} \tag{1}$$

$$\eta_L : L \to W \quad \text{represents the target} \tag{2}$$

$$\epsilon : W \to L \quad \text{represents the identity map} \tag{3}$$

$$\Delta : W \to W \otimes_L W \quad \text{represents composition} \tag{4}$$

The map $\eta_R$ embeds $L$ as constant polynomials in the $b_i$. The resulting structure is known as a "Hopf algebroid." An *even cobordism comodule* is a right comodule over this Hopf algebroid.

It is instructive to consider again the category **Lines** from this perspective. The category of "parametrized lines" over a ring $R$ is equivalent to the group of units $R^\times$ regarded as a groupoid with just one object. This functor from rings to groups is represented by the Hopf algebra $(\mathbb{Z}, \mathbb{Z}[e^{\pm 1}])$, in which $e$ is grouplike: $\Delta e = e \otimes e$. Given an abelian group $M$, the data of a comodule structure $\psi : M \to M[e^{\pm 1}]$ on $M$ is equivalent to a splitting of $M$ into summands indexed by the integers; that is, a *grading*:

$$M_n = \{x \in M : \psi x = x \otimes e^n\}\,.$$

In these terms, $\psi x = \sum x_n \otimes e^n$ where $x_n \in M_n$ and $x = \sum x_n$. The formal group law describing the Euler class of a tensor product in a complex oriented cohomology theory (for example an even periodic theory endowed with a choice of parameter) is *graded*, with $|s| = |t| = -2$. The Lazard ring $L$ admits a grading such that the universal formal group law over it is graded, and is the universal graded formal group law. In this grading $|a_{i,j}| = 2(1-i-j)$. The fundamental theorem of Quillen asserts that this graded ring is canonically isomorphic to the complex bordism ring $MU_*$. The groupoid of graded formal group laws and their *strict* graded isomorphisms is represented by a graded Hopf algebroid $(MU_*, MU_*MU)$ with $MU_*MU = MU_*[b_1, b_2, \ldots]$, and complex bordism determines a functor from spectra to the category of comodules over this Hopf algebroid. Our conventions make it natural to consider *right* comodules.

Not every $MU_*MU$-comodule is evenly graded, and to handle arbitrary gradings we extend the Hopf algebroid $(L, W)$ to $(L, W^s)$, in which

$$W^s = L[e^{\pm 1}, b_1, b_2, \ldots]\,.$$

$W$ maps to $W^s$ by sending $b_0 \mapsto e^2$; $e$ is grouplike, and again we can grade $(L, W^s)$-comodules using the comodule structure induced by the projection $W^s \to L[e^{\pm 1}]$.

DEFINITION 2.1. A *cobordism comodule* is a right comodule over the Hopf algebroid $(L, W^s)$.

The category of cobordism comodules is equivalent to the category of $(MU_*, MU_*MU)$-comodules. For example, $L$ itself is a cobordism comodule, with structure map $\eta_R : L \to W^s = L \otimes_L W^s$. The *suspension* of a cobordism comodule $M$ is the comodule $\Sigma M$ with the same underlying $L$-module but the coaction multiplied by $e$. Then

$$M_n = (\Sigma^{-n} M)_0 .$$

The geometrical object classified by the Hopf algebroid $(L, W^s)$ is captured by the following definition.

DEFINITION 2.2. A *spin formal group* over ring $R$ is a triple $(G, \lambda, i)$ where $G$ is a formal group over $R$, $\lambda$ is an invertible $R$-module, and $i : \lambda^2 \to \omega_G$ is an isomorphism.

Spin formal groups over $R$ form a category in which a morphism $(G, \lambda, i) \to (H, \mu, j)$ is an isomorphism of formal groups $\varphi : G \to H$ together with an isomorphism $\mu \to \lambda$ of invertible modules that is compatible with $i$ and $j$. Write $\mathbf{SFG}_R$ for this category. Clearly a ring homomorphism $f : R \to S$ determines a functor $\mathbf{SFG}_R \to \mathbf{SFG}_S$, and we can define the larger category $\mathbf{SFG}$ of spin formal groups over arbitrary rings in analogy with $\mathbf{FG}$.

The cotangent space of a formal group *law* over a ring $R$ is canonically trivialized, and there is, consequently, a canonical "trivial" spin structure with $\lambda = R$. Any spin formal group $(G/R, \lambda, i)$ becomes isomorphic to such a formal group law after passing to a faithfully flat extension that trivializes $\lambda$.

