Kervaire Invariant One

Following Mike Hill, Mike Hopkins,
and Doug Ravenel

Haynes Miller

Theorem (Hill, Hopkins, Ravenel, 2009)

The Kervaire invariant on framed manifolds is trivial in dimensions larger than 126.

Combined with earlier results, this gives:

The Kervaire invariant on framed manifolds is trivial except in dimensions 2, 6, 14, 30, 62, and possibly 126.

Dimension 126 remains open.

The back story -

The grand program of geometric topology in the 1960s was based on the method of "surgery," which attempts to simplify a manifold in a given bordism class.

Recall that closed n-manifolds are cobordant if their disjoint union forms the boundary of an (n+1)-manifold.

There may be extra structure demanded, such as a trivialization of the normal bundle of an embedding in a high-dimensional Euclidean space: this is framed bordism.

Framed bordism classes of closed n-manifolds form a group Ω_n^{fr}.

Given a framed manifold, surgery finds another cobordant framed manifold with less homology.

Even if surgery succeeds, you may not reach the standard n-sphere S^n -

In 1958 Milnor constructed smooth manifolds which were homotopy equivalent to S^7 - actually, homeomorphic to it - but not diffeomorphic to it.

So the optimal end-point of the surgery process is a homotopy sphere.

When n is odd there is no middle dimension; we get a homotopy sphere.

When $n \equiv 0 \bmod 4$ the surgery can be completed as well.

When $n \equiv 2 \bmod 4$ there is an obstruction.

Theorem (Kervaire & Milnor, 1963) -
Except in dimensions $n = 4k+2$, every framed cobordism class contains a homotopy sphere.
There is a homomorphism

$$\kappa : \Omega^{fr}_{4k+2} \to Z/2Z$$

such that [M] contains a homotopy sphere if and only if $\kappa[M] = 0$.

The HHR theorem shows that all but finitely many framed bordism classes contain homotopy spheres.

Brief on the Kervaire Invariant -

M a closed $(4k+2)$-manifold.

$H^{2k+1}(M;F_2)$ supports a symmetric bilinear form, nondegenerate by Poincaré duality.

A framing t determines a quadratic refinement,

$$q_t : H^{2k+1}(M;F_2) \to F_2$$

$$q_t(x+y) = q_t(x) + q_t(y) + x.y$$

$$\kappa[M,t] = \mathrm{Arf}(q_t)$$

In low dimensions Kervaire invariant one classes are not uncommon.

Begin with a parallelized sphere:

$$S^1 \qquad S^3 \qquad S^7$$

The normal bundle receives a trivialization, and we obtain framed bordism classes

$$\eta \in \Omega_1^{fr} \qquad \nu \in \Omega_3^{fr} \qquad \sigma \in \Omega_7^{fr}$$

These are the classes of "Hopf invariant one."

Their squares turn out to have Kervaire invariant one:

$\theta_1 = \eta^2 \in \Omega_2^{fr}$ $\theta_2 = \nu^2 \in \Omega_6^{fr}$ $\theta_3 = \sigma^2 \in \Omega_{14}^{fr}$

Kervaire & Milnor suggested that this might be the end, so that with these exceptions all framed bordism classes contain homotopy spheres. But ...

Homotopy theorists muddied the waters -

May, Mahowald, Tangora, Barratt, Jones:
$\kappa \neq 0$ in dimensions 30 and 62.

The bridge: Pontryagin-Thom construction -

$M^n \subset R^{n+k}$, framing t : (normal bundle) $\to R^k$

$$\Rightarrow \quad [S^{n+k} \to S^k] \in \pi_{n+k}(S^k)$$

For k large this is the n^{th} stable homotopy group:

