Notes on Clark Barwick’s operator categories

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Definition. An operator category is an essentially small category \( \Phi \) such that

1. Hom sets are finite.
2. There exists a terminal object, \( * \).
3. Fibers exist: for any map \( J \to I \) and any \( i : * \to I \), the pullback \( J_i \to J \) exists.

The set of points of an object \( I \) in \( \Phi \) is \( |I| = \Phi(*, I) \). This is a functor from \( \Phi \) to finite sets.

An operator morphism from \( \Psi \) to \( \Phi \) is a functor \( u : \Psi \to \Phi \) which preserves the terminal object and fibers, and is such that for every \( I \in \Psi \) the natural map \( |I| \to |u(I)| \) is surjective.

Lemma. If \( u : \Psi \to \Phi \) is an operator morphism then \( |I| \to |u(I)| \) is bijective for every \( I \in \text{ob} \, \Psi \).

Proof. Let \( i, j : * \to I \) in \( \Psi \) and assume that \( ui = uj : * = u(*) \to u(I) \). Let \( J \) be the fiber product of \( i \) and \( j \). Since \( u \) preserves fibers, \( uJ \) is the fiber product of \( ui \) and \( uj \). But these morphisms are equal, so \( uJ = * \). Now \( |J| \to |u(J)| \) is surjective, by assumption, so \( |J| \neq \emptyset \). Let \( k : * \to J \). Then we have a commutative diagram

\[
\begin{array}{ccc}
* & \to & * \\
\downarrow & & \downarrow \\
* & \to & I \\
\end{array}
\]

and this implies that \( i = j \).

Examples. The category \( \text{Fin} \) of finite sets is an operator category. The set of points functor for \( \text{Fin} \) is the identity functor. The category \( \text{Fin}' \) of nonempty finite sets and surjections is also an operator category.

The category \( */\text{Fin} \) of pointed finite sets is an operator category, but less interesting: for any \( I \), there’s only one map \( * \to I \), and the fiber of \( f : J \to I \) over it is the pre-image of \( * \in I \) as a pointed set.

The category \( \text{Ord} \) of finite totally ordered sets is an operator category. The set of points is the underlying set. The category \( \text{Ord}' \) of nonempty finite totally ordered sets and surjections is also an operator category.

For any operator category \( \Phi \), there is a natural operator map \( \Phi \to \text{Fin} \) given by the functor \( |-| \). This morphism is initial in the category of morphisms from \( \Phi \) to \( \text{Fin} \): for any \( u : \Phi \to \text{Fin} \), the effect of \( u \) on \( \Phi(*,-) \) induces a natural surjection

\[
|I| = \Phi(*, I) \to \text{Fin}(*, u(I)) = u(I) .
\]

This is the only natural transformation from \( |-| \) to \( u \): the image of \( (x : * \to I) \in |I| \) under a natural transformation is \( u(x) : * \to u(I) \).
One might call a map an “interval inclusion” if it is isomorphic to the “inclusion” of a fiber. Since any map from a terminal object is a monomorphism, and any pullback of a monomorphism is a monomorphism, the word “inclusion” is justified: it is a monomorphism. Note that in an operator category pullbacks along inclusions of intervals exist: in

\[
\begin{array}{ccc}
K_i & \xrightarrow{g|K_i} & J_i \\
\downarrow & & \downarrow i \\
K & \xrightarrow{g} & J \\
\end{array}
\]

the map \(g|K_i\) exists uniquely since the right hand square is a pullback, and the left hand square is a pullback because the outer rectangle is a pullback.

Note that given \(f : J \to I\),

\[
|J| = \prod_{i \in |I|} |J_i|
\]

**Definition.** Given an operator category \(\Phi\) and a symmetric monoidal category \((C, \otimes, 1)\), a \(\Phi\)-monoid in \(C\) consists of an object \(M \in \text{ob } C\) together with a morphism \(\varphi_I : M^{\otimes|I|} \to M\) for every \(I \in \text{ob } \Phi\), such that \(\varphi_* : M \to M\) is the identity map and for every \(f : J \to I\) in \(\Phi\) the diagram

\[
\begin{array}{ccc}
\bigotimes_{i \in |I|} M^{\otimes|J_i|} & \xrightarrow{\varphi_J} & M^{\otimes|J|} \\
\downarrow & & \downarrow \varphi^J \\
M^{\otimes|I|} & \xrightarrow{\varphi_I} & M
\end{array}
\]

commutes.

**Perfection and the Leinster category**

I think Barwick is interested in producing a version of the simplicial bar construction for such monoids, with the idea of getting at an \(A_\infty\) version. In order to construct the “\(\Phi\)-simplicial category,” he needs a little more structure on \(\Phi\).

**Definition.** Let \(\Phi\) be an operator category. A universal point in \(\Phi\) is a pair \((T, o)\), where \(o : * \to T\) is a point of \(T\), such that given any \(i : * \to I\), there is a unique map \(\chi_i : I \to T\) (the “characteristic map”) whose fiber over \(o\) is \(i\).