A *spin parameter* for a spin formal group $(G/R, \lambda, i)$ is a parameter $t \in I$ for $G/R$ together with an element $a \in \lambda$ generating $\lambda$ as a free $R$-module and such that $i(a^2) = dt \in \omega_G$. These data provide an isomorphism with a formal group law with its trivial spin structure.

A morphism between the spin formal groups associated in this way to formal group laws, $(G/R, R, 1) \to (H/S, S, 1)$, consists of not only a ring homomorphism $f : R \to S$ and an isomorphism of formal group laws $\varphi : H \to fG$, but also an element $a \in S$ such that $a^2 = \varphi'(0)$. For example, let $G_m$ be the multiplicative formal group law, $G_m(x, y) = x + y - xy$, with its trivial spin structure. As a spin formal group, it has an automorphism that is the identity on the formal group but $-1$ on the spin structure. This corresponds to the automorphism of topological $K$-theory that multiplies by $-1$ on odd degrees and $+1$ on even degrees. This automorphism is represented by an automorphism of the graded spectrum representing graded $K$-theory.

Write $\mathcal{F}^s_R$ for the category of spin formal group laws over $R$. This gives a functor from rings to groupoids, corepresented by the Hopf algebroid $(L, W^s)$. The ring $W^s$ supports *two* spin formal group laws, $G_L = \eta_L G_u$

and $G_R = \eta_R G_u$, and an isomorphism $G_R \to G_L$ given by

$$\varphi_u(t) = \sum_{i=1}^{\infty} b_{i-1} t^i \,, \quad a = e \,.$$

## 3. Cobordism sheaves

We now give an alternate "coordinate-free" description of the category of cobordism comodules, using the notion of spin formal groups, that is well-adapted to a discussion and proof of the Landweber exact functor theorem. We will often drop notation for the spin structure.

Definition 3.1. A *cobordism sheaf* is a functor $\mathcal{M} : \mathbf{SFG} \to \mathbf{Ab}$, together with a right $R$-module structure on $\mathcal{M}(G/R)$ for each $G/R \in \mathbf{SFG}$, such that for any morphism $G/R \to H/S$ of spin formal groups the induced map $\mathcal{M}(G/R) \to \mathcal{M}(H/S)$ is $R$-linear (where $R$ acts on the target via the ring homomorphism $R \to S$) and extends to an isomorphism

$$\mathcal{M}(G/R) \otimes_R S \xrightarrow{\cong} \mathcal{M}(H/S) \,.$$

These form a category with the evident morphisms. The term "cobordism sheaf" is short for "quasi-coherent sheaf over the stack of spin formal groups." If we omit the spin structure, we get the category of *even cobordism sheaves.*

Example 3.2. The functor assigning to $(G/R, \lambda, i)$ the $R$-module $\lambda$ and to the morphism $(f, \varphi, k) : (G/R, \lambda, i) \to (H/S, \mu, j)$ the composite

$$\lambda \longrightarrow \lambda \otimes_R S \xrightarrow{k} \mu$$

is evidently a sheaf, one we denote by $\sigma$. So also is the functor $\sigma^n$ assigning to $(G/R, \lambda, i)$ the $R$-module $\lambda^n$. In particular $\sigma^0$ simply assigns to $(G/R, \lambda, i)$ the underlying ring $R$. This might be called the *structure sheaf.*

Example 3.3. A spin formal group $(G/R, \lambda, i)$ determines a commutative ring $\mathcal{W}^s(G/R, \lambda, i)$ such that giving a homomorphism from it to a ring $T$ is the same as giving a ring homomorphism $g : R \to T$ together with a spin parameter on $g(G/R, \lambda, i) \in \mathbf{SFG}_T$. The functor $\mathcal{W}^s$ is a sheaf: Given $f : R \to S$, a map $\mathcal{W}^s(G/R, \lambda, i) \otimes_R S \to T$ consists of a pair of ring homomorphisms $g : R \to T$ and $h : S \to T$ such that $hf = g$, together with a spin parameter on $g(G, \lambda, i)$; but this is the same as a ring homomorphism $h : S \to T$ and a spin parameter on $hf(G, \lambda, i)$, i.e. a map $\mathcal{W}^s(f(G/R, \lambda, i)) \to T$.

The identity map on $\mathcal{W}^s(G/R, \lambda, i)$ gives us a ring homomorphism

$$\eta_R : R \to \mathcal{W}^s(G/R, \lambda, i)$$

together with a spin parameter $(t, a)$ on $\eta_R(G/R, \lambda, i)$. The parameter $t$ determines a formal group law over $\mathcal{W}^s(G/R, \lambda, i)$, which is classified by a map

$$\eta_L : L \to \mathcal{W}^s(G/R, \lambda, i) \,, \tag{5}$$

along with an isomorphism $\varphi : \eta_R G \to \eta_L G_u$. The map $\varphi$ descends to an isomorphism on cotangent spaces sending $i(a^2)$ to 1.