$$\Omega_n^{fr} \xrightarrow{\cong} \pi_n^S$$

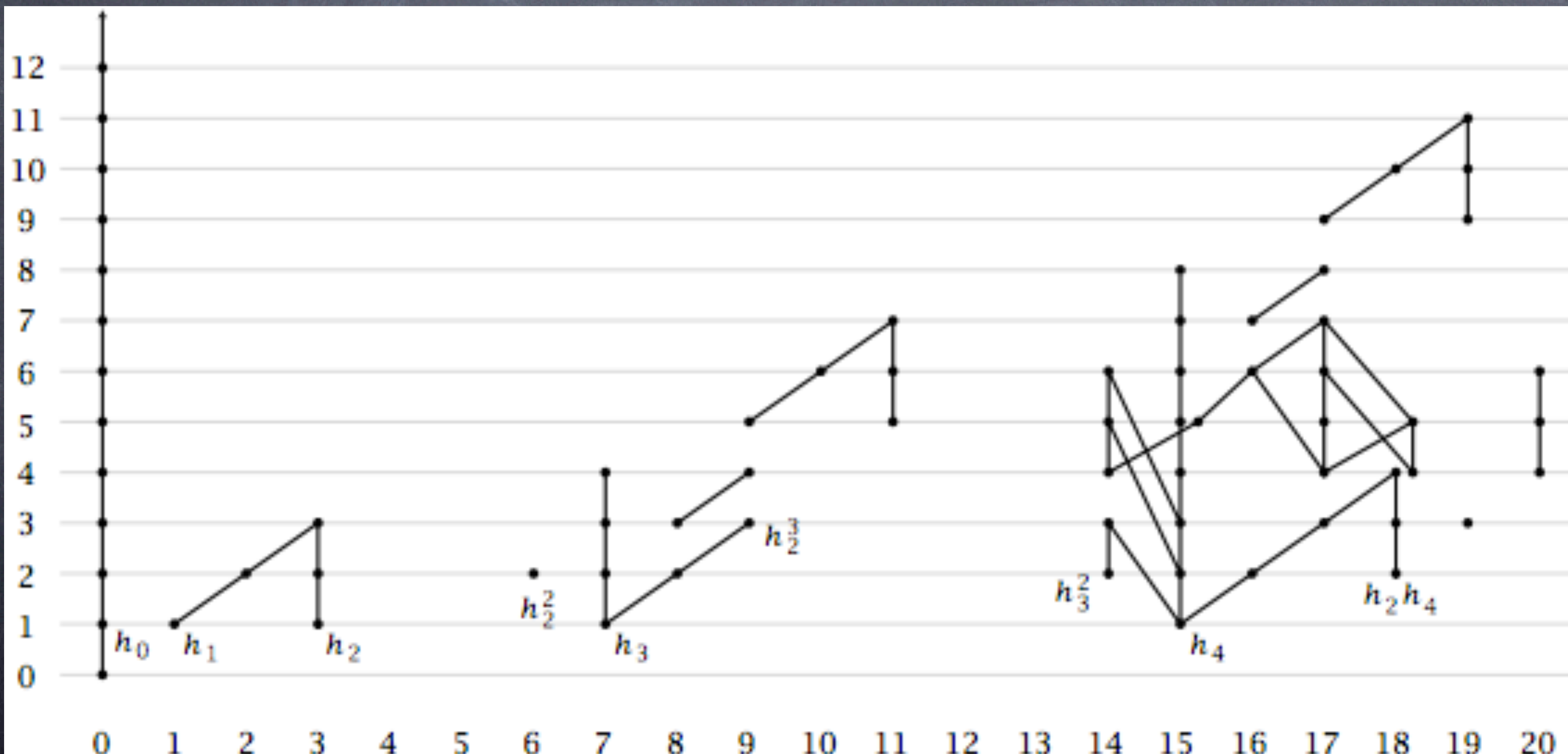
The tool - mod 2 Adams spectral sequence

The E_2 term contains "potential" homotopy classes - think of them as maps from a disk, waiting to be completed to homotopy classes by means of a map from a second disk agreeing on the boundary.

It converges to the 2-primary component of π_*^S .

