Computer-assisted Explorations in Mathematics:

Pedagogical Adaptations Across the Atlantic

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INTRODUCTION:

In the broader context of understanding change at the student level considered in this section, this chapter explores how experimental course structure, in particular the introduction of an unconventional mathematics project laboratory, can impact student development. Our discussion will be structured by considering a few questions:

What is the effect of a novel course structure (and we have two distinct models to study) on student attitudes towards the subject?

What are the impacts on student teamwork and interaction with course assistants, on the one hand, and of sequestered problem solving, on the other?

What is the variation of the impact of computational activities under these different instructional regimes?

These questions will be explored through an analysis of a pair of mathematics courses – the Project Laboratory in Mathematics at MIT and CATAM, Computer Assisted Teaching of All Mathematics, at Cambridge University – developed under a grant from the Cambridge MIT Institute. They share certain features – a focus on the application of computation in the understanding of mathematical contexts, learning activities which are very open-ended relative to their respective institutional standards, and some specific content – but differ markedly in others.

HISTORICAL CONTEXT

Early Days of CATAM

CATAM, whose initials first stood for *Computer Assisted Teaching of Applied Mathematics*, was considered revolutionary when it was introduced at Cambridge University in 1970. It was a course requiring students to write computer programs to investigate a variety of mathematics topics, launched at a very early stage in the development of modern computers. In fact, trends and advances in numerical tools, computing, and pioneering research in the field of applied mathematics at the time helped pave the way for CATAM's introduction into the Department of Applied Mathematics.

CATAM's first Director, Robert Harding[1], noted that the historical backdrop to CATAM was critical to its acceptance. During the late 1960's, UK universities were making more use of numerical solutions as lecturers in the Applied Mathematics Department increasingly found their research drawing on greater complex systems.

Cambridge, and the Mathematics Department at Cambridge in particular, was central in the early development of the electronic computer. Alan Turing made his fundamental work on the abstract theory of computation as a Mathematician at King's College.

Morris Wilkes, of the Mathematical Laboratory at Cambridge, led the team which developed EDSAC (Electronic Data Storage Automatic Computer), the first fully functional stored program electronic computer. He was also the first to teach mathematical computing, and developed the first assembler language.

A Departure from Tradition

The educational justification to teach a mathematics course requiring students to master computing techniques signaled a severe departure for traditionalists in the Cambridge Mathematics Faculty. Unlike the training in engineering or medicine, mathematics at Cambridge was intended to prepare the next generation of academics not practitioners or technicians, and many felt that computation was not part of that tradition. -Furthermore, the course was not examined by means of the traditional *Tripos* examination papers, but rather by papers written by students outside of exam conditions and submitted earlier in the year.

The rationale behind the course method was to gain a physical understanding of certain phenomena. This was attempted by introducing a mathematical model, experimenting with it, graphically representing it, and then running the model under different circumstances. In other words, by trying to get "inside the physical model," your understanding of the phenomenon would be enhanced.

Prior to the 1970's, only the most rudimentary of graphics tools were available to Cambridge students. The grid-lined paper with numbers used by students made for long and tedious calculations. The result was an inordinately slow, inefficient and demotivating experience. Other relevant advances in computing technology included the first "personal" computer at Cambridge: the "PDP 8 Machine" designed by DEC, hardly qualified as a desktop computer, being the size of a large desk itself, but it was relatively inexpensive, easy to program, came with a cathode-ray graphic output device, and, most important, one could work at it in person rather than submit jobs to technicians. All of these features were important prerequisites for the development of CATAM.

[1] Interview with Robert Harding, Lecturer, Department of Applied Mathematics & Theoretical Physics, 1/3/06

George Bachelor, one of the creators of modern fluid mechanics and founder of the Department of Applied Mathematics and Theoretical Physics, DAMTP, at Cambridge, was a strong advocate of integrating computation into the undergraduate curriculum. He secured funding from the Nuffield Foundation and Shell to support the initial work on CATAM.

The "Feedback Loop of Learning"

Bachelor's justification of the educational value of CATAM within the Department emphasized a *feedback loop of learning*. Students are given a problem, they create an appropriate mathematical model, they run it under a certain set of conditions, and then they study the output. They compare the results against reality, and this leads them to return to the original model, questioning correctness and assumptions that drove the model along the way. Students use experimentation to address various issues with the model. In the end they write up a report describing their journey and submit it for evaluation.

Harding asserts this basic *feedback loop* has remained unchanged over the years. CATAM students are still encouraged to follow this same line of active learning-- think, justify, test, fit, review, check predictions, and submit a report. Harding himself published ten papers on the pedagogical merits of CATAM in math education journals during the 1970's. Topics ranged from the use of computer graphics as a teaching aid to computer assisted learning in higher education to the impact of CATAM on mathematical problem solving abilities.