For example \(\text{Fin}\) admits a universal point, namely \(\{0, 1\}\), with \(o(*) = 1\). Similarly, a universal point for \(\text{Ord}\) is given by \(\{-, 0, +\}\), with \(o(*) = 0\). But neither \(\text{Fin}'\) nor \(\text{Ord}'\) admit universal points.

The universal point is unique up to unique isomorphism. Suppose that \(o : * \to T\) and \(o : * \to U\) are both universal points. Then \(o \in |T|\) determines a map \(T \to U\) pulling \(o\) back to \(o\), and \(o \in |U|\) determines a map \(U \to T\) pulling \(o\) back to \(o\). The composite
$T \to T$ is the unique map pulling $o$ back to itself—so is the identity on $T$. Similarly the composite $U \to U$ is the identity on $U$.

**Definition.** An operator category $\Phi$ is *perfect* if it admits a universal point and the functor $\text{Fib} : \Phi/T \to \Phi$, assigning to $I \to T$ the fiber $I_o$, has a right adjoint.

Barwick writes the source of the right adjoint as $T(-)$. The structure map $T(*) \downarrow T$ is an isomorphism, since a map from $J \downarrow T$ to $T(*) \downarrow T$ is equivalent to giving a map $J_o \to \ast$, which carries no information, and this implies that the structure map of the object over $T$ which we are mapping into is an isomorphism. Thus for any $I$, the unique map $p_I : I \to \ast$ induces

$$
\begin{array}{ccc}
T(I) & \xrightarrow{T(p_I)} & T(*) \\
\downarrow & & \downarrow \cong \\
T & & T
\end{array}
$$

This shows that the canonical map $p_I : I \to \ast$ induces the structure map $T(I) \to T$. For this reason, only the source of the object in $\Phi/T$ needs a symbol.

It might be worthwhile writing out the adjunction in terms of the functor $T$. For any $f : J \to T(*)$ we have a factorization

$$
\begin{array}{ccc}
J & \xrightarrow{\alpha_f} & T(\text{Fib}f) \\
\downarrow f & & \downarrow \quad T_{\text{Fib}f} \\
T(*) & & T(*)
\end{array}
$$

which is natural, and for any $I$ we have a map $\text{Fib}(T p_I) \to I$, which is a natural isomorphism; and the composites

$$
T(I) \xrightarrow{T(p_I)} T(\text{Fib}(T p_I)) \xrightarrow{T_{\beta_I}} T(I)
$$

and

$$
\text{Fib}f \xrightarrow{\text{Fib}_{\beta_I}} \text{Fib}(T p_{\text{Fib}f}) \xrightarrow{\beta_{\text{Fib}f}} \text{Fib}f
$$

are the identity maps. The left and right hand factors are thus inverse isomorphisms.

**Fin** is perfect. $T(I) = \{1\} \coprod I$ mapping to $T = \{0, 1\}$ by sending 1 to 1 and all of $I$ to 0.

**Ord** is perfect. $T(I) = \{-\} \coprod I \coprod \{+\}$ mapping to $T = \{-, 0, +\}$ by sending $-$ to $-$, $+$ to $+$, and all of $I$ to 0.

**Lemma.** The natural map $I \to \text{Fib}(T(I) \downarrow T)$ is an isomorphism.

**Proof.** We must show that

$$
\begin{array}{ccc}
I & \longrightarrow & \text{Fib}T(I) \\
\downarrow & & \downarrow \\
* & \longrightarrow & T
\end{array}
$$
is a pullback. A compatible pair of maps from $J$ into the corners of the diagram is the same as a map in $\Phi/T$ from $J\xrightarrow{o} T$ to $T(I)\rightarrow T$, which, by adjointness, is the same as a map $\text{Fib}(J\xrightarrow{o} T)\rightarrow I$. Now the fiber of the composite $J\xrightarrow{o} T$ is just $J$, so we have verified the pullback property.

The functor $T: \Phi \rightarrow \Phi$ has a canonical triple structure. The unit in the triple is the composite in the top line of the diagram above. To construct the multiplication, note first that since $T$ is a right adjoint, it carries the pullback diagram

$$
\begin{array}{c}
I \\
\downarrow \eta \\
T(I) \\
\downarrow \\
T(*)
\end{array}
\xrightarrow{\eta} \begin{array}{c} * \\
\downarrow \\
* \end{array}
$$

to a pullback, which we embed in the diagram of pullbacks

$$
\begin{array}{c}
I \\
\downarrow \eta \\
T(I) \\
\downarrow \\
T(*)
\end{array}
\xrightarrow{\eta} \begin{array}{c} * \\
\downarrow o \\
T(*)
\end{array}
\xrightarrow{\mu K} \begin{array}{c} T(*) \\
\downarrow T(o) \\
T^2(I)
\end{array}
\xrightarrow{\mu K} \begin{array}{c} T(*) \\
\downarrow T(o) \\
T^2(*)
\end{array}
\xrightarrow{T^2(*)} \begin{array}{c} T(*) \\
\downarrow \\
T(*)
\end{array}
$$

where the map $T^2(*)\rightarrow T(*)$ classifies the point $T(o)\circ o$ in $T^2(*).$ The multiplication $\mu_I: T^2(I)\rightarrow T(I)$ is the map over $T(*)$ which is adjoint to the identity map to $I$ from the fiber of $T^2(I)\rightarrow T(*)$.

