A LABORATORY COURSE IN MATHEMATICS

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Abstract. For the past ten years, the Mathematics Department at the Massachusetts Institute of Technology has offered an undergraduate course known as 18.821 Project Laboratory in Mathematics. The course is designed to give students a sense of what it is like to do mathematical research. It exhibits characteristics of undergraduate research programs and Inquiry Based Learning courses, but it does not fit comfortably in either camp. This paper describes the course and its reception by faculty and students, and explores its relationship to contemporary trends in learning theory.

1. Introduction

One of the degree requirements for undergraduates at MIT is completion of one of the twenty-four courses satisfying the Laboratory Requirement, described as follows:

> The Laboratory Requirement is not intended primarily to teach specific techniques for later experimental work, provide broad coverage of a particular field, or complement a specific subject. The laboratory subjects are planned to give each student . . . an opportunity to work on one or more experimental problems, exercising the same type of initiative and resourcefulness as a professional would in similar circumstances.

Up until 2004, MIT Mathematics majors had to satisfy this requirement outside the department. The most common choice was 15.310 Managerial Psychology Laboratory, offered by the MIT Sloan School of Management. For some of our majors, this was an appropriate choice; for most, it was not. In the spring term of 2004, the Mathematics Department launched its first—and thus far only—approved lab subject, 18.821 Project Laboratory in Mathematics. Our idea was to base the course on the scientific method as it is practiced in mathematics. For this, we had to devise a way to give students something of the experience of doing mathematical research, but in a sustainable manner, repeatable term after term with a wide variety of students. See [5] for an account of the creation of this course.

The course strategy was devised by Mike Artin and the second author, following ideas of Michael Brenner (now Professor of Applied Mathematics and Applied Physics at Harvard University). It has not changed in any essential way since its inception. Students work in teams of three on three open-ended projects, under the mentorship of experienced researchers. They explore puzzling and complex mathematical situations, generate and examine data, define research directions, search for regularities, and attempt to explain them mathematically. Students share their results through professional-style papers and presentations.

The expectations of the course are quite different from those of most others that students take. There is no set content for the students to master. Not only are there no right answers; there aren't even any right questions! The structure of the Project Lab is distinct from that of a true Research Experience for Undergraduates, and the goals are different as well. We might summarize the course goals as follows. As the course progresses, each student will

- develop research skills, including exploring examples, choosing and narrowing goals, and communicating findings;
- experience mathematics from the bottom up, starting with observing patterns and seeking to discover mathematical explanations for them;
- realize that mathematics is a living, evolving field to which the student can contribute;
- enhance mathematical understanding by putting earlier learning to use in new settings;
- improve written and oral communication skills, including use of language and argument in formal and informal settings;
- consolidate teamwork skills such as dividing tasks, being responsive, and taking responsibility for the work of the team.

While the Project Lab has obvious connections with Inquiry-Based Learning, it is separated from that movement by the absence of target disciplinary skills or understandings; the central course objectives are purely experiential. It is said that in the "Moore method" nothing is given—the student has to discover everything—but in a sense everything is given; the student is supposed to come to a predetermined conclusion. The Project Lab has an opposite philosophy.

Beyond the MIT context, the Project Lab adheres to a key recommendation of the forthcoming 2015 Curriculum Guide to Majors in the Mathematical Sciences. In this guide, the MAA Committee on the Undergraduate Program recommends that:

> Every major student should work, independently or in a small group, on a substantial mathematical project that involves techniques and concepts beyond the typical content of a single course. Students should present their results in written and oral forms. Institutions can provide this opportunity in various ways, or a combination thereof: undergraduate research experiences, courses driven by inquiry or open-ended problem-solving, capstone courses, internships or jobs with a substantial mathematical component, etc.

In this paper we describe the structure of the MIT Project Laboratory in Mathematics (Section 2) and instructor roles (Section 3). We then report on the demography of students taking the course (Section 4), the projects, and how the projects have been tailored to the course and student population (Section 5). In Section 6, we review student reception of the Project Lab, and we end in Section 7 with some comments about transferability of this type of course to other educational environments.