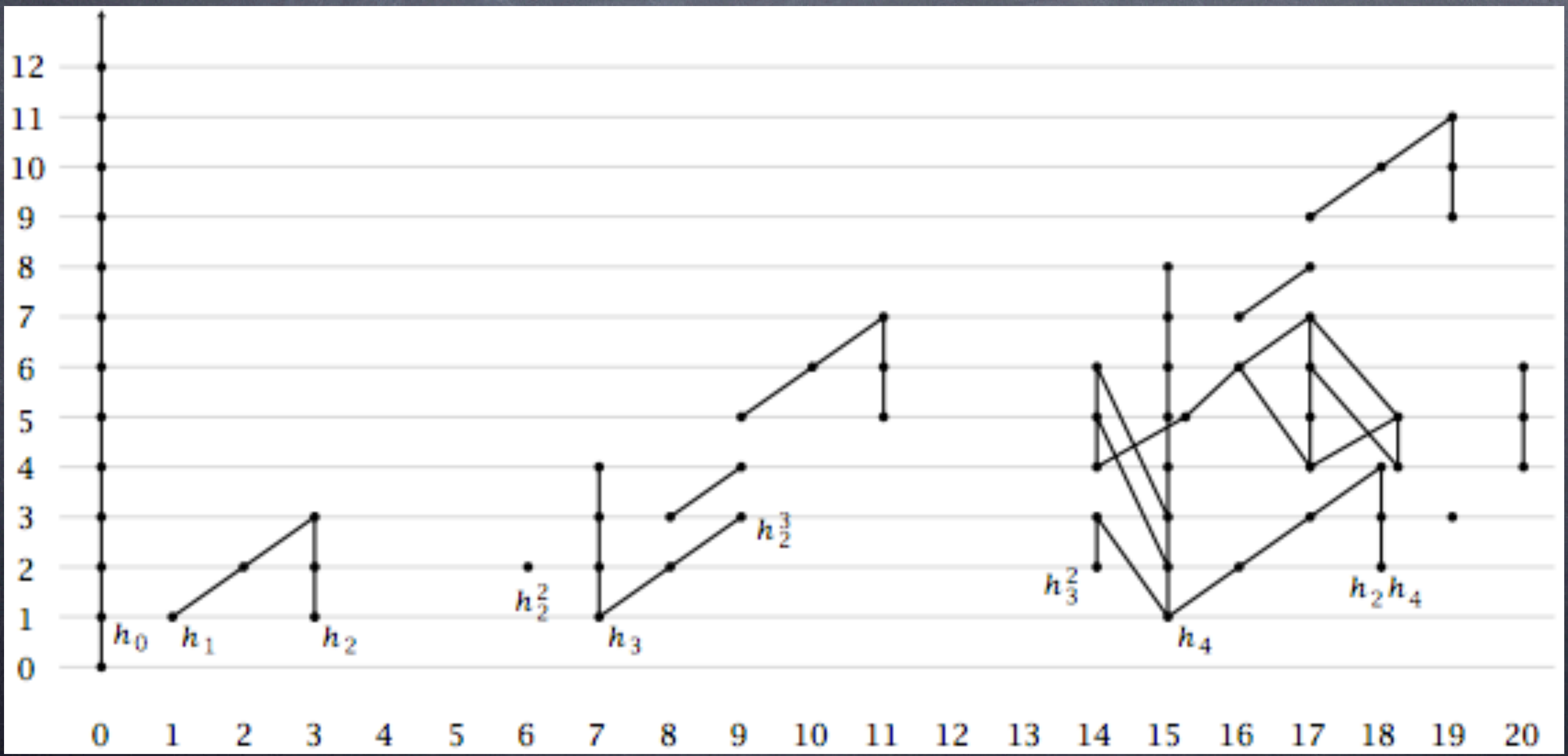
Adams spectral sequence

Part of E_2 -