A choice of spin parameter on $(G/R, \lambda, i)$ determines an isomorphism

$$W^s \otimes_L R \xrightarrow{\cong} \mathcal{W}^s(G/R, \lambda, i)$$

as the pushout of the map $\eta_R : R \to \mathcal{W}^s(G/R, \lambda, i)$ and the map $W^s \to \mathcal{W}^s(G/R, \lambda, i)$ classifying the isomorphism $(t, a)$ (which is now an isomorphism of spin formal group laws). Under this isomorphism the map $\eta_L$ comes from $\eta_L : L \to W^s$ classifying the target of the universal isomorphism, and the map $\eta_R$ is the evident map to the tensor product.

This construction preserves flatness:

Lemma 3.4. Let $G/R$ be any formal group and $f : R \to T$ a (faithfully) flat ring homomorphism. Then the induced map $f : \mathcal{W}^s(G/R) \to \mathcal{W}^s(fG/T)$ is again (faithfully) flat.

Proof. This can be checked after faithfully flat extension, so we may choose a parameter for $G$ and identify $\mathcal{W}^s(G/R)$ with $W^s \otimes_L R$. The map $\mathcal{W}^s(G/R) \to \mathcal{W}^s(fG/T)$ identifies with $1 \otimes_L f$. The result then follows from the fact that $\eta_R : L \to W^s$ is faithfully flat (in fact free). $\square$

The sheaf condition implies the usual "faithfully flat descent" property:

Lemma 3.5. Let $\eta : R \to S$ be a ring homomorphism, and let $\eta_L, \eta_R : S \rightrightarrows S \otimes_R S$ be the left and right $S$-linear maps. Note that $\eta_L \eta = \eta_R \eta$, and write $\eta^2$ for this common composite. Then for any cobordism sheaf $\mathcal{M}$ we have a map

$$\mathcal{M}(G/R) \to \mathrm{eq}\left(\mathcal{M}(\eta G/S) \rightrightarrows \mathcal{M}(\eta^2 G/S \otimes_R S)\right)$$

that is an isomorphism provided that $\eta$ is faithfully flat.

Proof. Using the sheaf condition, the equalizer diagram may be rewritten

$$\mathcal{M}(G/R) \to \mathcal{M}(G/R) \otimes_R S \rightrightarrows \mathcal{M}(G/R) \otimes_R S \otimes_R S\,. \tag{6}$$

Now tensor this again with $S$ and observe that the resulting diagram is a split equalizer. If $\eta$ is faithfully flat this implies that (6) is an equalizer diagram. $\square$

This lemma admits a converse. The formula (6) uses the values of $\mathcal{M}$ on $\mathcal{F}^s$ to define an extension to all of **SFG**. We leave the details of the following lemma to the reader.

Lemma 3.6. A sheaf on $\mathcal{F}^s$ uniquely determines a sheaf on the whole of **SFG**.

There is an evident symmetric monoidal structure on the category of cobordism sheaves: Given sheaves $\mathcal{M}$ and $\mathcal{N}$, define

$$(\mathcal{M} \otimes \mathcal{N})(G/R) = \mathcal{M}(G/R) \otimes^R \mathcal{N}(G/R),$$

where here we are denoting the tensor product of two right $R$-modules by $\otimes^R$.

Proposition 3.7. There is an equivalence of symmetric monoidal categories between the categories of cobordism sheaves and cobordism comodules.

Proof. The comodule associated to a cobordism sheaf $\mathcal{M}$ has as its underlying $L$-module $M = \mathcal{M}(G_u/L)$, where $G_u$ is the universal formal group law over the Lazard ring $L$ with its trivial spin structure. The coaction comes from the interaction of two morphisms $\mathcal{F}^s$:

$$\psi_L = (\eta_L, \varphi_0, 1) : G_u/L \to G_L/W^s$$

(where $\varphi_0(t) = t$) and

$$\psi_R = (\eta_R, \varphi_u, e) : G_u/L \to G_L/W^s.$$

These induce the diagonal maps in the diagram

$$\begin{tikzcd}
\mathcal{M}(G_u/L) \arrow[r] \arrow[dr, "\psi_{L*}"'] & \mathcal{M}(G_u/L) \otimes_L W^s \arrow[d, "\cong"] & \mathcal{M}(G_u/L) \arrow[l, dashed, "\psi"'] \arrow[dl, "\psi_{R*}"] \\
& \mathcal{M}(G_R/W^s) &
\end{tikzcd} \tag{7}$$

in which the vertical isomorphism is the $W^s$-linear extension of $\psi_{L*}$. The resulting map $\psi$ gives a $W^s$-coaction on $\mathcal{M}(G_u/L)$.