**Definition.** The Leinster category $\mathcal{L}(\Phi)$ of a perfect operator category $\Phi$ is the Kleisli category of the triple $T$.

Thus the objects of $\mathcal{L}(\Phi)$ are just the objects of $\Phi$, while $\mathcal{L}(\Phi)(I,J) = \Phi(I,T(J))$. The identity map of $I$ is $\eta_I : I \rightarrow T(I)$, and the composite of $f : I \rightarrow T(J)$ and $g : J \rightarrow T(K)$ is the composite

$$
I \xrightarrow{f} T(J) \xrightarrow{T(g)} T^2(K) \xrightarrow{\mu K} T(K)
$$

The Leinster category will be the opposite of the $\Phi$-analogue of the simplicial category.

For example, the Leinster category of $\text{Fin}$ has finite sets as objects, and maps from $I$ to $J$ given by maps from $I$ to $T(J)$, that is, $J$ with a point added. Adjoining a basepoint to $I$ and sending it to the basepoint of $J$ shows that $\mathcal{L}(\text{Fin})$ is the category of finite pointed sets.
Similarly, $\mathcal{L}(\text{Ord})$ is the category of finite totally ordered sets with distinct maxima and minima which are preserved by morphisms. This is isomorphic to the opposite of the simplicial category, $\Delta^\text{op}$.

I think that if $\Phi$ is a perfect operator category, there is a functor

$$\text{Mon}^\Phi(C) \to \text{Fun}(\mathcal{L}(\Phi), C)$$

whose essential image is described by a certain “Segal” condition.

**Wreath product**

An important construction is the “wreath product.” I suppose this is a special case of a Grothendieck construction. Let $\Phi$ and $\Psi$ be two operator categories. The wreath product $\Phi \wr \Psi$ has objects $(I : |J| \to \text{ob} \Phi, J \in \text{ob} \Psi)$. A morphism $(I', J') \to (I, J)$ consists of a morphism $g \in \Psi(J', J)$ and for each $j \in |J'|$ a morphism $f_j \in \Phi(I'(j), I(g \circ j))$.

A terminal object in $\Phi \wr \Psi$ is given by $(\ast, \ast)$, where the second $\ast$ denotes a terminal object of $\Psi$ and the first $\ast$ denotes the function $|\ast| \to \text{ob} \Phi$ sending $1 \in |\ast|$ to a terminal object in $\Phi$. A point in $(I, J)$ is thus a point $j$ in $J$ and a point $i$ in $I(j)$;

$$|(I, J)| = \bigsqcup_{j \in |J|} |I(j)|$$

Given $(f, g) : (I', J') \to (I, J)$, the fiber over a point $(i, j) \in |(I, J)|$ is given by $(j' \mapsto I'(j')_i, J'_j)$, where $I'(j')_i$ is the fiber of $g_{j'} : I'(j') \to I(j)$ over $i \in |I(j)|$, and $j' \in |J'|$, i.e. $j' : \ast \to J'$ such that $f \circ j' = j$.

The forgetful functor $\Phi \wr \Psi \to \Psi$ is an operator morphism: given $(I, J)$ and a point $j \in |J|$, any choice of $i \in |I(j)|$ provides a point in $(I, J)$ over $j$.

The functor $\Phi \to \Phi \wr \Psi$ sending $I$ to $(I : \ast \to \text{ob} \Phi, \ast)$ is an operator morphism.

If $(T_\Phi, o)$ and $(T_\Psi, o)$ are universal points in $\Phi$ and $\Psi$, then we can construct a universal point in the wreath product. It is given by $(F, T_\Psi)$, where $F : |T_\Psi| \to \text{ob} \Phi$ by sending $o$ to $T_\Phi$ and the other points to $o$.

The category $\text{Ord}'_n$ defined inductively by $\text{Ord}'_1 = \text{Ord}'$, the category of nonempty finite ordered sets, and $\text{Ord}'_{n+1} = \text{Ord}'_{n-1} \wr \text{Ord}'$ is isomorphic to Batanin’s category of “$n$-ordinals.” He gives the following description of that category. An $n$-ordinal is a level-tree of uniform height $n$ (so the levels are numbered 0 through $n$, level 0 is the root, and all branches grow up to level $n$) together with a total ordering of the leaves, with the property that if $a \leq b \leq c$ then $b$ is at least as closely related to $a$ and to $c$ as $a$ and $c$ are to each other. This says that we have a planar tree with the leaves numbered consecutively. You could also describe this by giving a nested sequence of order-preserving relations, given by a composable sequence of order preserving surjections. Morphisms may not be order preserving (though Batanin defines a map to be order preserving if it is a morphism). To describe them, think of the object as a phylogenetic tree, expressing how closely related the various leaves are. Then a morphism never increases the distance between leaves, and if it reverses the order of a pair of leaves then it makes them more closely related.
When \( n = 2 \), the isomorphism assigns to a 2-ordinal \( T \) the object \((I, J)\) of \( \text{Ord}' \wr \text{Ord}' \) with \( J \) given by the set of nodes of level 1 (with its given order) and \( I(a) \) the set of leaves above \( a \) (with the subset order). The inverse is given by sending \((I, J)\) to \( \prod_{j \in J} I(j) \), with the lexicographic order, and tree structure putting \( J \) at level 1.