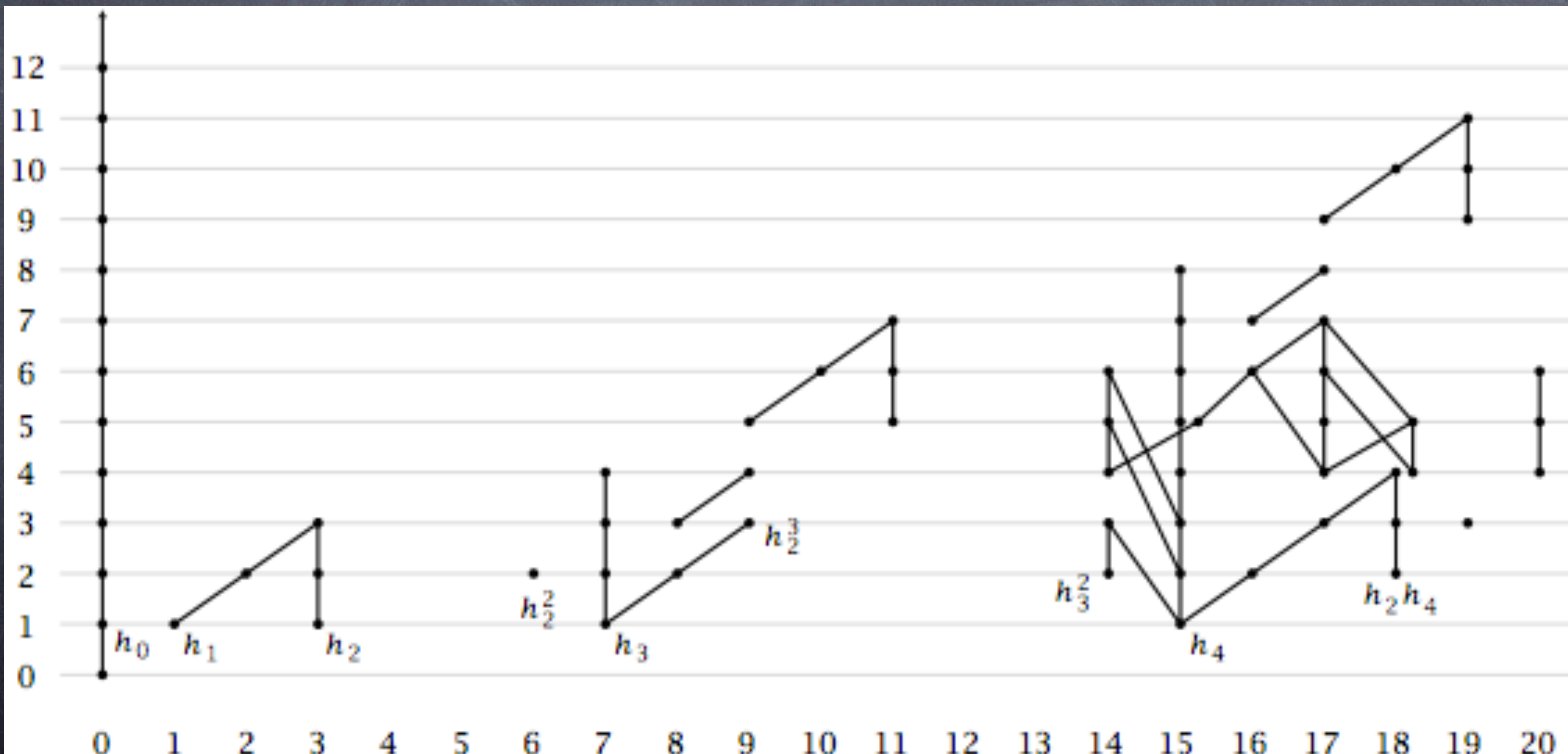
Classes of Hopf invariant one represented by h_j