Enrollment Trends

With the excitement surrounding the introduction of computing into undergraduate work at Cambridge, CATAM undergraduate enrollment was initially quite high. However, "once the shine wore off"[2] enrollment began to decline. Another factor depressing enrollments stemmed from opposition to computing within the Cambridge Mathematics Faculty. To some, using computers to solve problems was not what constituted doing "real mathematics." However, with Cambridge's own faculty engaged in ground breaking research which relied heavily on computing methods, attitudes began to change. For example, John Conway's work on group theory, it has been argued[3], would not have been possible without the aid of the computer. The use of computers in the context of mathematical research was increasingly being modeled by Cambridge faculty themselves.

Traditional assessment schemes would also prove a natural obstacle for CATAM to hurdle. Traditionalists questioned how 'CATAM knowledge' could be tested. As computer programming far exceeded the standard length and conditions of Cambridge

[2] Ibid, 1/3/06.
[3] Ibid, 1/3/06

exams, an alternative arrangement would need to be approved in order to legitimize its role in the mathematics curriculum. In its early years, CATAM problems could typically take students anywhere from eight to ten hours to complete (Today's more complex problems take students many more hours to complete.) Therefore, project reports were to be the only reasonable forms of assessment. This too stirred talk of unfair advantages, but ultimately it was accepted that credit was deserved for tackling challenges in computer programming, de-bugging, identifying incorrect output and more generally, deciding what to do when things go wrong.

The awareness of the possibility of things going wrong still concerns Harding, the original CATAM Director. With significant advances in computer technology, Harding worries whether today's students are equally savvy about what can break down and why. Looking for problems and bugs, noticing spurious results or even seeing effects which don't make sense are all instincts which Harding had always hoped would improve in the context of computer aided mathematical problem solving.

MODERN DAY CATAM

Evolution of CATAM

Today, Dr. Robert Hunt, Lecturer in the Department of Applied Mathematics & Theoretical Physics at Cambridge, leads the CATAM initiative. Roughly two hundred students work on CATAM in their second and third years, as part of their Mathematics *Tripos*. Although considered optional, CATAM project marks are added to students' end of year exams, and for this reason, most students opt to do the CATAM projects. In the third year program, students choose five projects from a booklet containing thirty project ideas, which they are expected to explore, write up and submit at the end of the year. Each project write up tends to range from twenty to thirty pages in length.

Students at Cambridge are also expected to work individually on their projects, without the aid of instructors, teaching assistants or their peers. In terms of guidance, students are permitted to receive assistance in programming only. Assessments are based entirely on written work and final marks are given for mathematical content and for getting the results. They are graded using a very specific rubric, which includes special marks for noticing features revealed by the computer output but not specifically mentioned in the instructions. While graphs must be readable and referenced, the expected literary style might be called telegraphic.

Naturally, in terms of assessment, high student enrollments translate into massive year end marking demands. Marking schemes then are facilitated by highly detailed project questions. The rationale runs as follows: The more detailed the project question, the less need for guidance and the more straightforward the marking. Extra assistance from instructors is considered cheating within Cambridge University's general academic policy as is the prohibition of teamwork.

Seeing it in Front of Their Eyes: Cambridge Students' Perspectives

For Cambridge students, CATAM represents a marked departure from their staple diet of lectures and recitations. It offers a much more independent, exploratory experience which not only exposes many of them for the first time to computer programming, as evidenced from end of year interviews,[4] but also challenges their depth of understanding. The ability to see functions unfold before their eyes was reported as a particularly valuable experience. The actualization of theories and models afforded by computer programming motivated one Cambridge student (who received a 'first' degree in mathematics) to express the following:

> "…I mean, the lecturer says if you do this, then you get this kind of bifurcation, you get that kind of bifurcation, but then when you program something, you can see it happening in front of your eyes and that's really, really a huge advantage. Then you sort of believe it more, if you produced it. That was very, very useful."

Despite the overwhelming, agreed value in learning computer programming for its practical application in their future careers, some students expressed great frustration over debugging programs. This often led to project abandonment altogether.

Nearly all students interviewed explained that they tend to work on their CATAM projects during their university holidays rather than during term time. Nearly all students interviewed also reported taking longer than originally anticipated. Because of this unique course feature, students say they require long periods of uninterrupted time—a luxury rarely found during term time. With the steady rhythm of supervisions, papers and example sheets, students complain that there is never enough time to do justice to CATAM projects during term.