One can also say that an object of \( \text{Ord}_2' \) is a nonempty totally ordered set together with an order-preserving relation on it; and that a morphism \((J, \sim) \to (I, \sim)\) is a function \( f : J \to I \) which respects the equivalence relations, is order preserving on each equivalence class, and such that if \( f \) reverses the order of \( j \) and \( j' \) then \( f(j) \sim f(j') \).

If we add the empty set back in, we can identify the category \( \text{Ord}_2 \). Its objects are the arrows in \( \text{Ord} \). A map from \( p \downarrow q \to p' \downarrow q' \) is an order preserving map \( q \to q' \) together with a map \( p \to p' \) which is order preserving on fibers. These are trees which are not required to grow up to the top level.

Similarly, \( \text{Ord}_n \) is the category of \((n - 1)\)-chains in \( \text{Ord} \),

\[ p_0 \xrightarrow{\alpha_1} p_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} p_{n-1} \]

and a morphism is a sequence of maps of underlying sets each of which is order preserving on the fibers of the \( \alpha \)'s.

**Operads**

**Definition.** Given an operator category \( \Phi \) and a symmetric monoidal category \((C, \otimes, 1)\), a \( \Phi \)-operad in \( C \) consists in

1. An object \( P(I) \in C \) for every \( I \in \Phi \),
2. A map \( \eta : 1 \to P(*) \), and
3. A map \( \theta_f : P(I) \otimes P(f) \to P(J) \) for every morphism \( f : J \to I \) in \( \Phi \),

where

\[ P(f) = \bigotimes_{i \in |I|} P(J_i), \]

such that for every \( I, J \in \text{ob} \Phi \) the diagrams

\[
\begin{align*}
1 \otimes P(I) & \xrightarrow{\eta \otimes 1} P(*) \otimes P(p_I) \xrightarrow{\theta_{p_I}} P(I) \\
1 \otimes P(J) \otimes 1 & \xrightarrow{1 \otimes \eta \otimes 1} P(J) \otimes P(1_J) \xrightarrow{\theta_{1_J}} P(J)
\end{align*}
\]

commutes, where \( p_I : I \to * \) is the unique map to the terminal object (so that \( P(I) = P(p_I) \)) and

\[
\eta_J : 1 = \bigotimes_{j \in |J|} 1 \xrightarrow{\otimes \eta} \bigotimes_{j \in |J|} P(*) = P(1_J),
\]
and for every \( f : J \to I \) and \( g : K \to J \) the diagram

\[
\begin{array}{c}
P(I) \otimes P(f) \otimes P(g) \\
\downarrow 1 \otimes \mu_{f,g} \\
P(I) \otimes P(fg)
\end{array} \rightarrow \begin{array}{c}
P(J) \otimes P(g) \\
\downarrow \theta_g \\
P(K)
\end{array}
\]

commutes, where \( \mu_{f,g} \) is the composite

\[
\left( \bigotimes_{i \in |I|} P(J_i) \right) \otimes \left( \bigotimes_{j \in |J|} P(K_j) \right) \cong \bigotimes_{i \in |I|} \left( P(J_i) \otimes \bigotimes_{j \in |J_i|} P(K_j) \right)
\]

\[
= \bigotimes_{i \in |I|} (P(J_i) \otimes P(f|K_i)) \otimes_{\theta_{f|K_i}} \bigotimes_{i \in |I|} P(K_i)
\]

A morphism of \( \Phi \) operads, \( \alpha : P \to P' \), is a collection of maps \( \alpha_I : P(I) \to P'(I) \) for every \( I \in \text{ob} \Phi \) which commute with the structure maps.

**Example.** The one-morphism category is an operator category. A \( * \)-operad in \( C \) is a \( \otimes \)-monoid in \( C \).

**Example.** Let \( \text{Fin} \) be the operator category of finite sets. If \( f : I \to I \) is a permutation then \( P(f) \cong \bigotimes_{i \in I} P(*) \) is endowed with a canonical map from \( 1 \), and composition with the structure map \( P(I) \otimes P(f) \to P(I) \) yields an automorphism of \( P(I) \). Thus a \( \text{Fin} \) operad determines a symmetric sequence, and the rest of the \( \text{Fin} \) operad structure determines on \( P \) the structure of an operad in the traditional sense.

**Example.** Let \( \text{Ord} \) be the category of finite ordered sets. Now the intervals are precisely intervals in the usual sense. An \( \text{Ord} \) operad is a “non-symmetric” operad.