Adams, 1962: $d_2 h_{j+1} = h_0 h_j^2 \neq 0$ for $j > 3$

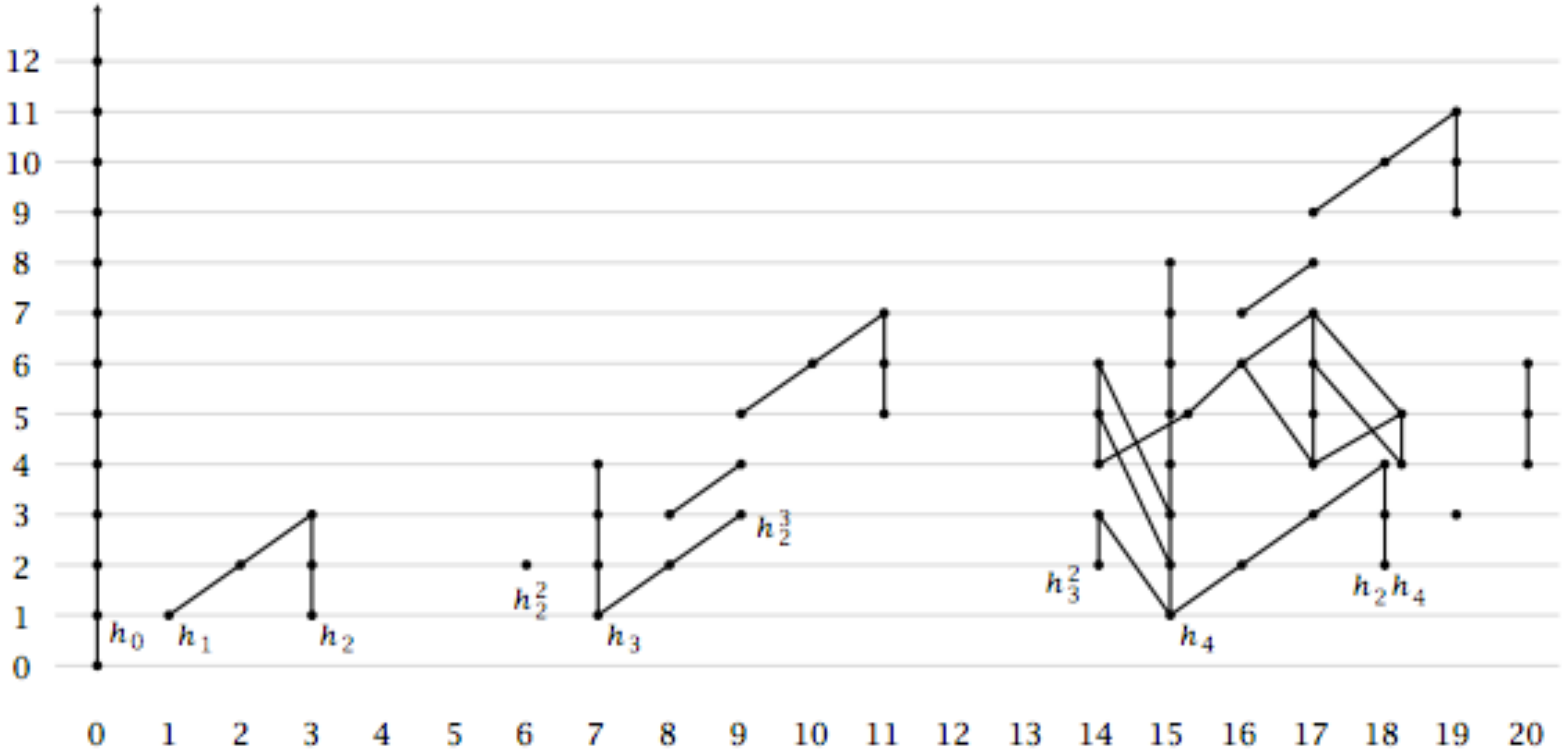


Browder: Classes of Kervaire invariant one: h_j^2

(so they occur only in dimensions $2^{j+1} - 2$).



The "easy" h_1^2, h_2^2, and h_3^2, survive because they are squares of permanent cycles.



The "easy" h_1^2, h_2^2, and h_3^2, survive because they are squares of permanent cycles.