This paper expands on a talk by the second author at the conference "New Directions for Mathematics REUs" at Mt. Holyoke College on June 21, 2013. Haynes would like to thank the organizers of this conference for providing such a stimulating venue for disseminating new ideas about and roles for undergraduate research. The first author's involvement in this report arose from her work as the lead architect of a new category of publications by MIT OpenCourseWare (OCW) called OCW Educator. This collection of webpages seeks to portray not so much *what* we teach at MIT (the traditional OCW focus) as *how* we teach; the target audience is educators. One of the first MIT subjects to receive in-depth exposure on OCW Educator was the Project Lab [7]. Kathy is particularly well prepared to report on this subject, having taken it as an undergraduate. She would like to thank her OCW co-worker Kimberly Li, who worked with her on publishing Project Lab on OCW. We both thank Susan Ruff of the MIT Writing Across the Curriculum Office. Susan has been a constant with this course over the years and has been instrumental in its development, especially the communication component and the workshops. We also thank Michel Goemans for providing us with much of the enrollment data we report on below, and Jeff Lagarias for providing information about his variation on this course at the University of Michigan.

2. Structure of the Project Laboratory

Unlike most research experiences for undergraduates, which typically take place outside of class and often in the summer, the Project Lab is a semester-long course that takes place during the school year. This enables students to experience mathematical research while satisfying a degree requirement, and without the commitment of a summer.

Each semester begins with an introductory class, a teamwork workshop, a presentation workshop, and a writing workshop.

Over the course of the 14-week semester, students work in teams of three on three open-ended research projects. The project cycles are deliberately overlapped to allow time for feedback and revision. Each project cycle lasts five to six weeks and includes the following:

- *Topic selection*: Each team selects a project topic from a list of over forty options. In most semesters each project topic can be chosen by at most one team.
- *Mathematical work*: Students work together in their teams to explore their project topics. Students may work out examples, run computer simulations, read relevant literature, refine or redefine the focus of the project, make conjectures, and attempt to construct proofs.
- *Mentor meetings*: For every project, each student team is assigned to a mathematics instructor or "mentor." Each team meets with its mentor once a week for the duration of the project. The communication instructor for the course works with all of the teams as needed.
- *First draft*: Each team submits a first draft of the paper ten days before the final draft due date. The paper is then read and commented upon by the team's mentor and sometimes also the communication instructor, while the students begin work on their next project.
- *Debriefing meetings*: The team meets with a group including the lead instructor, the team's mentor, the team's mentor for the next project, and often the communication instructor. The team gives a brief presentation of their research, and the whole group discusses both the mathematics and the writing.
- *Final draft*: Students finish a project cycle by revising their paper following instructor feedback and submitting a final draft.

Requiring three projects in a single semester is a deliberate choice. This quick turn-around is consistent with the rhythm of students' experiences in their other courses, which often involve weekly homework assignments and monthly exams. Having three iterations also gives students fresh learning opportunities: to choose more appropriate projects, conduct better research, and write better papers.

Each team delivers a presentation to the class on one of their projects during the semester. The formal presentation is always preceded by a practice presentation, which is attended by course instructors and includes extensive feedback. The team then refines its presentation before presenting to the entire class.

Communication is an essential component of the course experience. Project Lab satisfies MIT's Communication Intensive in the Major (CI-M) requirement, which mandates that all undergraduates take at least two courses designated as CI-M. CI-M courses are described as follows:

> CI-M subjects (Communication Intensive in the Major) teach the specific forms of communication common to the field's professional and academic culture. Students may write in teams, prepare and present oral and visual research reports for different audiences, learn audience analysis and peer review, or go through the experience of proposing, writing, and extensively revising a professional journal article.

The focus on communication provides a wider variety of assessed skills than is typical in a mathematics course. It is widely recognized that writing improves understanding. Moreover, it can be argued that the communication skills learned in this and other CI-M courses are among the skills we offer training in that are most easily transferred to the typical career paths of our students.

3. Staffing the Project Lab

The Project Lab is a labor-intensive operation. We limit enrollment to 27 students; once we tried it with 36 and the burden on the lead instructor was too heavy. The course is typically staffed with a lead instructor (a faculty member), two co-instructors (typically graduate students or post-doctoral researchers), and a communication instructor. Collectively, the team spends about 450 man-hours over the semester on the course. This serious commitment of resources allows us to provide students with personal mentorship, with encouragement and support as they confront the uncertainties of research, and with feedback and guidance on their presentations and papers. This student-to-instructor ratio is roughly in line with that of many of MIT's other upper-level mathematics courses.