**Sequences.** A \( \Phi \) operad doesn’t determine a functor from \( \Phi \) to \( C \), but it does determine a contravariant functor \( \text{Seq} \Phi \to C \), where \( \text{Seq} \Phi \) is subcategory of \( \Phi \) consisting of morphisms all of whose fibers are points. Pulling back along such a morphism \( f : J \to I \) induces a map \( |I| \to |J| \) which is inverse to \( |f| \): so the functor of points takes quasi-isomorphisms to bijections. The subcategory \( \text{Seq} \Phi \) is the analogue of the category of sets and bijections in the traditional development of the theory of operads, so a functor from it to \( C \) is a “\( \Phi \) sequence.”

Intervals pull back (to intervals). Using the fact that fiber inclusions are monomorphisms, it is easy to show that if \( \phi : J \to I \) is a quasi-isomorphism and \( F \to I \) is a fiber inclusion, then the map \( \phi^{-1} F \to F \) is again a quasi-isomorphism.

Let \( \phi : J \to I \) be a quasi-isomorphism and \( P \) a \( \Phi \) operad. Let \( f : K \to J \). For any \( i \in |I| \), let \( j \in |J| \) be the unique point such that \( \phi j = i \). Then there is a canonical isomorphism of fibers \( K_j \to K_i \). Write \( g = \phi f \). Combined with the functoriality \( \phi^* : \)
$P(I) \to P(J)$, we get a map along the top of the diagram

$$
P(I) \otimes \bigotimes_{i \in |I|} P(K_i) \to P(J) \otimes \bigotimes_{j \in |J|} P(K_j)
$$

which commutes by the associativity diagram of the operad.

This suggests that we try to define a monoidal structure on $\Phi$ sequences by

$$(P \circ Q)(K) = \colim_{\phi: K \to J} P(J) \otimes \bigotimes_{j \in |J|} Q(K_j)$$

where the colimit is taken over the category $K/\text{Seq } \Phi$.

Note that if $\Phi = \text{Fin}$, we get

$$(P \circ Q)(0) = \coprod_{n \geq 0} P(n) \otimes_{\Sigma_n} Q(0)^\otimes_n$$

In a $\text{Fin}$ operad, $P(0)$ is a $P$-algebra. If we assume that $Q(0) = o$, then the terms with $\phi$ not surjective don’t contribute. If $\phi$ is surjective, then there are no automorphisms of it in $K/\text{Seq } \Phi$, so

$$(P \circ Q)(K) = \coprod_{\sim} P(K/ \sim) \otimes \bigotimes_{j \in K/\sim} Q(j)$$

where the coproduct runs over equivalence relations on $K$.

**Pulling back operads.** Let $u : \Psi \to \Phi$ be an operator morphism, and $P$ a $\Phi$-operad in $C$. The pullback of $P$ along $u$ is the $\Psi$-operad $u^*P$ in $C$ with

$$(u^*P)(K) = P(u(K))$$

and structure maps given as follows. Let $f : J \to I$ in $\Psi$. For each $i \in |I|$, $(uf)^{-1}(ui) = u(f^{-1}(i))$ since $u$ respects fibers, and $u$ induces a bijection $|I| \to |u(I)|$ (by the Lemma). Thus the structure map

$$\theta_{uf} : P(uI) \otimes \bigotimes_{k \in |uI|} P((uf)^{-1}(k)) \to P(uJ)$$

precisely determines a map

$$\theta_f : (u^*P)(I) \otimes \bigotimes_{i \in |I|} (u^*P)(f^{-1}(i)) \to (u^*P)(J)$$

Also, $\eta : 1 \to P(*)$ is the same as a map $\eta : 1 \to (u^*P)(*)$, since $u^* = *$. These define the structure of a $\Psi$-operad.
For example, suppose \( u : \Psi \to \text{Fin} \) is given by \( uI = |I| \). Then a \( \text{Fin} \)-operad \( P \)—that is, an operad in the usual sense—gives rise to a \( \Psi \)-operad \( u^*P \) given by \( u^*P(I) = P(|I|) \), and structure map given by

\[
P(|I|) \otimes \bigotimes_{i \in |I|} P(|f^{-1}(i)|) \to P(|J|),
\]

using \( |f^{-1}(i)| = |f|^{-1}(i) \).

**Algebras over operads**

**Definition.** Let \( P \) be a \( \Phi \)-operad in \( C \). A \( P \)-**algebra** is an object \( A \) in \( C \) together with a map

\[
\varphi_I : P(I) \otimes A^{|I|} \to A
\]

for each \( I \in \text{ob } \Phi \) such that the diagram

\[
\begin{array}{ccc}
1 \otimes A & \xrightarrow{\eta \otimes 1} & P(*) \otimes A \\
& \searrow & \downarrow \varphi^* \\
& & A
\end{array}
\]

commutes and for each \( f : J \to I \) in \( \Phi \) the diagram

\[
\begin{array}{ccc}
P(I) \otimes \bigotimes_{i \in |I|} P(J_i) \otimes A^{|J_i|} & \xrightarrow{1 \otimes \otimes \varphi_j} & P(I) \otimes A^{|I|} \\
& \searrow & \\
& \left( P(I) \otimes \bigotimes_{i \in |I|} P(J_i) \right) \otimes A^{|J|} & \xrightarrow{\varphi_I} & A \\
& \downarrow \theta_f \otimes 1 & \\
& P(J) \otimes A^{|J|} & \xrightarrow{\varphi_J} & A
\end{array}
\]

commutes.