Conversely, given a cobordism comodule $M$ we define a sheaf, which we denote by $\mathcal{M}$, as follows. First define its value on the formal group *law* $G/R$, with its trivial spin structure, by

$$\mathcal{M}(G/R) = M \otimes_L R.$$

Next consider functoriality for maps of spin formal groups laws. Let

$$(f, \varphi, a) : G/R \to H/S$$

be a morphism of spin formal group laws, so $f : R \to S$, $\varphi : H \to fG$, and $a \in S$ is such that $a^2 = \varphi'(0)$. Represent $G/R$ by $g_G : L \to R$ and $H/S$ by $g_H : L \to S$. The morphism is represented by a map $h : W^s \to S$ such that $h\eta_R = fg_G$, $h\eta_L = g_H$, and $he = s$. The first identity gives us a ring map $h \cdot f : W^s \otimes_L R \to S$, and the second tells us that $(h \cdot f)(\eta_L \cdot g_G) = g_H$, where $\eta_L \cdot g_G : L \to W^s \otimes_L R$ is the evident map. Thus we can form the composite

$$(f, \varphi, i)_* : M \otimes_L R \xrightarrow{\psi \otimes 1} M \otimes_L W^s \otimes_L R \xrightarrow{1 \otimes (h \cdot f)} M \otimes_L S\,.$$

This is the map induced by $(f, \varphi, a)$. It is compatible with the ring map $f : R \to S$ since $h \cdot f$ is, and it clearly extends to an isomorphism $M \otimes_L R \otimes_R S \to M \otimes_L S$. We leave to the reader the check of functoriality.

This construction suffices, by virtue of Lemma 3.6. We leave it to the reader to check that the symmetric monoidal structures match up. $\square$

EXAMPLE 3.8. The main topological example is the cobordism sheaf associated to the $MU_*MU$-comodule given by the $MU$-homology of a spectrum $X$. Write $\mathcal{M}_X$ for this; we get a functor from spectra to cobordism sheaves.

EXAMPLE 3.9. The cobordism sheaf $\sigma^n$ evaluates at $(G_u/L, L, 1)$ to $L$, with the $W^s$-comodule structure given by

$$e^n\eta_R : L \to W^s = L \otimes_L W^s \,.$$

Note that in this example, the map $\psi_{L*}$ in (7) is a ring homomorphism, but $\psi_{R*}$ is not (unless $n = 0$).

EXAMPLE 3.10. The sheaf $\mathcal{W}^s$ corresponds to the cobordism comodule $W^s$, and the maps $\eta_R$ and $\eta_L$ are as described in (1,2).

## 4. Height

We have so far considered only isomorphisms of formal groups over a ring $R$. The category of formal groups over $R$ and homomorphisms between them (not just isomorphisms) is a pre-additive category; so in particular any formal group $G$ has an endomorphism $[k]$ for any $k \in \mathbb{Z}$. The homomorphism $[k]$ corresponds to a Hopf algebra map $[k]^* : \mathcal{O}_G \to \mathcal{O}_G$.

DEFINITION 4.1. Let $G$ be a formal group over a ring of characteristic $p$. The *height* $\operatorname{ht} G$ of $G$ is the largest integer (or $\infty$) $n$ such that $[p]^*(I) \subseteq I^{p^n}$.

DEFINITION 4.2. Let $G/R$ be a formal group and $p$ a prime number. Define an increasing sequence of ideals in $R$ as follows. $I_{p,0} = 0$. $I_{p,1} = (p)$. For $n > 1$, let $I_{p,n}$ be the minimal ideal such that over $R/I_{p,n}$ the formal group $G$ has height at least $n$.

The ideals $I_{p,n}$ are in fact subsheaves of the structure sheaf on $\mathbf{FG}$.

LEMMA 4.3. Suppose $R$ is a ring with $pR = 0$ and $G$ a formal group over $R$, and let $f : R \to S$ be a ring homomorphism. Then $\operatorname{ht}(fG) \geq \operatorname{ht} G$, with equality if $f$ is faithfully flat.