For each project, each course instructor mentors three of the nine student groups. The mentor typically meets with each group once a week. These meetings are similar to meetings that a faculty member might have with a graduate student. Often, mentors do not know any more about the project than the students do, and they think through the issues together. Instructors try to be honest and sincere, and not feign ignorance. It is important that mentors try to help the students while at the same time giving them enough leeway to find their own direction, since the whole point of the course is for students to experience research themselves. If an instructor dominates the conversation or sets the research direction, the instructor can rob the students of the research experience. The meetings with mentors and the debriefings are designed to provide timely, formative feedback that encourages learning and spurs continued progress.

At the same time, most students are truly novice researchers. Sometimes they need help understanding the research process, identifying ways to tackle a problem, and feeling comfortable defining a research direction. Instructors help students cope with the challenge and uncertainty of confronting a question that does not have a known answer. Sometimes a week goes by with no visible progress, and instructors might offer a gentle push in some direction. Instructors try to be sensitive to students' worries, encourage the students, and help them overcome obstacles.

After a team submits a first draft of a paper, the mentor and sometimes the faculty lead instructor and the communication instructor read and mark up the draft. Then a "debriefing" meeting is held. This meeting is attended by the students on the team, their mentor, the lead instructor, and sometimes the communication instructor and the mentor who will work with that team on the next project. At this meeting, the students give an informal description of their research, and then everyone discusses both the mathematics and the writing. This meeting serves several purposes. It provides students with some practice at presenting mathematical ideas. It serves as an opportunity for the instructors and students to discuss the students' research and explore possible ways to push the students' work further or in new directions. And it gives the instructors a chance to talk through improvements for the paper rather than depending entirely upon written commentary.

The students as well as the instructors can appreciate this close working relationship. Many students have never worked with a faculty member, postdoc, or graduate student this closely before, and the course can give them a personal connection to all three. For the instructors, one of the most enjoyable aspects of the course can be getting to know a particular group of students very well.

The course lead is always a faculty member, who, along with our experienced writing instructor, offers guidance and feedback to the other two instructors. The course thus provides an excellent and exciting training ground for the mentors who are less experienced postdocs or graduate students. In fact we feel that the qualities of good mentorship found, for example, in [6], provide good orientation for the course instructors. Their INSPIRE model suggests that the mentor should be: Intelligent, Nurturant, Socratic (not diactic), Progressive, Indirect, Reflective, and Encouraging.

Educational initiatives stand or fall on how well they are adopted by individuals other than the initiators. On these grounds the Project Lab has been a signal success. It has never been difficult to staff. The following twelve faculty members have led the course, aided by quite a large number of postdocs and graduate students.

Mike Artin, Spring 2004, Spring 2006, Fall 2006
Haynes Miller, Spring 2005, Spring 2013
Bjorn Poonen, Fall 2007
Paul Seidel, Spring 2008, Spring 2009, Fall 2009
David Vogan and James McKernan, Fall 2008
David Jerison, Spring 2010, Spring 2014
JuLee Kim, Fall 2010, Fall 2011
Scott Sheffield, Spring 2011
Tomasz Mrowka, Spring 2012
Richard Stanley, Fall 2012
Larry Guth, Fall 2013

In each case, the lead faculty made additions or improvements to the project list and brought new ideas about workshops or other elements of the course process and grading method.

A certain level of constancy of staffing is also beneficial, and in the case of the Project Lab this has been provided by Susan Ruff of the MIT Writing Across the Curriculum office. Susan has worked with the course almost since its beginning. She knows the history of the course and what has or has not worked, and she helps faculty who are new to the course understand and structure the course. We have also been helped by the MIT Mathematics CI Space, designed by Sami Assaf, Mia Minnes, and Susan Ruff, and maintained by Susan. This is an instance of the Educational Collaboration Space, a publicly available WordPress package found at ecs.mit.edu. We have used it since 2010 to harvest and disseminate ideas and course material pertaining to the communication-intensive courses in the Mathematics Department.

