1 Review of our strategy

Our goal is to prove

Main Theorem. The Arf-Kervaire elements \( \theta_j \in \pi_{2j+1-2}(S^0) \) do not exist for \( j \geq 7 \).

Our strategy is to find a map \( S^0 \to M \) to a nonconnective spectrum \( M \) with the following properties.

(i) It has an Adams-Novikov spectral sequence in which the image of each \( \theta_j \) is nontrivial. This is the Detection Theorem discussed by Hopkins here on July 8.
(ii) \( \pi_{-2}(M) = 0 \). This is the Gap Theorem discussed by Hill here on July 15.
(iii) It is 256-periodic, meaning \( \Sigma^{256}M \cong M \). This is the Periodicity Theorem.

Our strategy (continued)

(ii) and (iii) imply that \( \pi_{254}(M) = 0 \).

If \( \theta_7 \) exists, (i) implies it has a nontrivial image in this group, so it cannot exist.

The argument for \( \theta_j \) for larger \( j \) is similar, since \( |\theta_j| = 2^{j+1} - 2 \equiv -2 \mod 256 \) for \( j \geq 7 \).

2 The spectrum \( M \)

The spectrum \( M \)

As explained previously, there is an action of the cyclic group \( C_8 \) on the 4-fold smash product \( MU^{(4)} \). It is derived using a norm induction from the action of \( C_2 \) on \( MU \) by complex conjugation.

We show that its homotopy fixed point set \( (MU^{(4)})^{hC_8} \) and its actual fixed point set \( (MU^{(4)})^{C_8} \) are equivalent. It is an \( E_\infty \)-ring spectrum, and \( M \) is obtained from it by inverting an element \( D \in \pi_{256} \) which we will identify below.

The homotopy of \( (MU^{(4)})^{hC_8} \) can be computed using the homotopy fixed point spectral sequence, for which

\[
E_2 = H^*(C_8; \pi_*(MU^{(4)}))
\]

In this case it concides with the Adams-Novikov spectral sequence for \( \pi_*(MU^{(4)})^{hC_8} \). The algebraic methods described by Hopkins can be used to show that it detects the \( \theta_j \)s. \( D \) has to be chosen so that this is still true after we invert it.
The spectrum $M$ (continued)

The homotopy of $(MU^{(4)})^G$ and $M = D^{-1}(MU^{(4)})^G$ can be also computed using the slice spectral sequence described by Hill. It has the convenient property that $\pi_{-2}$ vanishes in the $E_2$-term. In fact $\pi_k$ vanishes for $-4 < k < 0$.

This is our main motivation for developing the slice spectral sequence. We do not know how to show this vanishing using the other spectral sequence.

In order to identify $D$ we need to study the slice spectral sequence in more detail.

3 The slice spectral sequence

The slice spectral sequence

Recall that for $G = C_8$ we have a slice tower

$$
\cdots \longrightarrow P_{G}^{n+1}MU^{(4)} \longrightarrow P_{G}^{n}MU^{(4)} \longrightarrow P_{G}^{n-1}MU^{(4)} \longrightarrow \cdots
$$

with

- the inverse limit is $MU^{(4)}$,
- the direct limit is contractible and
- $G P_{n}^{n}MU^{(4)}$ is the fiber of the map $P_{G}^{n}MU^{(4)} \rightarrow P_{G}^{n-1}MU^{(4)}$.

$G P_{n}^{n}MU^{(4)}$ is the $n$th slice and the decreasing sequence of subgroups of $\pi_{*}(MU^{(4)})$ is the slice filtration. We also get slice filtrations of the $RO(G)$-graded homotopy $\pi_{*}(MU^{(4)})$ and the homotopy groups of fixed point sets $\pi_{*}((MU^{(4)})^H)$ for each subgroup $H$.

The slice spectral sequence (continued)

This means the slice filtration leads to a slice spectral sequence converging to $\pi_{*}(MU^{(4)})$ and its variants.

One variant has the form

$$
E_{2}^{s,t} = \pi_{t-s}(G P_{s}^{s}MU^{(4)}) \Longrightarrow \pi_{t-s}^{G}(MU^{(4)}).
$$

Recall that $\pi_{*}^{G}(MU^{(4)})$ is by definition $\pi_{*}((MU^{(4)})^G)$, the homotopy of the fixed point set.