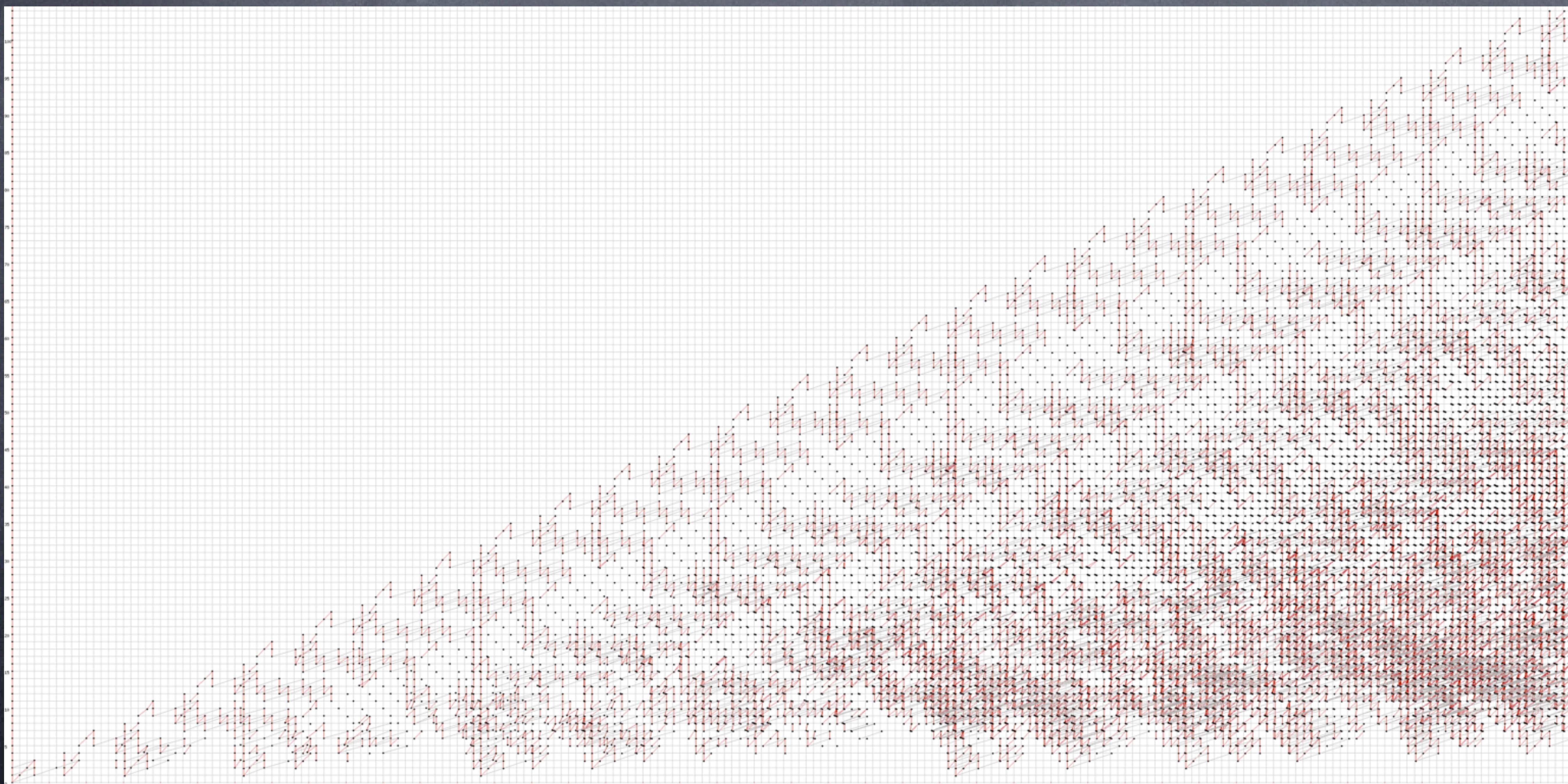
Hard work: h_4^2 and h_5^2 also survive (to θ_4 and θ_5, in dimensions 30 and 62).

Theorem (HHR) - h_j^2 does not survive for $j > 6$.

Question - What differentials kill them?

Adams spectral sequence

A bigger part of E_2 -



Something about the proof ...