Although most CATAM 3rd year students had also previously taken the course during their 2nd year, procrastination behaviors saw very little improvement. Time management skills and project pacing throughout the year were sorely lacking across the board. One student reflected upon her mismanagement of time and its perceived painful repercussions:

> "Some of the proofs I think I spent quite a lot of time on so they were quite useful mathematically. But I think I probably spent the wrong proportion of time doing the programming relative to the maths… Whereas most of the mathematical parts of the questions were 'Explain why' or 'Talk about this' or 'Discuss' which was really woolly. It made it really difficult to know what you had to write and how far your investigation had to go, and I think having spent so much time on the programming, I probably cut the explanations shorter and didn't research as much of the mathematics as I should have done."

For many CATAM students, the appeal of this course over others in their *tripos* is the freedom to choose projects. Oftentimes, students who have taken appropriate

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[4] CATAM End of Year Student Interviews, 8/2004 (n=18), 8/2005 (n=20).

coursework end up choosing the projects which allow them opportunities to apply these competencies. Other times, students who identify themselves as falling into one camp or another - either "applicable maths" or "pure maths" - will choose appropriate projects accordingly.

Most of the students did not report a sense of feeling greatly challenged by the uncertain nature of some of the problems, but some felt deeply frustrated. Dealing with uncertainty seemed to require greater problem solving confidence and a definite departure from one's 'comfort zone', as expressed here by one student:

> "In maths, it's totally not like that, like you can always predict if you've got it right, if you've got it wrong. But with CATAM, it's like 'That's a total black box. I don't know what is going on there.' I mean that's a bit more like research. It was a useful exercise."

Despite its time-intensive computing challenges and an unfamiliar open-endedness, CATAM seemed to most Cambridge students to provide a unique contribution to their overall levels of mathematical understanding. Despite the complaints over its disproportionate credit or "markings" value, most students admitted the learning benefits paid off:

> "… I mean it's quite annoying … because it takes up a disproportionate amount of time to what it's worth. Its total value is worth less than a single 24 lecture course. It's a tradeoff between guaranteed marks and taking up far more time. [Nevertheless,] in retrospective, it's definitely worthwhile."

CATAM Crosses the Pond: The Mystery behind MIT's Math Lab

In the spring semester, 2004, instructors from MIT's Department of Mathematics won a substantial grant to develop an experimental "Math Lab," loosely based on the Cambridge University CATAM course material. Known in the local vernacular as 18.821, the Mathematics Project recently became the first course in the Mathematics Department to satisfy the undergraduate laboratory requirement. It also satisfies the newly established Communications Requirement in Mathematics. The course involves one instructor, three to four teaching assistants, and a capped enrollment of thirty students. Students break into teams of three, constant over the semester, which then negotiate the selection of three open-ended projects from a menu of over twenty selections.

Haynes Miller and Mike Artin directed the creation of material for this course. Many of the Math Lab projects originated as CATAM scripts, but many others were solicited from MIT faculty. These scripts and ideas were worked through by MIT undergraduates – about a dozen altogether - to get a sense of what was really involved from the perspective of an MIT student. An attempt was made to systematically strip away the prescriptive, directed elements, leaving only a treasure map to a mathematical context.

The typical desired pattern of student behavior is this: The students find themselves confronted with a mathematical question; they do some experimentation, typically with a computer; they find regularities, which they then try to explain mathematically. They have a long conference with their course assistant at least once a week. The course assistant has been instructed to offer feedback of a restricted type only. Since there is no specific content objective in these projects, there are many fruitful avenues which the students may explore. The course assistant may have preconceptions about which will be more or less fruitful, but unless the students are stuck or really off on an unproductive tangent, the course assistants are under orders not to spoil the experience of the challenges of research by giving away too much information.

At the end of three weeks or so, they have to have produced a paper on their findings. They turn in a draft of this paper, and a day or so later give a briefing to the faculty in charge and the course assistant, who offer constructive criticism of both their mathematical work and their written and oral presentations. They then have a short additional time in which to revise and resubmit the paper. Meanwhile, the next project gets underway. At the end of the term there is a course conference, at which each group presents one of their three projects in a public 40 minute talk.

The creation of "Math Lab," or MIT's variation on CATAM, raises larger questions about cultural adaptations of imported pedagogic models:

How do curricula travel across cultural boundaries?

In what ways do educational philosophies shape the adaptation itself?

Are there ways for the imported curricular model to feed back and influence the further development of the original?

In the case of CATAM, for example, Cambridge mathematics instructors have been trying to integrate some of MIT's new project ideas, including their characteristic *open-ended* problem solving approach. In fact, CATAM project menus have even started including a handful of MIT projects among its original menu of thirty.