**Definition.** Let \( \Phi \) be an operator category, \( C \) a closed symmetric monoidal category, and \( P \) a \( \Phi \) operad in \( C \). The May-Thomason construction provides us with a category enriched over \( C \), with the same object set as \( \Phi \). Its object of morphisms from \( J \) to \( I \) is

\[
\bigotimes_{f : J \to I} P(J_i)
\]
and the composition is given by the structure map for the operad. This is perhaps better viewed as a “Φ-graded” category.

**Example.** Suppose $C$ contains a zero object $\emptyset$. Define a Φ operad $Z$ in $C$ by declaring that $Z(*) = 1$ and $Z(I) = \emptyset$ if $I \not\cong \ast$. An algebra for this operad is precisely a Φ-sequence. This is because of the

**Lemma.** $f : J \to I$ is a quasi-isomorphism if and only if $J_i = \ast$ for every $i \in |I|$.

**Multicategories and modules over them**

**Definition.** Let $\Phi$ be an operator category and $C$ a symmetric monoidal category. A $\Phi$ multicategory $H$ enriched over $C$ consists in the data of:

1. A set $S$ of “objects” of $H$;
2. For each $I \in \text{ob } \Phi$, object $x \in S$, and map $y : |I| \to S$, an object $H_I(y, x) \in C$;
3. For each $x \in S$, a map $\eta : 1 \to H_\ast(x, x)$;
4. For each morphism $f : J \to I$ in $\Phi$, $w \in S$, $x : |I| \to S$, and $y : |J| \to S$, a map $\theta_f : H_I(x, w) \otimes H_f(y, x) \to H_J(y, w)$

where

$$H_f(y, x) = \bigotimes_{i \in |I|} H_{j_i}(y_{|j_i|}, x_i)$$

To give the axioms, note that if $p_J : J \to \ast$ then $H_{p_J}(y, x) = H_J(y, x)$ and

$$H_{1_J}(y, x) = \bigotimes_{j \in |J|} H_\ast(y_j, x_j)$$

Define $\mu_{f,g} : H_f(y, x) \otimes H_g(z, y) \to H_{fg}(z, x)$ as the composite

$$\left( \bigotimes_{i \in |I|} H_{j_i}(y_{|j_i|}, x_i) \right) \otimes \left( \bigotimes_{j \in |J|} H_{K_j}(z_{|K_j|}, y_j) \right) \cong \bigotimes_{i \in |I|} \left( H_{j_i}(y_{|j_i|}, x_i) \otimes \bigotimes_{j \in |J_i|} H_{K_j}(z_{|K_j|}, y_j) \right)$$

$$= \bigotimes_{i \in |I|} \left( H_{j_i}(y_{|j_i|}, x_i) \otimes H_{g_i}(z_{|K_i|}, y_{|K_i|}) \right) \otimes \theta_{g_i} \to \bigotimes_{i \in |I|} H_{K_i}(z_{|K_i|}, x_i)$$

where $g_i$ is the unique map making

$$K_i \xrightarrow{g_i} J_i$$

$$K \xrightarrow{g} J$$

commutative. We require that

$$\begin{align*}
1 \otimes H_K(z, y) & \xrightarrow{\eta_J \otimes 1} H_\ast(y, y) \otimes H_{p_K}(z, y) \xrightarrow{\mu_{1,p}} H_K(z, y) \\
H_f(y, x) \otimes 1 & \xrightarrow{1 \otimes \eta_f} H_f(y, x) \otimes H_{1_J}(y, y) \xrightarrow{\mu_{f,1}} H_f(y, x)
\end{align*}$$
and

\[ H_I(x, w) \otimes H_f(y, x) \otimes H_g(z, y) \xrightarrow{\theta_f \otimes 1} H_f(y, w) \otimes H_g(z, y) \]
\[ \xrightarrow{1 \otimes \mu_{f,g}} H_I(x, w) \otimes H_g(z, x) \xrightarrow{\theta_g} H_K(z, w) \]

commute.

For example, if \( S = \{ \ast \} \), then we have a \( \Phi \)-operad in \( C \). If \( \Phi = \ast \), then we have a category enriched over \( C \) with object set \( S \).

Given any \( \Phi \) multicategory in \( C \) with object set \( S \), and any \( x \in S \), we have the “endomorphism operad” \( \text{End}_x \) with

\[ \text{End}_x(I) = H_I(x|I|, x) \]

where \( x|I| \) denotes the constant function from \( |I| \) with value \( x \).