PROOF. The inequality follows from $I_{fG}^{p^n} = I_G^{p^n} \otimes_R S$. We get an equality for $f$ faithfully flat because $f$ is then injective. $\square$

Thus we can compute the height in general by forming a faithfully flat ring extension over which $\omega_G$ trivializes. If $G$ admits a parameter, we can express the height in terms of the associated formal group law. The endomorphism $[p]$ is represented by a formal power series

$$[p](t) = \sum_{i \geq 1} a_{i-1} t^i \,, \quad a_0 = p \,.$$

Then

$$\operatorname{ht} G = \max\{n : a_i = 0 \text{ for } i < p^n\}$$

and

$$I_{p,n} = (p, a_1, \ldots, a_{p^n-2}) \,.$$

Definition 4.4. A formal group $G$ over a ring $R$ with $pR = 0$ is of *exact height* $n$ provided that it is of height $n$ and the diagonal in

$$\begin{tikzcd} I \arrow[r, "{[p]^*}"] \arrow[d] \arrow[dr] & I^{p^n} \arrow[d] \\ \omega \arrow[r] & \omega^{p^n} \end{tikzcd}$$

is surjective.

Thus a formal group of height (at least) $n$ is of exact height $n$ if and only if the induced map $\omega \to \omega^{p^n}$ is an isomorphism. It follows that in this case $\omega^{p^n-1}$ is canonically trivialized.

A formal group law of height $n$ is of exact height $n$ precisely when the leading coefficient $a_{p^n}$ in $[p](t)$ is a unit in $R$.

We end this section with two basic facts about the notion of height.

Proposition 4.5. Let $G/R$ be a formal group law, $p$ a prime, and $[p](t) = \sum a_{i-1}t^i\,, a_0 = p$. Then

$$I_{p,n} = I_{p,n-1} + Ra_{p^{n-1}-1}$$

for any $n \geq 1$.

Sketch of proof: It suffices to show this in the universal $p$-typical case, that is, over $BP_* = \mathbb{Z}_{(p)}[v_1, \ldots]$. The generator $v_i$ can be taken to be $a_{p^i-1}$. □

Proposition 4.6 (Dieudonné and Lazard [**7**]). Let $G/R$ and $H/S$ be formal group laws over rings of characteristic $p$, both of exact height $n$. Then there is a ring $T$ and faithfully flat homomorphisms $f : R \to T$ and $g : S \to T$ such that $fG$ and $gH$ are isomorphic formal groups.

Sketch of proof: The extensions $R \to R \otimes S$ and $S \to R \otimes S$ are both faithfully flat, so we may assume that the two formal groups lie over the same ring. Now inductively construct an isomorphism between them. At each stage, one must solve a certain equation. Formally adjoining a root of that equation gives another faithfully flat extension. This is the same as Lazard's proof, which classified formal groups over algebraically closed fields by solving these equations in the field. □

## 5. Landweber exactness

Definition 5.1. A spin formal group $(G/R, \lambda, i)$ is *Landweber exact* if the functor from cobordism sheaves to $R$-modules given by evaluation at $(G/R, \lambda, i)$ is exact.

This definition is important in topology because a Landweber exact spin formal group $(G/R, \lambda, i)$ defines an even periodic ring spectrum $\mathbf{R}$ with $\pi_0(\mathbf{R}) = R$, $G_{\mathbf{R}} = G$, and

$$R_n(X) = \mathcal{M}_X(G/R, \lambda, i)_n\,.$$

This is the content of the Landweber exact functor theorem, combined with Brown representability.

Landweber exactness is invariant under flat base change:

Lemma 5.2. Let $f : R \to S$ be a ring homomorphism. If $(G, \lambda, i)$ is Landweber exact and $f$ is flat then $f(G, \lambda, i)$ is Landweber exact. If $f(G, \lambda, i)$ is Landweber exact and $f$ is faithfully flat then $(G, \lambda, i)$ is Landweber exact.

Proof. By the sheaf condition, $\mathcal{M}(G, \lambda, i) \otimes_R S \xrightarrow{\cong} \mathcal{M}(f(G, \lambda, i))$. □

We will use the following criterion for Landweber exactness, due, as far as I can tell, to Gerd Laures, in terms of the sheaf $\mathcal{W}^s$ and the natural transformation $\eta_L : L \to \mathcal{W}^s(G, \lambda, i)$ of (5).

Lemma 5.3. A spin formal group $(G, \lambda, i)$ is Landweber exact if and only if $\eta_L : L \to \mathcal{W}^s(G, \lambda, i)$ is a flat ring homomorphism.