Three new branches of topology used by HHR -

(1) Chromatic homotopy theory

(2) The theory of structured ring spectra

(3) Equivariant stable homotopy theory

... but there is almost no computation at all!

Idea of proof - Find a generalized cohomology theory strong enough to detect θ_j , and then show that for $j > 6$ it vanishes in the dimension where this detection occurs.

Example - $K^*(X)$ = topological complex K-theory

$K^*(*)$ -

...	-2	-1	0	1	2	3	...
...	Z	0	Z	0	Z	0	...

The coefficient ring is $K^*(*) = Z[\beta,\beta^{-1}]$,

$$\beta \in K^0(S^2,*) = K^{-2}(*)$$

"Dectection"?

If $L^*(-)$ is multiplicative,

$$1 \in L^0(*) = L^0(S^0,*) = L^k(S^k,*)$$

The L-degree of $f : S^{k+n} \to S^k$ is its image -

$$f^* : L^k(S^k,*) \to L^k(S^{k+n},*) = L^{-n}(*)$$

$$1 \longmapsto d_L(f)$$

$L^*(-)$ detects f if $d_L(f) \neq 0$. Then $f \neq *$.

Example: $d_L(\theta_j) \in L^{2-2^{j+1}}(*)$.

Clearer idea of proof - There is a generalized cohomology theory $LO^*(-)$ such that

(1) (detection) If θ_j exists then it is detected by $LO^*(-)$

(2) (gap) $LO^2(*) = 0$

(3) (periodicity) $LO^*(-) \cong LO^{*+256}(-)$ $256 = 2^8$

Proof that θ_j does not exist for $j > 6$ -

$$LO^{2-2^{j+1}}(*) = 0 \text{ for } j > 6$$

What is $LO^*(-)$? It can't be $K^*(-)$:

- $K^*(-)$ doesn't detect θ_j for any j , and

- $K^{2i}(*)$ is never zero.

The second problem is easy to fix: $KO^*(-)$ -

$KO^*(*)$ -

...	-4	-3	-2	-1	0	1	2	3	...
...	Z	0	Z/2	Z/2	Z	0	0	0	...

$$KO^{2-2^{j+1}}(*) = 0 \quad \text{for} \quad j > 2$$

because $KO^2(*) = 0$ and KO is 8-periodic.

In fact the first problem is partly fixed too -

$$d_{KO}(\eta^2) \neq 0 \quad \text{in} \quad KO^{-2}(*)$$

To find a good candidate for $LO^*(-)$, we need to
(a) understand how $K^*(-)$ & $KO^*(-)$ are related,
(b) find a better cohomology theory than $K^*(-)$.

Ad (a) - C_2 acts on $K^*(X)$ as $Gal(C/R)$,
but $KO^*(X)$ is not just the fixed points.

Any generalized cohomology theory is
represented by "spectrum" - $K^*(-)$ by K, etc.

C_2 acts on K, and with respect to this action

$$Fix(C_2,K) = KO$$

Atiyah - Equivariance makes periodicity easier

The Bott class $\beta \in K^0(S^2,*)$ is represented by and equivariant map

$$\beta : S^\rho \to K$$

ρ = regular representation of C_2

S^ρ = 1-point compactification of ρ
= Riemann sphere

The product in $K^*(X)$ is represented by a map of spectra

$$K \wedge K \to K$$

which induces an equivariant equivalence

$$S^\rho \wedge K \xrightarrow{\beta \wedge 1} K \wedge K \to K$$

This implies periodicity for both K and KO .

Dan Dugger: there is a connective equivariant spectrum k with $\beta: S^\rho \to k$ such that

$$K = \beta^{-1} k$$

Ad (b) - To improve on $K^*(-)$ we need spectra that "see" more. A good supply is obtained starting from complex cobordism -

$$\Omega_U^n(X) = MU^n(X)$$

When $X = *$ this is the ring of cobordism classes of closed manifolds with a complex structure on the normal bundle.