Slice Theorem. In the slice tower for $MU^{(4)}$, every odd slice is contractible and $P_{2n}^{2n} = \hat{W}_n \wedge HZ$, where $HZ$ is the integer Eilenberg-Mac Lane spectrum and $\hat{W}_n$ is a certain wedge of the following three types of finite $G$-spectra:

- $S^{(n/4)p_8}$, where $p_8$ denotes the regular real representation of $C_8$,
- $C_8 \wedge C_4 S^{(n/2)p_4}$ and
- $C_8 \wedge C_2 S^{np_2}$.

The same holds after we invert $D$, in which case negative values of $n$ can occur.

3.1 Slices of the form $S^{mp_h} \wedge HZ$

Here is a picture of some slices $S^{mp_h} \wedge HZ$.

Slices of the form $S^{mp_h} \wedge HZ$ (continued)

- Note that all elements are in the first and third quadrants between certain black lines with slopes 0 and orchid lines with slope 7, and are concentrated on diagonals where $r$ is divisible by 8.
- Bullets, circles and diamonds indicate cyclic groups of order 2, 4 and 8, and boxes indicate copies of the integers.
- A similar picture for $S^{mp_4} \wedge HZ$ would be confined to the regions between the black lines and blue lines with slope 3 and concentrated on diagonals where $t$ is divisible by 4.
- A similar picture for $S^{mp_2} \wedge HZ$ would be confined to the regions between the black lines and green lines with slope 1 and concentrated on diagonals where $r$ is divisible by 2.

3.2 Implications for the slice spectral sequence

These calculations imply the following.

- The slice spectral sequence for $MU^{(4)}$ is concentrated in the first quadrant and confined by the same vanishing lines.
- Later we will invert elements in $\pi_{mp_8}(MU^{(4)})$. The fact that
  \[ S^{-mp_8} \wedge (C_8 \wedge H S^{mp_8}) = C_8 \wedge H S^{(m-8)/h}\rho_h \]
  means that the resulting slice spectral sequence is confined to the regions of the first and third quadrants shown in the picture.

4 Geometric fixed points

Geometric fixed points

In order to proceed further, we need another concept from equivariant stable homotopy theory.

Unstably a $G$-space $X$ has a fixed point set,

$$X^G = \{ x \in X : \gamma(x) = x \forall \gamma \in G \}.$$
This is the same as \( F(S^0,X_+)^G \), the space of based equivariant maps \( S^0 \to X_+ \), which is the same as the space of unbased equivariant maps \(* \to X\).

The homotopy fixed point set \( X^hG \) is the space of based equivariant maps \( EG_+ \to X_+ \), where \( EG \) is a contractible free \( G \)-space. The equivariant homotopy type of \( X^hG \) is independent of the choice of \( EG \).

**Geometric fixed points (continued)**

Both of these definitions have stable analogs, but the fixed point functor is awkward for two reasons:

- it fails to commute with smash products and
- it fails to commute with infinite suspensions.

The geometric fixed set \( \Phi^G X \) is a convenient substitute that avoids these difficulties. In order to define it we need the isotropy separation sequence, which in the case of a finite cyclic 2-group \( G \) is

\[
EC_{2+} \to S^0 \to \hat{EC}_2.
\]

Here \( EZ/2 \) is a \( G \)-space via the projection \( G \to Z/2 \) and \( S^0 \) has the trivial action, so \( \hat{EC}_2 \) is also a \( G \)-space.

**Geometric fixed points (continued)**

Under this action \( EC^G_2 \) is empty while for any proper subgroup \( H \) of \( G \), \( EC^H_2 = EC_2 \), which is contractible. For an arbitrary finite group \( G \) it is possible to construct a \( G \)-space with the similar properties.

**Definition.** For a finite cyclic 2-group \( G \) and \( G \)-spectrum \( X \), the geometric fixed point spectrum is

\[
\Phi^G X = (X \wedge \hat{EC}_2)^G.
\]

**Geometric fixed points (continued)**

This functor has the following properties:

- For \( G \)-spectra \( X \) and \( Y \), \( \Phi^G(X \wedge Y) = \Phi^G X \wedge \Phi^G Y \).
- For a \( G \)-space \( X \), \( \Phi^G \Sigma^\infty X = \Sigma^\infty(X^G) \).
- A map \( f : X \to Y \) is a \( G \)-equivalence iff \( \Phi^H f \) is an ordinary equivalence for each subgroup \( H < G \).