Proof. Both conditions are invariant under faithfully flat extensions, so we may assume that $\lambda$ is trivial. Pick a spin parameter for $G$, so $\mathcal{W}^s(G, \lambda, i) \cong W^s \otimes_L R$. Suppose that $\eta_L : L \to W^s \otimes_L R$ is flat, and let $M$ be a cobordism comodule. The coaction map $\psi : M \to M \otimes_L W^s$ is an $L$-module splitting of the map $\epsilon \otimes_L 1 : M \otimes_L W^s \to M \otimes_L L = M$, natural in $M$. Thus $\mathcal{M}(G, \lambda, i) = M \otimes_L R$ is naturally a retract of $(M \otimes_L W^s) \otimes_L R = M \otimes_L (W^s \otimes_L R)$. By hypothesis $W^s \otimes_L R$ is flat as an $L$-module, so this and hence its retract $\mathcal{M}(G, \lambda, i)$ is exact as a functor of $M$.

Conversely, suppose that $\mathcal{M} \mapsto \mathcal{M}(G, \lambda, i)$ is exact in the sheaf $\mathcal{M}$. Let $N$ be an $L$-module. Then

$$N \otimes_L (W^s \otimes_L R) = (N \otimes_L W^s) \otimes_L R = (N \otimes_L W^s)(G/R).$$

Since $W^s$ is flat (free, in fact) over $L$ the functor $N \mapsto N \otimes_L \mathcal{W}^s$ from $L$-modules to sheaves is exact; and the functor evaluating on $(G, \lambda, i)$ is exact by hypothesis; so the composite is exact as required. □

By Lemma 5.2, we may restrict our attention to formal group laws with trivial spin structure. The proof of the exact functor theorem uses the two properties stated in (4.5) and (4.6), so we begin by establishing some general ideas about sequences of ideals.

Definition 5.4. An increasing sequence $0 = I_0 \subseteq I_1 \subseteq \cdots$ of ideals in a ring $R$ is a *scale* if there exist elements $v_0, v_1, \ldots$ in $R$ such that for each $n \geq 0$,

$$I_{n+1} = I_n + Rv_n\,.$$

Such a sequence of elements is a *defining sequence* for the scale.

For example, the sequence $I_{p,0} \subseteq I_{p,1} \subseteq \cdots$, is a "sheaf of scales," called the *height scale* at $p$.

Here are two easy lemmas. For the second, recall that a sequence $v_0, v_1, \ldots$ in a ring $R$ is said to act regularly on the $R$-module $M$ provided $v_n$ acts monomorphically on $M/(v_0, \ldots, v_{n-1})$ for all $n \geq 0$.

Lemma 5.5. Let $v_0, v_1, \ldots$ and $w_0, w_1, \ldots$ define the same scale $0 = I_0 \subseteq I_1 \subseteq \cdots$ in $R$. Then for each $n$ the localizations $v_n^{-1}R/I_n$ and $w_n^{-1}R/I_n$ are isomorphic $R/I_n$-algebras.

Proof. If $u, v \in S$ generate the same ideal then $u^{-1}S = v^{-1}S$ as $S$-algebras. Apply this with $S = R/I_n$. $\square$

Lemma 5.6. Suppose two sequences, $v_0, v_1, \ldots$ and $w_0, w_1, \ldots$, define the same scale in $R$, and let $M$ be an $R$-module. Then $v_0, v_1, \ldots$ acts regularly on $M$ if and only if $w_0, w_1, \ldots$ does.

Proof. An element $s \in S$ acts injectively on an $S$-module $N$ if and only if $N \to s^{-1}N = s^{-1}S \otimes_S N$ is injective. Apply this with $S = R/I_n$. $\square$

Definition 5.7. A scale acts *regularly* on $M$ provided some (hence any) defining sequence acts regularly on $M$. If $M = R$ we will call the scale regular. The scale is *finite* on $M$ if $I_n M = M$ for large $n$, and finite if it is finite on $R$.

Here is the main theorem, first proven by a different method in [**6**].

Theorem 5.8 (P. S. Landweber). Let $G/R$ be a spin formal group such that for every prime $p$ the scale $I_{p,0}(G/R) \subseteq I_{p,1}(G/R) \subseteq \cdots$ is finite and regular. Then $G/R$ is Landweber exact.

Example 5.9. Any spin formal group over $\mathbb{Q}$ is Landweber exact.

Example 5.10. The multiplicative formal group law $G_m(x, y) = x + y - xy$ over $\mathbb{Z}$ is Landweber exact. We compute $[p](t) = 1 - (1 - t)^p$. $p$ is a non-zero-divisor in $\mathbb{Z}$; and the coefficient of $t^p$ in $[p](t)$ is 1, so $I_{p,2}(G_m/\mathbb{Z}) = \mathbb{Z}$.