This resource, and most of the MIT material, is now publicly available through the MAA-supported website mathcomm.org.

4. Student demography

The Project Lab is offered every semester, and it is open to all MIT students. Because the course is the only mathematics course that fulfills the MIT Undergraduate Laboratory Requirement, and it also fulfills one of MIT's communication requirements, the demand for the course is generally high; usually 40 to 50 students register to take the course each semester, well over the cap of 27 students.

Regrettably, the enrollment cap effectively limits the course to mathematics majors and to upperclassmen. Only 12 of the 489 students who completed the course by spring 2014 were not mathematics majors. On average, 72% of 18.821 students are seniors when they take the course, and 26% are juniors. While it has not been possible to accommodate many students from outside the Mathematics Department, it has been possible to guarantee that all mathematics majors can take the course if they choose to use it to fulfill their Laboratory Requirement. The drop rate has been extraordinarily low: about 1.3%.

The only enrollment requirement for students is that they first complete at least two mathematics courses beyond the basic courses in calculus, differential equations, and linear algebra. Many students enter the course with no substantial research experience, while others enter the course having already published papers in professional journals. This course is intended to be broadly accessible, within the MIT context.

Mathematics majors at MIT go on to a huge diversity of careers. About 40% continue after graduation in some academic pursuit, and 50% take jobs of various types. From the first group, about a third enter an engineering masters degree program, a third go into a Mathematics PhD program, and the rest go into PhD programs in some other discipline. Among the non-academic career group, about half go into investment banking and financial services; consulting and software development attract about 15% each; and the rest move on into a wide range of other activities.

A cross-section of mathematics majors do take the course, as indicated by the following statistics (from the past fourteen semesters). Many MIT undergraduates are "double majors," and the Mathematics department hosts more second majors than any other department: over the past seven years about 11% of our majors have listed Mathematics as their second major. Many other majors at MIT require a laboratory subject in the department as a graduation requirement, but nonetheless fully 19% of Project Lab students list Mathematics as their second major. Among them, more than a third are Electrical Engineering and Computer Science majors, more than a quarter are Physics majors, and the rest are scattered over ten other majors, tracking closely with overall trends.

Institute grade point average and rank within the Mathematics major for the Project Lab students closely matches the departmental averages. The mean GPA of majors graduating over the past decade is 4.53/5.00; among the mathematics majors who took the Project Lab it is 4.49/5.00. An arcane departmental ranking system uses grades and a measure of the difficulty of the class to assign a score to each senior, ranging, in the class of 2014, from a top score of 82 down to the low of −8, with a mean of 21.0 and standard deviation of 19.7. The mean score of Project Lab students was 23.6, one seventh of a standard deviation above the mean.

Mathematics majors at MIT can choose from a wide variety of mathematics courses; they can elect to specialize their program as they like. In order to understand whether the the Project Lab attracts students from one or another corner of the major, we determined, for each undergraduate subject (except ones carrying Communication Intensive credit, which are subject to other pressures) the number of students graduating with a mathematics degree by May, 2014, who passed that subject, and the number passing that subject who also took the Project Lab. We then aggregated these data by summing over courses in similar areas, and took the ratio of the resulting sums. This gives a crude indication of whether the course is more popular among some groups than others. The table below presents this summary. So for example 56% of the seats occupied by undergraduate mathematics majors in analysis subjects were occupied by students who by the time they graduated had also taken the Project Lab.

Analysis	56%
Topology, geometry	53%
Algebra	50%
Combinatorics, physical appl. math.	47%
Computer science, probability	42%

Students taking classes in the last category availed themselves of the Project Lab about 75% as often as those taking classes in the first category. This somewhat lower participation of students interested in combinatorics and computer science is at least partly accounted for by the large course 18.310 Principles of Discrete Applied Mathematics, which also carries communication-intensive credit, and the availability of Lab courses in computer science.

Grading a class like this presents challenges to those used to basing a grade on numerical exam and homework scores. Some thoughts about this can be found at the OCW Educator website [7]. For now we just report on the observed grade distribution. In the semesters since fall, 2006, the grades have been distributed as follows (percentage):

A+	A	A-	B+	B	B-	C+	C	C-	D
5	31	20	14	18	7	2	2	1	1

This distribution is somewhat higher than the overall departmental grade distribution, but we feel that generally students devote themselves to this course and these grades have seemed fair to us.