From the suspension property we can deduce that

\[
\Phi^G(ZU(4)) = MO,
\]

the unoriented cobordism spectrum.

**Geometric Fixed Point Theorem.** Let \( \sigma \) denote the sign representation. Then for any \( G \)-spectrum \( X \), \( \pi_*(EC_2 \wedge X) = a_\sigma^{-1} \pi_*(X) \), where \( a_\sigma : S^0 \to S^\sigma \) is the element defined in Hill’s lecture.

**Geometric fixed points (continued)**

Recall that \( \pi_*(MO) = Z/2[|y_i| > 0, i \neq 2^k - 1] \) where \( |y_i| = i \). It is not hard to show that

\[
\pi_*(ZU(4)) = Z[r_i, \gamma(r_i), \gamma^2(r_i), \gamma^3(r_i) : i > 0]
\]

where \( |r_i| = 2i \), \( \gamma \) is a generator of \( G \) and \( \gamma^4(r_i) = (-1)^i r_i \). In \( \pi_{|p_5|}(ZU(4)) \) we have the element

\[
N r_i = r_i \gamma(r_i) \gamma^2(r_i) \gamma^3(r_i).
\]

Applying the functor \( \Phi^G \) to the map \( N r_i : S^{p_5} \to ZU(4) \) gives a map \( S^i \to MO \).

**Lemma.** The generators \( r_i \) and \( y_i \) can be chosen so that

\[
\Phi^G N r_i = \begin{cases} 
0 & \text{for } i = 2^k - 1 \\
y_i & \text{otherwise}.
\end{cases}
\]
5 Some slice differentials

Some slice differentials

It follows from the above that the slice spectral sequence for $MU^{(4)}$ has a vanishing line of slope 7. We will describe the subring of elements lying on it.

Let $f_i \in \pi_i(MU^{(4)})$ be the composite

$$S^i \xrightarrow{a \rho_i} S^{\rho_i} \xrightarrow{N n} MU^{(4)}.$$

The following facts about $f_i$ are easy to prove.

- It appears in the slice spectral sequence in $E_2^{7,8}$, which is on the vanishing line.
- The subring of elements on the vanishing line is the polynomial algebra on the $f_i$.

Some slice differentials (continued)

- Under the map $\pi_i(MU^{(8/2)}) \to \pi_i(\Phi^G MU^{(8/2)}) = \pi_{g_i}(MO)$ we have

$$f_i \mapsto \begin{cases} 0 & \text{for } i = 2^k - 1 \\ y_i & \text{otherwise} \end{cases}$$

- Any differential landing on the vanishing line must have a target in the ideal $(f_1, f_3, f_7, \ldots)$. A similar statement can be made after smashing with $S^{2^k}\sigma$.

Some slice differentials (continued)

Recall that for an oriented representation $V$ there is a map $u_V : S^{|V|} \to \Sigma^V HZ$, which lies in $\pi_{|V|}(HZ)$.

Slice Differentials Theorem. In the slice spectral sequence for $\Sigma^{2^k}\sigma MU^{(4)}$ (for $k > 0$) we have $d_r(u_{2^k}\sigma) = 0$ for $r < 1 + 8(2^k - 1)$, and

$$d_{1+8(2^k-1)}(u_{2^k}\sigma) = a_{2^k}\sigma f_{2^k-1}.$$

Inverting $a_{2^k}\sigma$ in the slice spectral sequence will make it converge to $\pi_{g_i}(MO)$. This means each $f_{2^k-1}$ must be killed by some power of $a_{2^k}\sigma$. The only way this can happen is as indicated in the theorem.

Some slice differentials (continued)

Let

$$\overline{a}_{2^k}^{(8)} = Nr_{2^k-1} \in \pi_{(2^k-1)\rho_i}(MU^{(4)}).$$

We want to invert this element and study the resulting slice spectral sequence. As explained previously, it is confined to the first and third quadrants with vanishing lines of slopes 0 and 7.