Remark 5.11. In fact Landweber proves something slightly different. Recall that an $L$-module $M$ is *coherent* if it is finitely generated and every finitely generated submodule is finitely presented. Landweber restricts himself to the category of comodules which are coherent as $L$-modules, and shows that the functor $M \mapsto M \otimes_L R$ is exact on this category if and only if the height scale in $R$ is regular for every prime. A result from [**9**] shows that every comodule is a filtering direct limit of finitely presented comodules, and it follows that one may dispense with the coherence condition here. The theorem we prove is thus less general; for us, the height scales have to be finite at every prime. It is possible to modify the proof to give the general result ([**8**], Lecture 16).

The proof of Theorem 5.8 will use the criterion of Lemma 5.3: we must show that

$$\eta_L : L \to W^s \otimes_L R$$

is flat. We will describe an appropriate flatness condition.

So suppose $R$ is any ring containing a scale $0 = I_0 \subseteq I_1 \subseteq \cdots$ and let $M$ be an $R$-module. Pick a defining sequence $v_0, v_1, \ldots$ for the scale. We have a diagram of $R$-modules

$$\begin{array}{ccccc} M & \xrightarrow{v_0} & M & \longrightarrow & v_0^{-1}M \\ & & \downarrow & & \\ M/I_1 & \xrightarrow{v_1} & M/I_1 & \longrightarrow & v_1^{-1}M/I_1 \\ & & \downarrow & & \\ M/I_2 & \xrightarrow{v_2} & M/I_2 & \longrightarrow & v_2^{-1}M/I_2 \\ & & \downarrow & & \\ \cdots & & \vdots & & \cdots \end{array}$$

in which the vertical maps are the cokernels of the left horizontals, and the right horizontal maps are the localization homomorphisms. Assume the scale acts regularly on $M$. The left horizontal maps are then monic, so we have short exact sequences

$$0 \longrightarrow M/I_{n-1} \xrightarrow{v_{n-1}} M/I_{n-1} \longrightarrow M/I_n \longrightarrow 0. \tag{8}$$

If the scale is finite on $M$, the modules in the diagram are eventually zero.

LEMMA 5.12. Assume that the scale $I_0 \subseteq I_1 \subseteq \cdots$ acts regularly and finitely on $M$. If in addition

$$\mathrm{Tor}^R_{n+1}(v_n^{-1}M/I_n, -) = 0$$

for all $n \geq 0$, then $M$ is flat over $R$.

PROOF. We prove that $\mathrm{Tor}^R_{n+1}(M/I_n, -) = 0$ by downward induction on $n \geq 0$. The induction ends with $\mathrm{Tor}^R_1(M, -) = 0$, which is equivalent to the flatness of $M$. The assumption that $M/I_n = 0$ for large $n$ grounds the induction, so suppose $\mathrm{Tor}^R_{n+1}(M/I_n, -) = 0$. The short exact sequence (8) leads to a long exact sequence in $\mathrm{Tor}^R$, which reads in part

$$\mathrm{Tor}^R_{n+1}(M/I_n, -) \longrightarrow \mathrm{Tor}^R_n(M/I_{n-1}, -) \xrightarrow{v_{n-1}} \mathrm{Tor}^R_n(M/I_{n-1}, -).$$

By assumption the first term is zero, so $v_{n-1}$ acts injectively on $\mathrm{Tor}^R_n(M/I_{n-1}, -)$. It follows that $\mathrm{Tor}^R_n(M/I_{n-1}, -)$ embeds in

$$v_{n-1}^{-1}\mathrm{Tor}^R_n(M/I_{n-1}, -) = \mathrm{Tor}^R_n(v_{n-1}^{-1}M/I_{n-1}, -)$$

which is zero by assumption. This gives us the next step in the induction: $\mathrm{Tor}^R_n(M/I_{n-1}, -) = 0$. $\square$

Write $R_n$ for the $R$-algebra $v_n^{-1}R/I_n$. In particular, the ring $L_n$ supports the universal formal group law over $\mathbb{Z}_{(p)}$-algebras of exact height $n$.

LEMMA 5.13. If the scale $I_0 \subseteq I_1 \subseteq \cdots$ is regular, then $R_n$ has flat dimension at most $n$ over $R$; that is, $\mathrm{Tor}^R_k(R_n, -) = 0$ for $k > n$.

