SOME REMARKS ON THE KERVAIRE INVARIANT PROBLEM FROM THE HOMOTOPY POINT OF VIEW

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The object of this note is to discuss some results which were obtained in an effort to settle the Kervaire invariant conjecture.

There is a secondary cohomology operation based on the Adem relation which expands $Sq^2Sq^2$. Call this $\varphi_{j,j}$ after Adams.

**Conjecture A.** There exists a two cell complex $S^a \cup e^{a+N+1} (N = 2^{j+1} - 2)$ with $\varphi_{j,j}$ nonzero. We call any such element $\theta_j$. (Note that this defines only a coset.)

Browder has shown that this conjecture is equivalent to the existence of a framed manifold with nonzero Kervaire invariant.

There are many statements which imply the conjecture. Most are conjectured to be equivalent. Let us begin with the simplest.

Suppose $X = S^0 \cup_{e^2} e^2$, that is, the space in the stable category which represents $\Sigma^{-1}RP^2$.

**Theorem 1.** An element of Hopf invariant 1 in $\Pi_{j,j-1}(X)$ implies A in dim $2^{j+1} - 2$.

**Proof.** An element of Hopf invariant one implies a three cell complex so that $Sq^{2^{j+1}} \neq 0$. Adams has shown that $Sq^{2^{j+1}} = \sum a_{i,k,l} \varphi_{i,k}$. We apply this to the Spanier-Whitehead dual of the complex and conclude $\varphi_{j,j} \neq 0$ and $a_{j,j,1} = Sq^{1}$.

**Corollary 2.** If $[e_1, e_2] = 2\alpha$, $N = 2^{j+1} - 1$, then $\alpha = \theta_j$.

We can use Theorem 1 to try and construct the $\theta_j$'s inductively. Indeed,

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1 These notes are based on the joint work of M. G. Barratt and M. E. Mahowald.

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suppose there is an element in dimension $2^j - 1$ of Hopf invariant 1. Let us call it $\{h_j\}$.

**Proposition 3.** \[ 2\{h_j\} = \eta \theta_{j-1} \]

This follows immediately from Theorem 1 and a standard relationship in the homotopy of $X$.

Thus $\{h_j\}$ admits an extension by 2 modulo $\eta$. This gives a map

\[ \eta \]

This clearly defines a map

\[ \eta \]

and a four cell space

\[ \alpha \]

with attaching maps $\alpha = \theta_{j-1}^2$ and $\beta = \langle \theta_{j-1}, 2t, \theta_{j-1} \rangle$ such that $\varphi_{j,j}$ is nonzero from the bottom to the top cell. Summarizing, we have shown:

**Theorem 4.** There is a null-homotopy defined on $S^{2^j}$ of

\[ \eta \theta_{j-1}^2 + 2t\langle \theta_{j-1}, 2t, \theta_{j-1} \rangle \]

which carries a $\varphi_{j,j}$.

**Corollary 5.** If there is a non-$\varphi_{j,j}$ carrying null-homotopy of

\[ \eta \theta_{j-1}^2 + 2t\langle \theta_{j-1}, 2t, \theta_{j-1} \rangle \]

then $\theta_j$ exists.

Relevant to this is the following canonical relation.

**Proposition 6.** If $\alpha \in \Pi_k$, $2\alpha = 0, k \equiv 2 \pmod{4}$, then $\langle \alpha, 2t, \alpha \rangle = 0$. 
Thus Corollary 5 has a weaker version.

**Corollary 5'**. If there is a non-$\varphi_{j,j}$ carrying null-homotopy of $n\theta_{j-1}^2$, then $\theta_j$ exists.

Thus we have the first part of the following theorem.

**Theorem 7**. If $\theta_{j-1}$ exists, $2\theta_{j-1} = 0 = \theta_{j-1}^2$ then $\theta_j$ exists and $2\theta_j = 0$.

The second part follows by a similar analysis.