The differential $d_r$ on $u_{2^k+1}\sigma$ described in the theorem is the last one possible since its target, $a_{2^k+1}\sigma f_{2^k+1-1}$, lies on the vanishing line. If we can show that this target is killed by an earlier differential after inverting $\overline{a}_{2^k}^{(8)}$, then $u_{2^k+1}\sigma$ will be a permanent cycle.
Some slice differentials (continued)

We have

\[ f^{2k+1}_{2k+1-1} \Delta^{(8)} = a^{(2k+1-1)} r_{2k+1-1} \Delta^{(8)} \]
\[ = a^{2k+1} \Delta^{(8)} \]
\[ = \Delta^{(8)} d_{r'}(u_{2\sigma}) \text{ for } r' < r. \]

**Corollary.** In the RO\((G)\)-graded slice spectral sequence for \((\Delta^{(8)}_{k+1})^{-1} \text{MU}^{(4)}\), the class \(u_{2\sigma}^{2} \) is a permanent cycle.

### 6 The proof of the Periodicity Theorem

**The proof of the Periodicity Theorem**

The corollary shows that inverting a certain element makes a power of \(u_{2\sigma}\) a permanent cycle. We need a similar statement about a power of \(u_{2\rho_{8}}\).

We will get this by using the norm property of \(u\), namely that if \(V\) is an oriented representation of a subgroup \(H \subset G\) with \(V^H = 0\) and induced representation \(V'\), then the norm functor \(N^G_H\) from \(H\)-spectra to \(G\)-spectra satisfies \(N^G_H(uV) = u(V')\).

From this we can deduce that \(u_{2\rho_{8}} = u_{8\sigma_{1}}N^{8}_4(u_{4\sigma_{2}})N^{8}_2(u_{2\sigma_{1}})\), where \(\sigma_{m}\) denotes the sign representation on \(C_{2m}\).

**The proof of the Periodicity Theorem (continued)**

We have \(u_{2\rho_{8}} = u_{8\sigma_{1}}N^{8}_4(u_{4\sigma_{2}})N^{8}_2(u_{2\sigma_{1}})\).

By the Corollary we can make a power of each factor a permanent cycle by inverting some \(\Delta^{(2m)}_{k_m}\) for \(1 \leq m \leq 3\). If we make \(k_m\) too small we will lose the detection property, that is we will get a spectrum that does not detect the \(\theta_j\). It turns out that \(k_m\) must be chosen so that \(8/2^m k_m\).

- Inverting \(\Delta^{(2)}_{k_{1}}\) makes \(u_{3\sigma_{1}}\) a permanent cycle.
- Inverting \(\Delta^{(4)}_{k_{2}}\) makes \(u_{8\sigma_{2}}\) a permanent cycle.
- Inverting \(\Delta^{(8)}_{k_{4}}\) makes \(u_{4\sigma_{4}}\) a permanent cycle.
- Inverting the product \(D\) of the norms of all three makes \(u_{32\rho_{8}}\) a permanent cycle.

**The proof of the Periodicity Theorem (continued)**

Let

\[ D = \Delta^{(8)}_{k_{1}} N^{4}_4(\Delta^{(4)}_{k_{2}}) N^{8}_2(\Delta^{(2)}_{k_{4}}) \]

The we define \(\tilde{M} = D^{-1} \text{MU}^{(4)}\) and \(M = \tilde{M} C_{8}\).

Since the inverted element is represented by a map from \(S^{m\rho_{8}}\), the slice spectral sequence for \(\pi_{*}(M)\) has the usual properties:

- It is concentrated in the first and third quadrants and confined by vanishing lines of slopes 0 and 7.
- It has the gap property, i.e., no homotopy between dimensions \(-4\) and 0.
The proof of the Periodicity Theorem (continued)

Preperiodicity Theorem. Let $\Delta_1^{(8)} = u_2 p_h (\Delta_1^{(8)})^2 \in E_2^{16,0} (D^{-1}MU^{(4)})$. Then $(\Delta_1^{(8)})^{16}$ is a permanent cycle.

To prove this, note that $(\Delta_1^{(8)})^{16} = u_3 2 p_h (\Delta_1^{(8)})^{32}$. Both $u_3 2 p_h$ and $\Delta_1^{(8)}$ are permanent cycles, so $(\Delta_1^{(8)})^{16}$ is also one.

Thus we have an equivariant map $\Sigma^{256} D^{-1}MU^{(4)} \rightarrow D^{-1}MU^{(4)}$ and a similar map on the fixed point set. The latter one is invertible because $u_3 2 p_h$ restricts to the identity.

Thus we have proved

Periodicity Theorem. Let $M = (D^{-1}MU^{(4)})^{C_8}$. Then $\Sigma^{256} M$ is equivalent to $M$. 