5. About the projects

The Project Lab is taken by 54 students per year, with students' backgrounds and interests covering a broad span. In many research programs for undergraduates, the end-goal is novel, publishable results. This is not the case with the Project Lab. Instead, we aim for a "research-like" experience. There is no expectation that the teams discover previously unknown mathematics, though sometimes they do; the key is that they discover mathematics previously unknown *to them*, through a research-like process. Thus the Project Lab is a mathematics course, not what mathematics educators would typically call an "undergraduate research experience." This goal makes the experience accessible to students who do not yet have enough mathematical background to work at the forefront of mathematical research. It also enables the course to be repeatable, since projects can be reused from year to year.

The project descriptions have been crafted with this perspective in mind. Students begin each project by choosing from a list of over forty projects that has been developed by the mathematics faculty over the past decade. Each project description presents an open-ended mathematical situation, suggests some relevant questions, and allows students to define and pursue a range of research directions. The success of the course depends upon a list of good project topics.

While we prefer not to publicize the project descriptions, a couple of examples can be found at the OpenCourseWare website [7], and the second author invites the reader to contact him for more information. The following are characteristics that we have found to be important features of a good project topic:

- *Open-ended, with a variety of possible research directions.* In mathematical research, we rarely aim at one specific target. Instead, we look at a general area and try to find the parts of it that are both interesting and accessible. We want students to experience the challenge of defining and pursuing their own specific research directions.
- *Accessible to students across a spectrum of backgrounds, interests, and research abilities.* The projects span pure mathematics, applied mathematics, and combinatorics. It is important that the projects appeal to students with different levels of preparation or areas of expertise. Many projects work as well with less well-prepared groups as they do with very well-prepared groups. It has been quite remarkable to see what different teams have done with the same projects and to appreciate the diversity of results that students have produced. This brings home the message that mathematics has a broad entry path, which people of many different backgrounds can access. We feel that this lesson is an important component of the course. Successful papers and presentations require a substantial investment of time and energy, for everyone. This course, with its very heterogeneous enrollment, harmonizes with the sense that too much is made of intrinsic genius and not enough of hard work and motivation [3].
- *Manageable in a one-month period.* We want the students to have a taste of what it is like to do mathematical research, but we only have about one month for each project, and students often enter the course with little to no research experience. In contrast, professional mathematicians often spend months or years on a project and submit a paper only after substantial progress has been made. Our research topics are open-ended but focused enough that we believe most teams can make reasonable progress within the short timeframe.
- *Difficult to resolve via external resources.* We have found that absorbing existing research can be distracting or overly time-consuming for students, and we recommend that they focus on exploring the mathematics themselves instead of diving into a full-blown literature search. Most of our topics are not associated with extensive literature and are not easily searchable on the web. When there is a giveaway term, we often rephrase and disguise the problem in a way that makes it much harder to find. It is fine if students find existing research on a topic, but

> they are then expected to build upon what they find or pursue a different direction.

The fact that student teams choose their own projects gives them a sense of ownership over their work. Moreover, the projects are conceived to allow a variety of interpretations, and the students take them in their own remarkably diverse directions. This sense of ownership is critical in motivating students to push on through the research, and especially through the writing. One of the big challenges in teaching technical writing is finding subjects that the learner is committed to; if the subject is the student's own original idea and work, the commitment is much easier.

Every student is expected to make progress that is commensurate with his or her background and experience. Some students barely make any progress on their first project but eventually become more confident and capable at approaching research problems and at describing their findings. Because the course is so open-ended, students can find a project that is accessible to them and define a sub-question or attempt an approach within their grasp. Our most accomplished students can learn to be more patient and improve their teamwork skills. They can gain experience tackling new research projects, and sometimes they find brilliant solutions. Many projects lead right to the frontier of current research, and students gain an understanding of the effect of deeper mathematics on these fairly naïve questions. These are all important experiences for budding mathematicians.