**Definition.** Let \( H \) be a \( \Phi \) multicategory enriched over \( C \). A *module* for \( H \) consists in

1. a function \( M : S \to \text{ob } C \), and
2. for each \( J \in \text{ob } \Phi \) and each \( y : |J| \to S \) and \( x \in S \), a morphism

\[ \varphi : H_J(y, x) \otimes \bigotimes_{j \in |J|} M(y_j) \to M(x) \]

such that the diagrams

\[
\begin{array}{ccc}
1 \otimes M(x) & \xrightarrow{\eta \otimes 1} & M(x) \\
\downarrow & = & \downarrow \\
H_{1x}(x, x) \otimes M(x) & \xrightarrow{\varphi} & M(x)
\end{array}
\]
and

\[
H_I(x, w) \otimes \bigotimes_{i \in |I|} \left( H_{J_i}(y|_{J_i}, x_i) \otimes \bigotimes_{j \in |J_i|} M(y_j) \right) \xrightarrow{1 \otimes \varphi} H_I(x, w) \otimes \bigotimes_{i \in |I|} M(x_i)
\]

\[
= \bigotimes_{i \in |I|} H_{J_i}(y|_{J_i}, x_i) \otimes \bigotimes_{j \in |J|} M(y_j) \xrightarrow{\theta \otimes 1} H_J(y, w) \otimes \bigotimes_{j \in |J|} M(y_j) \xrightarrow{\varphi} M(w)
\]

commute.

For example, if \( S = \{\ast\} \) this is the notion of an algebra over the operad.

**Definition.** Suppose that \( C \) has an initial object \( \emptyset \) such that \( \emptyset \otimes X \cong \emptyset \) for any \( X \).

Fix a universal point \( o : \ast \to T \) in an operator category \( \Phi \). A \( \Phi \)-chirality in \( C \) is a \( \Phi \)-multicategory \( H \) in \( C \) with object set \( |T| \) such that:

1. For any \( I \in \text{ob } \Phi \), \( H_I(x, o) = \emptyset \) unless \( x = |\chi_i| : |I| \to |T| \) for some \( i \in |I| \).
2. If \( y \neq o \) in \( |T| \) then for any \( I \in \text{ob } \Phi \), \( H_I(x, y) = \emptyset \) unless \( x : |I| \to |T| \) is the constant function with value \( y \).

This is actually what Barwick calls a “pure chirality.” In the general definition, the various different “generic” points of \( |T| \) are allowed to interact with each other: (2) is replaced by the requirement that \( o \notin \text{im } x \). I hope that pure chiralities will suffice.

First let’s study the objects \( H_I(x, y) \).

By condition (2), when \( y \) is generic—i.e. \( y \neq o \)—the only nontrivial \( H_I(x, y) \)'s are the objects making up the endomorphism operad \( P_y \) of \( y \),

\[
P_y(I) = H_I(y|_I, y).
\]

The other objects determined by a chirality are, for \( J \in \text{ob } \Phi \) and \( j \in |J| \),

\[
M(J, j) = H_J(|\chi_j|, o).
\]

All the other \( H_J(x, o) \)'s are trivial.

To analyze the endomorphism operad \( P_o \) we use:

**Lemma.** Let \( J \in \text{ob } \Phi \) and \( j \in |J| \). The only element of \( |J| \) mapped to \( o \) by \( \chi_j \) is \( j \) itself.
Proof. Let \( f : J \to I \), and suppose that \( f \) pulls \( i : * \to I \) back to \( j : * \to J \). Since \(|-|\) preserves pullbacks, the inverse image of \( i \in |I| \) under \(|f|\) is \( \{j\} \subseteq |J| \). Apply this with \( J = T \).

Thus if \( \chi_j \) is a constant map then \(|J| = \{j\} \). Conditions (1) and (2) thus combine to show that the objects \( H_f(o|I|, o) \) are trivial unless \( I = * \). The endomorphism operad of \( o \) thus reduces to the monoid

\[
H_s(o, o) = P_o(*) = M(*, \text{id}_*)
\]

which we denote by \( A \).

We analyze what data is given by the structure maps

\[
\eta : 1 \to H_s(x, x)
\]

for \( x \in |T| \) and

\[
\theta_f : H_f(x, w) \otimes H_f(y, x) \to H_f(y, w)
\]

for \( f : J \to I \), \( w \in |T| \), \( x : |I| \to |T| \), and \( y : |J| \to |T| \).

When \( w = o \) and \( I = * \), the map is trivial unless \( x = o \) and the structure map gives us

\[
H_s(o, o) \otimes H_J(\chi_j, o) \to H_J(\chi_j, o)
\]

that is, an action by the monoid \( A \) on \( M(J, j) \).

I’m finding it hard to interpret what the other structure morphisms give you. It seems to me that the following lemma should be useful.

Lemma. Let \( \Phi \) be an operator category with a universal point \( o : * \to T \), let \( f : J \to I \) be a monomorphism (for example a fiber inclusion), and let \( j \in |J| \). Then

\[
\chi_{fj} \circ f = \chi_j : J \to T
\]

Proof. Let \( g : K \to J \) be such that \((\chi_{fj}f)g = op_K\). We want to show that \( g = jp_K \). From the definition of \( \chi_{fj} \), \( \chi_{fj}(fg) = op_K \) implies that \( fg = (fj)p_K \), so what we want follows from associativity of composition and monomorphicity of \( f \).