PROOF. We use upward induction on $n$ to show that $R/I_n$ has flat dimension at most $n$ over $R$. The long exact sequence associated to

$$0 \longrightarrow R/I_n \xrightarrow{v_n} R/I_n \longrightarrow R/I_{n+1} \longrightarrow 0$$

reads in part

$$\mathrm{Tor}_k^R(R/I_n, -) \longrightarrow \mathrm{Tor}_k^R(R/I_{n+1}, -) \longrightarrow \mathrm{Tor}_{k-1}^R(R/I_n, -).$$

The inductive assumption is that the left term is zero for $k > n$ and the right term is zero for $k-1 > n$, so the middle term is zero for $k > n+1$, establishing the induction.

Now since localization is exact $\mathrm{Tor}_k^R(R_n, -) = v_n^{-1}\mathrm{Tor}_k^R(R/I_n, -)$, so the flat dimension of $R_n$ is at most $n$. □

PROPOSITION 5.14. If a regular scale $I_0 \subseteq I_1 \subseteq \cdots \subseteq R$ acts regularly and finitely on the $R$-module $M$, and $M_n = v_n^{-1}M/I_n$ is flat as an $R_n$-module for each $n \geq 0$, then $M$ is flat as an $R$-module.

PROOF. (Simplified proof, following a suggestion of the referee) Let $R \to S$ be a ring homomorphism and let $N$ be a flat $S$-module. Then there is a natural isomorphism

$$N \otimes_S \mathrm{Tor}_*^R(S, -) \cong \mathrm{Tor}_*^R(N, -)\,.$$

Apply this with $S = R_n$ and $N = M_n$. □

To begin the proof of the exact functor theorem (5.8) note that we can proceed one prime at a time, by virtue of the following standard lemma, in which $A_{(p)}$ denotes the localization of an abelian group $A$ at the prime number $p$: $A_{(p)} = A \otimes \mathbb{Z}_{(p)}$.

LEMMA 5.15. Let $R$ be a ring. An $R$ module $M$ is flat over $R$ if and only if $M_{(p)}$ is flat over $R_{(p)}$ for all prime numbers $p$. □

PROOF. See [**1**] II Exercises §3.8(b), p. 138. □

To show that $\eta_L : L \to W^s \otimes_L R$ is flat it will thus suffice to show that $\eta_{L(p)}$ is flat for each prime number $p$. This map is the factorization through $L_{(p)}$ of the map $\eta_L$ associated to the formal group $G$ base-changed to $R_{(p)}$. The following proposition gives the input needed to apply Proposition 5.14, using the height scale at $p$ and the fact that $L_n$ carries the universal spin formal group of exact height $n$.

PROPOSITION 5.16. For any spin formal group law $G$ of exact height $n$ over a $\mathbb{Z}_{(p)}$-algebra $R$, the map $\eta_L : L_n \to W^s \otimes_L R$ is flat.

PROOF. To begin with, notice that we at least have one example of a formal group law for which this claim is true: the universal formal group law of exact height $n$ itself. So we are claiming that $\eta_L : L_n \to W_n^s$ is flat. Now $\eta_R$ embeds $L$ into $W^s$ as the constants in $W^s = L[e^{\pm 1}, b_1, \ldots]$, so $(\eta_R)_n : L_n \to W_n^s$ is certainly flat. But the anti-automorphism $c : W^s \to W^s$ swaps $\eta_R$ and $\eta_L$, so $(\eta_L)_n$ is flat as well.

This single example in fact suffices. Let $G$ be any formal group law of exact height $n$ over a $\mathbb{Z}_{(p)}$-algebra $R$. Let $j : R \to T$ and $k : L_n \to T$ be faithfully flat maps as in Proposition 4.6, and let

$$G/R \xrightarrow{j} jG/T \xleftarrow{\tilde{k}} G_u/L_n$$

be the morphisms as guaranteed by that proposition. Using naturality of the map $\eta_L$, together with the fact that $L \to L_n$ is an epimorphism in the category of rings, we obtain a commutative diagram

$$\begin{tikzcd}
 & L_n \arrow[dl, "(\eta_L)_{G/R}"'] \arrow[d, "(\eta_L)_{jG/T}"] \arrow[dr, "(\eta_L)_{G_u/L_n}"] & \\
\mathcal{W}^s(G/R) \arrow[r, "j_*"] & \mathcal{W}^s(jG/T) & \mathcal{W}^s(G_u/L_n)\,. \arrow[l, "\tilde{k}_*"']
\end{tikzcd}$$

The map $(\eta_L)_{G_u/L_n}$ is flat, as we have seen, and $\tilde{k}$ is flat by Lemma 3.4, so the composite $(\eta_L)_{jG/T}$ is flat. The map $j$ is faithfully flat by Lemma 3.4, and it follows that $(\eta_L)_{G/R}$ is flat. $\square$

This completes the proof of Theorem 5.8.

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