Education theory offers an interesting perspective on students' experience in the Project Lab. The course provides a context in which the student can engage in a form of "deliberate practice," described by Ericsson et al. [4] in these terms: The subjects experience

> "...motivation to attend to the task and exert effort to improve their performance. In addition, the design of the task should take into account the preexisting knowledge of the learners so that the task can be correctly understood after a brief period of instruction. The subjects should receive informative feedback and knowledge of results of their performance. The subjects should repeatedly perform the same or similar tasks."

The learning process in the Project Lab conforms well to the constructivist paradigm (springing from the work of Piaget), as described for example by Micheline Chi ([2], pp 85–86):

> "Constructive activities ... allow the learners to infer new ideas, new insights, new conclusions, from making

> deductions and inductions, from reasoning analogically through comparisons, from integrating new knowledge with old knowledge, or linking information from disparate sources. In short, these various creating processes of comparing, connecting, inducing, analogizing, generalizing, etc., allow the learners not only to infer new knowledge but also to repair and improve their existing knowledge. How would these creating processes enhance learning? Inferring new relations, new conclusions, and new insights obviously makes one's knowledge more rich, and repairing one's knowledge also makes it more coherent, more accurate, and better-structured, and so forth. These changes can deepen one's understanding of new materials and have been show to improve learning."

Chi refines the constructivist hypothesis in an interesting way, which seems relevant to the Project Lab. She distinguishes between the terms "active," "constructive," and "interactive." Active processes are likely to engage students more than purely passive ones, but may still bypass the brain cells. Constructive activities require more engagement, as illustrated in the quote above. But activities which respond to and stimulate a response from the learner's environment form a separate category. Chi describes several varieties of interactive process, including joint dialogues with peers.

> "When a learner interacts with a peer, such interactions can sometimes characterize a pattern of *joint dialogues*, which occur when both peers make substantive contributions to the topic or concept under discussion, such as by building on each other's contributions, defending or arguing a position, challenging and criticizing each other on the same concept or point, asking and answering each other's questions. . . . Thus *joint dialogues* refer to a pattern of interactions in which both partners make substantive contributions to the topic or concept under discussion . . . ."

The power of this form of knowledge-building is part of what makes the teamwork aspect of the Project Lab so important.

6. Student reception

We have tracked student attitudes quite closely. The following data are assembled from responses to the spring 2013 departmental opinion survey of our graduating seniors. One question we ask is: "What

Mathematics course did you find most beneficial?" Given the non-syllabus nature of the Project Lab, it is perhaps surprising that Project Lab ranks third in the rate of mention (mentions per student taking the class), with six of the 62 respondents mentioning it. The top two are Algebraic Combinatorics and Theory of Computation. So the Project Lab ranks above [Modern] Algebra I, Real Analysis, and Probability and Random Variables (which come next in this ranking). It was also very interesting to discover that five of these six students were not bound for an academic mathematics career; in fact only two of them were going on in academics at all.

Here are some student comments mentioning the Project Lab as the mathematics course they benefited from the most, followed by the student's post-graduation plan.

- "It gave me a lot of practice in working on a challenging problem and writing and presenting about it. It definitely got me thinking about how to communicate about my work, much more so than the seminars I took." (Statistics PhD)
- "The most beneficial class that I have taken is [the Project Lab]. Although my teammates and I struggled a lot when working on the papers, we learned how to approach an open-ended problem and discovered new problems." (Financial sector)
- "This was the first time I had to give a formal presentation and write a formal math paper. These skills are invaluable and I have used them countless times since taking that class." (Software)
- "Working in a group, working on writing and papers, more faculty interaction than other classes. It was good!" (Consulting)

We also ask students about their experiences with our "Communication Intensive-Major" offerings. In response to "What is the best thing about the Mathematics CI-M program?", the Project Lab was mentioned almost twice as often as any other subject. Here are some comments in reaction to this question, followed by the student's post-graduation plan.