If I continue to take \( w = o \), and expect to get something nontrivial in the source of \( \theta_f \), I had better take \( x = |\chi_i| \) for some \( i \in |I| \) and also \( y = |\chi_j| \) for some \( j \in |J| \), so the map is

\[
\theta_f : M(I, i) \otimes H_f(|\chi_j|, |\chi_i|) \to M(J, j)
\]

Now

\[
H_f(|\chi_j|, |\chi_i|) = \bigotimes_{k \in |I|} H_{J_k}(|\chi_j||J_k|, \chi_i k)
\]

We have to understand some things about the values of the characteristic functions of points. First, \( \chi_k k = o \) if and only if \( k = i \). So that tensor factor is a special case. The factor is nontrivial if and only if \( |\chi_j||J_k| = |\chi_l| \) for some \( l \in |J_i| \). If this occurs, then \( \chi_j l = o \); but the only element carried to \( o \) by \( \chi_j \) is \( j \), so \( l = j \). Conversely, by the lemma, if there is \( l \in |J_i| \) such that \( j = gl \) (where \( g : J_i \to J \) is the interval inclusion) then \( \chi_j \circ g = \chi_l \), and it follows that \( |\chi_j||J_i| = |\chi_l| \).
The upshot is that the $k = i$ factor $H_f([\chi_i]|_{|I|}, o)$ is nontrivial only if $i = fj$. When
this factor is trivial, it kills the entire object $H_f([\chi_i], |I|)$: so $\theta_f$ is interesting only when
$i = fj$. In that case, the $k = i$ factor is $H_{J_fj}([\chi_j], o) = M(J_fj, j)$, so the structure
morphism has the form

$$\theta_f : M(I, fj) \otimes M(J_{fj}, j) \otimes \otimes_k P_{\chi_fj,k}(J_k) \rightarrow M(J, j)$$

as long as all points $k$ of $I$ other than $fj$ have the property that $|\chi_j|$ is constant on $J_k$
with value $\chi_fj,k$. If this condition fails for any value of $k$, then the source is $\emptyset$ and the
structure map has no content.

So the question is: Let $f : J \rightarrow I$, $k \in |I|$, and $j \in |J|$. Is $|\chi_j|$ constant on $J_k$
with value $\chi_fj,k$? This is true if $f$ is a monomorphism. If $k = fj$ then $J_k = \ast$, mapping to $J$
by $j$, so it is true then too.

But it certainly isn’t true in general. Suppose $I = \ast$ for example. Then $J_k = J$, and
$\chi_j$ is certainly not constant (as long as $J$ has more than one point). In this case $fj = k$,
so one definitely needs to assume that $fj \neq k$ for the hypothesis to stand a chance.

Clark mentions that there are (pathological) perfect operator categories for which $T$
contains intervals which don’t contain $o$ but do contain more than one point. Perhaps
$\chi_j$ is always interval-preserving; if so, then Clark’s hypothesis would imply that $\chi_j$ is
constant on intervals not containing $j$.

If $w \neq o$, then for $H_f(x, w)$ to be nontrivial we must have $x = w|_{|I|}$, and for $H_f(y, w)$
to be nontrivial we must have $y = w|_{|J|}$. So now we want to understand

$$H_f(w|_{|J|}, w|_{|I|}) = \bigotimes_{i \in |I|} H_{J_i}(w|_{J_i}, w) = \bigotimes_{i \in |I|} P_w(J_i)$$

so the structure map is

$$P_w(I) \otimes \bigotimes_{i \in |I|} P_w(J_i) \rightarrow P_w(J)$$

which is just the operad structure on $P_w$.

Notes on points: Suppose $f : I \rightarrow \ast$ is a monomorphism. If $i \in |I|$, then the composite
$fi$ is the identity (since it’s a self-map of the terminal object), and the composite $if$ is
too (since $f \circ if = fi \circ f = f = f \circ id_I$, and $f$ is a monomorphism). Thus the proper
subobjects of $\ast$ have no points. A pointless object doesn’t have to be initial, in interesting
cases. For example, in $\mathbf{Ord}_2$ we have the object $(J, I)$, where $I = \{1\}$ and $J(1) = \emptyset$.

In $\mathbf{Ord} \cup \mathbf{Ord}$, we claim that the universal point is the central point in the object with
base 3 and fibers $1, 3, 1$. But suppose I want to classify 1 in the object with base 2 and
both fibers 1. It seems to me that there are three maps which pull $o$ back to 1: send 1
to $o$, and send 2 to either of the other points in the middle fiber or to the unique point
in the right fiber.

The fallacy is that when 2 gets sent to the wrong point, the pullback is not the
terminal object $\square_1 \square 1$ but rather $\square_1 \square 2$. 

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Given a \( \Phi \)-operad \( P \) there will generally be many chiralities with \( P_x = P \) for all \( x \), but there is a preferred one. It has

\[
M(J, j) = P(J)
\]

for all \( J \in \text{ob} \Phi \) and all \( j \in |J| \). The structure map corresponding to \( f : J \to I, j \in |J| \) is

\[
\theta_f : P(I) \otimes \bigotimes_{k \in |I|} P(J_k) \to P(J)
\]

as long as all \( k \in |I| \) other than \( k = f j \) have the property that \( |\chi_j| = (\chi_{fj} k)_{|J_k|} \). This is a piece of the operad structure of \( P \) (and if the condition holds for all \( k \), it is precisely the operad structure map).