Doing Mathematics is Discovering Mathematics: MIT Instructors' Perspectives

MIT Math Lab's pioneers note CATAM's "radically different educational goal"[5] in the context of both Cambridge and MIT. Students at MIT are used to frequent, directive problem sets where *plugging and chugging* sometimes takes the place of deeper, riskier, experimental engagement. The Math Lab offers students "license to wonder."[6] Students

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[5] Interviews with H. Miller, M. Artin, 1/06.
[6] Ibid, 1/06.

who when given the license to navigate the far reaches of these projects, say Math Lab Instructors, "can sometimes surprise you with how far they'll go."[7] Experiencing the "wide open spaces" of mathematics is one of the unique virtues of Math Lab, as confirmed by one of the students:

> "I like them (Math Lab projects) a lot more (than problem sets). I don't like problem sets. I don't like it when someone tells me what I have to learn. So, that was sort of a problem for me at MIT, because that's sort of the whole thing that they do here…When someone doesn't tell you what to learn, but like [says], 'Here's an interesting problem,' I like that."
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> --MIT Math Lab Student, Spring, '05

One of Math Lab's Instructors, Professor Michael Artin, taught the first cohort of 18.821 at MIT. The following year, Professor Artin also taught the same course at University of Michigan during a sabbatical semester. Artin noted little difference between the best students at MIT and Michigan, but observed how the work ethic and intensity at MIT gave other students the persistence necessary to struggle and stick with Math Lab projects to the bitter end. In this curricular context, talent plus effort proved a winning formula.

Artin also observed both at MIT and at Michigan that students did not seem to take as many risks in choice of projects as he would have preferred. Sticking to topics which seem safe is a common pattern. However, according to Artin's educational philosophy, exploring the open-ended, unfamiliar questions, like diving into unchartered waters, enhances the discovery process in the context of problem solving.

Because of this high 'market value' placed on discovery, Artin sees CATAM, the progenitor, as overly directive and therefore under-nourishing. The open-ended approach, as Artin sees it, does not dictate to students what to notice, what to do, or even what a certain algorithm is named (so they can find clues on Google, for example). In this process, says Artin, students sometimes "flounder," sometimes go down "dead ends," but ultimately, on their own doing, they find a path. On this path, students do their own thought experiments, get somewhere, make their own conjectures, sometimes get creative, but ultimately, they "learn to go deeply into some particular area." This, explains Artin, is the essence of "doing mathematics."

While this clarity of vision is communicated to the Lab's teaching staff, student Course Assistants express a palpable dilemma between letting students go off on their own experimental "journeys" and guiding them "toward the light."

One graduate Teaching Assistant, Michael Ching, himself a veteran of CATAM as a former undergraduate at Cambridge, reported sharp contrasts between the original and the adapted curricular models. Without the group component, says Ching, CATAM felt like "more of a chore" that you had to complete by the end of the year. In contrast, Ching also would have preferred the license to explore projects in greater depth, a characteristic

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[7] Ibid, 1/06.

feature of the MIT adaptation. The projects themselves, noted Ching, then take on a much more interesting challenge.

As a Teaching Assistant for Math Lab, though, Ching says he experienced his biggest learning curve at the point where he'd gotten to understand the group interactions of his student teams well enough to "just let them do it on their own," with perhaps a general suggestion here and there.[8] Having served as a Teaching Assistant, a recitation leader and a grader for other courses at MIT, Ching noticed the opportunity to interact more with the students themselves generated the most gratifying results. For example, Ching found that in the context of Math Lab, different teams had very different needs. Some groups needed ideas and extra direction, while some needed none at all. Striking the balance, according to Ching, was the toughest yet most illuminating part of the teaching challenge.

Getting their Feet Wet: MIT Students' Perspectives

> "It's the way that I like to learn…the fact that these (projects) are open…I like that."
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> --MIT student, 18.821 Project Lab, 2005

Overall, the Math Lab students from the first two cohorts at MIT reported overwhelming satisfaction with the course according to a number of measures. The biggest appeal by far was the discovery process. This, they said, offered them an authentic connection to the material unlike any other course instruction or problem set routine. But some enjoyed the opportunities for independent research and choice of projects above all else, with the access to guidance and structure (e.g. deadlines, required meetings with Instructors and Teaching Assistants, regular team meetings, oral presentation rehearsals, team pre-selection, writing assistance, and best practices of math research papers.)