Chromatic homotopy theory -

Quillen established a deep connection between MU and the theory of one dimensional commutative formal group laws:

$$MU^*(*) = \text{Lazard ring}$$

This has led to an explosion of progress in understanding stable homotopy theory.

This connection showed HHR that a spectrum strong enough to detect Θ_j requires an action of the group $G = C_8$.

C_2 acts by complex conjugation on MU as well as on K.

The norm or multiplicative induction provides a C_8 action on

$$MU \wedge MU \wedge MU \wedge MU$$

$$\gamma : (a,b,c,d) \longmapsto (\bar{d},a,b,c)$$

The analogue of the Bott class will be a map

$$D : S^{k\rho} \to MU^{\wedge 4}$$

where $k\rho$ denotes k copies of the regular representation of C_8. Define

$$L = D^{-1} MU^{\wedge 4}$$

This is a very sloppy analogue of the spectrum K. The sloppy analogue of KO is

$$LO = Fix(C_8, L)$$

Theorem - There is a choice of "Bott class"

$$D : S^{k\rho} \to MU^{\wedge 4}$$

such that the generalized cohomology theory represented by

$$LO = (D^{-1} MU^{\wedge 4})^{C_8}$$

satisfies the following properties:

(1) (detection) If θ_j exists then it is detected by $LO^*(-)$.

(2) (gap) $LO^2(*) = 0$.

(3) (periodicity) $LO^*(-)$ is periodic of period 256.

Proof of the gap and periodicity properties

These follow from a "purity" theorem for the C_8 spectrum $MU^{\wedge 4}$.

Spaces, or spectra, admit (up to homotopy) decreasing filtration whose quotients are Eilenberg Mac Lane objects:
the Postnikov system.

Postnikov system -

For example $\pi_t(K) = Z$ for t even ,
0 for t odd

so in the Postnikov filtration

$$P^tK \,/\, P^{t+1}K = S^t \wedge HZ \quad \text{for } t \text{ even}$$
$$* \quad \text{for } t \text{ odd}$$

Slice filtration: an equivariant enhancement -

Equivariantly (Dugger):

$P^t K \,/\, P^{t+1} K \;=\; S^{k\rho} \wedge H\underline{Z}$ for $t = 2k$

$*$ for t odd

Here $H\underline{Z}$ denotes an appropriate equivariant Eilenberg Mac Lane spectrum.

Not every spectrum admits such a filtration.

The key theorem in this work is -

The Slice Theorem -

The $G = C_8$ spectrum $MU^{\wedge 4}$ admits a "slice" filtration in which the quotients are sums of equivariant spectra of the form

$$(\mathrm{Ind}_H^G\, S^{k\rho_H}) \wedge H\underline{Z}$$

where $H \leq G$ is a nontrivial subgroup and Ind_H^G denotes the "additive induction"

$$\mathrm{Ind}_H^G\, X \;=\; G_+ \wedge_H X$$

On the Slice Theorem -

The analogue for the C_2 spectrum MU is due to Po Hu and Igor Kriz.

A motivic analogue was known to Hopkins and Morel.

In the end it boils down to an equivariant version of Quillen's theorem: coefficients of the formal group law generate $MU^*(*)$.

Proofs of the properties of LO -

The gap property follows quickly from the Slice Theorem since it holds for the quotients

$$(\mathrm{Ind}_H^G S^{k\rho_H}) \wedge H\underline{Z}$$

The underlying geometric fact is that the orbit space S^{ρ}/C_2 is contractible.

Proof of the periodicity property requires the computation of some differentials in the spectral sequence associated to the slice filtration.

The detection property is proved by studying the analogue of the Adams spectral sequence based on MU . The connection with formal groups provides homomorphisms to

$$H^*(G;R)$$

when G acts on a formal group over R . The relevant action is one by $G = C_8$ on a formal group of height 4 .

Questions -

- What Adams differential kills h_j^2 ?
- What is the geometry underlying this proof?

Questions -

- What Adams differential kills h_j^2 ?
- What is the geometry underlying this proof?

Thank you!