- "The chance to present mathematics and receive feedback on presentations." (Mathematics PhD)
- "I thought the Project Lab was fun, the problems were just the right level, so they were challenging but not impossible." (Mathematics PhD)
- "The Project Lab was a great course, as it involved both original work and communicating about it." (Non-mathematics PhD)

- "I liked working on Math Projects in the Project Lab. It was very interesting and the scope of the projects was about right." (Software)
- "Giving presentations - I learned a lot from them. [X] was an awesome teacher; he put a lot of time into meeting with our group, and really thought about our project at the board alongside us. I wished we could have more experiences like this!" (Non-mathematics PhD)
- "Learning to give long lectures/talks was great. I really loved the problems presented in the Project Lab, we had to work collaboratively and produce results in a fixed amount of time." (Software)
- "[The Project Lab] was great. Faculty interaction, good setup to learn writing, group work, open ended problems." (Consulting)
- "For the Project Lab, working a long time on a particular problem was fun, especially since we were able to pick problems that were particularly interesting to us. I was also working with my roommate and a close friend so that's always nice." (Software)

In response to "What is the worst thing about the Mathematics CI-M program" there was just one complaint about the Project Lab: "A specific project advisor in project lab was awful to deal with." (Mathematics PhD)

The MIT end of term subject survey also always generates quite a bit of commentary by students, on all aspects of the course. A principal theme of these comments is the student appreciation of the mentorship they received in the course. Here are two interesting responses reflecting this, and exhibiting the self-directed learning that takes place.

> "This has been one of the most memorable classes I've taken at MIT. It was more work than I initially imagined, but very gratifying to be able to write 3 full papers and give a full-length presentation. The amount of feedback we received on all assignments was so detailed and so helpful, and I have definitely grown as a writer and presenter. Thank you very much."

The next quote responds to a request for commentary on the projects. We mention it not to highlight one or another of the projects, but rather to illustrate the student control over the learning process that this course can engender.

> "Project 1: I knew from past experience that this question would be painful to tackle. However, once proper

> questions (as Professor . . . repeatedly notes) were carefully mathematically put, either results would follow immediately or the question transforms into a much deeper one. [Our mentor] was very helpful in guiding us on whether the questions we were asking were properly put and if our formulation would just lead to further complications (along with other guidance). Although I got results with the basic algebraic approach, I went on to investigate for a long time the more abstract algebraic approach. So, I learned from this first project the important skill of asking questions properly, but I did not learn the important skill of discarding complicating formulations early on. Project 2: What I learned and what I didn't learn from the first project carried on here. A problem that arose here was that I thought the natural way to look at this project was the algebraic one. However, I should have discarded this approach early on for the sake of time. Project 3: By this time in the semester, I've mastered the second skill I mentioned above, namely, that I should discard complicating paths early on. This skill enabled me to explore different ways to look at this problem in the time allowed."

Practice presentations turn out to be a very important and popular part of the course. Here are some comments reflecting that.

- "Most valuable part of the class."
- "Learning to give mathematics presentations is a different skill than that of giving general talks. I needed this skill, and the presentation part of the course gave me this skill. It was very helpful to have one and only one practice presentation. That way, we get prepared and very prepared early. The comments were very helpful and applicable. The presentation experience was very nice."
- "The practice presentation was an incredibly useful couple hours for us because we had the wrong approach and were not aware of it. We worked hard on improving our presentation and it felt very satisfying when we finally presented, delivering what seemed like a clear message (based on students' comments)."

Other student comments have centered on two issues: It is tricky to communicate the grading rubric to students, and in some semesters there have been significant complaints about this. Teamwork can be problematic, and students whose teams become dysfunctional were not shy about reporting the resulting problems.

7. Other instances of the Project Lab

Variations on this course have been offered elsewhere. The course Math 389 Explorations in Mathematics has been offered at the University of Michigan for a number of years under the direction of Professor Jeff Lagarias, as part of the Mathematics Department's Center for Inquiry Based Learning. The problem list began as a copy of the MIT list, but has evolved somewhat differently. There are interesting differences between the UM and MIT courses. The UM course is regarded as a toolkit course, training in the nuts and bolts of constructing and writing up proofs. It's addressed to students early in their undergraduate career. Lab space is provided, where students can find a mentor. There are two practice presentations rather than one, and each team presents twice in the course of the semester. It enjoys an enrollment of about a dozen student each year, despite fulfilling no institutional requirements.

Another version of the course was offered for a few years at the University of California, Berkeley, again using the MIT problem list.

We feel that courses of this type provide important elements of an undergaduate education in mathematics not well addressed by traditional curricular courses. The OpenCourseWare website [7] is designed to provide guidance to others interested in setting up such a course.

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