In addition to discovery, independence and choice, students also valued the opportunity to gain depth of understanding in Math Lab. Clarification through explanation dominated students' records of their experiences with this course. Theirs was a powerful description of active as well as deeper exploration of topics. Relevant activities noted included 'explaining,' 'starting from scratch,' 'clarifying,' 'backtracking,' 'thinking,' 'correcting,' and, last but not least, 'throwing everything you know at it.' One student even commented on the effect this course had on his "confiden(ce)…to do real research." And finally, another student sarcastically noted that he "liked coming up with new ideas and watching them fail."

No comparative discussion between CATAM and Math Lab would be complete without raising perspectives on Math Lab's adapted feature of teamwork at MIT. In Math Lab at MIT, teams typically consisted of three students. Division of labor naturally evolved from student interaction. Tasks tended to divide among problem solving, computing, writing, revising, and communicating. Having to deal with different skill levels,

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[8] Interview with M. Ching, 6/05.

schedules and academic priorities apparently colored individuals' 'teamwork' experiences in both positive and negative lights.

While teamwork was far from perfect across all Math Lab groups, most students took naturally to this style of working and problem solving. Upon first breaking out into teams, MIT students reported that they almost instantly divided up the labor according to individual strengths. One could argue that in addition to confirming their own areas of expertise, their awareness of other's strengths and weaknesses improved and optimized teamwork. This included making more effective use of meeting times as well as capitalizing on peer teaching and learning opportunities.

HUNT'S HYPOTHESIS

In the summer of 2004, student papers were compared across MIT and Cambridge University focusing on parallel lab projects. First, the focus of the assessment looked at the presentation of the original problems, asking:

To what extent did the presentation of the problem affect the nature of the student's solutions?

Standards of presentation, including similarities and differences, were also considered and assessed within this context. Secondly, outcomes, quality and final marks were compared:

How does the intrinsic mathematical quality modulate across the ability range?

Investigators also hoped to see whether projects of equivalent mathematical quality were valued equally at MIT as well as at Cambridge.

Finally, an overall summary of strengths and weaknesses for both courses were reviewed and discussed in order to improve future course materials and methods at both campuses.

Prior to reviewing samples of student work in the summer of 2004, Professor Robert Hunt, Lecturer in the Department of Applied Mathematics & Theoretical Physics at Cambridge University hypothesized that the quality of mathematics produced by the Cambridge CATAM students would be "deeper" than those of their MIT counterparts. Three main factors influenced Hunt's Hypothesis:

1) Cambridge University's relatively detailed project approach, Hunt argued, 'push(ed) them toward the deeper bits';

2) Since presentation and grammatical correctness did not factor into students' final marks at Cambridge, Hunt argued that students, theoretically, were left free to focus *more* on the mathematical content than their MIT counterparts, and finally;

3) Cambridge students, according to Hunt, were mathematically more advanced because they represented 2nd & 3rd year students (the two final years) compared to a mixture of levels at MIT.

The evidence to prove or disprove Hunt's 'Hypothesis' lay in the student papers themselves, subject to comparative scrutiny.

Concluding Remarks

When asked about the greatest long-term benefits of both CATAM and Math Lab, students mention the variety of 'real world,' practical skill sets--oral presentation, teamwork, writing papers, and computer modeling to name a few. On the whole, students noted they perceived these skills to be extremely beneficial for them to gain experience and practice before entering the 'real world' or further education upon graduation.

Time management, arguably a vital skill in the 'real world' as well as the 'academic world,' proved challenging to the MIT Math Lab student, just as it had to the Cambridge CATAM student. But in the context of teamwork at MIT, some students found it even more challenging they needed to orchestrate their work time with that of their teammates in order to meet team project deadlines.

However, despite these logistical constraints, students resoundingly supported project-based work, in general, over piecemeal problem sets. Overall, students reported preferring these opportunities to 'think' and be more 'creative,' as well as face the more challenging 'why' questions.

Furthermore, in the context of a team, rather than in standard, solitary problem solving mode, students wrestle with projects together, dividing and conquering the labor by individual strengths. As they experience first hand, complex project tasks can run the gamut, calling upon a host of team members' talents, from computing to communicating the results, from writing to revising. Understandably, working in teams requires team members to cope with each others' different skill levels, schedules, and personally defined course objectives. Arguably, this challenges students beyond the dimensions of intellectual development as discussed in Chapter Three, and into the 'murkier waters' of personal and social development as discussed in Chapter Four.

In closing, the independent, exploratory nature of CATAM and Math Lab speaks volumes to students who thrive as *discoverers* rather than *inheritors* of knowledge. By offering an authentic connection to the material, via a 'field' for 'wandering' with one's peers, project laboratories such as these can open up opportunities for greater in-depth, conceptual understanding and publicly shared explanation.