Another proof of Theorem 4 is based on the smash product. Let $S^{2j-1} \to X$ represent $\{h_j\}$. Then $S^{2j-1} \wedge S^{2j-1} \to X \wedge X$ represents $\{h_j^2\}$ and this gives a map

\[
S^{2j+1} - 1 \to \eta_{j} \to \eta
\]

as before. This proof is a version of the central idea of our approach. The philosophy is to use the existence of $\theta_{j,j}^*$ and apply a functorial construction which hopefully gives $\theta_j$. The $\Gamma$ construction discussed earlier by Barratt is an example. It contains the "quadratic" construction and higher symmetries. In particular, explicit construction of the 30-manifold using $\mathcal{S}_4$, the symmetric group on four letters, has been given.

Milgram, using $\mathcal{S}_4$ symmetries, has proved the following:

**Theorem 8 [Milgram]**. With the hypothesis of Theorem 7, $\theta_{j+1}$ exists.

**Remark**. It can be shown that $\theta_{j}^2 = 0$ and thus Milgram's theorem implies $\theta_j$ exists.

A more delicate argument about $\theta_{j-1}$ shows

**Proposition 9**. $\langle \theta_{j-1}, 2i, \theta_{j-1} \rangle_n = [\iota_n, \beta_{j-1}]$, where $n = 2^{j+1} - \varphi(j - 1) - 1$, $\beta_{j-1}$ is the generator of the im $f$ in stem $\varphi(j - 1) - 1$ and $\varphi(j - 1)$ is the Adams function.

**Corollary 10**. $\Sigma^{-2} \langle \theta_{j-1}, 2i, \theta_{j-1} \rangle_n = [\iota_{2j+1}^{-1} - \varphi(j - 1), \beta_j]$ where $\epsilon = \varphi(j) - \varphi(j - 1).

Thus

**Theorem 11**. If $\theta_{j-1}$ exists and has order 2 and $\theta_j$ exists, $\theta_j$ appears on the $2^{j+1} - \varphi(j)$ sphere with Hopf invariant $\beta_j$.

There is a slightly less direct approach. First observe that there is a map $\lambda: RP \to S^n$ in the stable category. $\lambda$ on each cell is the Whitehead product. To be precise, there is a map $\Sigma^p S^{n-1} \to S^n$ and $S^{2n-1} \to \Sigma^p S^{n-1}$ where $d_n$ is the natural map $S^{n-1} \to P^{n-1}$. The composite $\lambda_n \Sigma^p a_n = [\iota_n, \iota_n]$. Thus

**Proposition 12**. If $\Sigma^p a_n$ can be halved for $n = 2^{j+1} - 1$ then $\theta_j$ exists.
There is a similar statement for $CP$ and $QP$.

Consider the situation with just a single suspension. We have the following fibration:

$$ P \ast P \to \Sigma RP \to K(Z_2, 2). $$

**Proposition 13.** If there is a map $f : S^{2j+1-1} \to P \ast P$ so that

$$ f^*(\sigma^{2j-1} \ast \sigma^{2j-1}) \neq 0 $$

then $\theta_j$ exists.

A weaker version is also true.

**Proposition 13'.** If there is a stable map $f : S^{2j-1-2} \to P \wedge P$ so that

$$ f^*(\sigma^{2j-1} \wedge \sigma^{2j-1}) \neq 0 $$

then $\theta_j$ exists.

Even weaker versions than this are possible.

**Proposition 13''.** Let $v_j$ be a cohomology class in $SO$ which transgresses to $w_j$. Then $v_j \otimes v_j$ being spherical in $SO \ast SO$ or in $S(SO \wedge SO)^*$ implies Conjecture $A$.

Another approach stems from the effort to construct large brackets.

**Theorem 14 (Hoffman).** If $\langle \sigma, 2\sigma, 2\sigma, \ldots, 2\sigma, \sigma \rangle$ can be defined then $\theta_j$ is in it.

This is verified for $j = 4$. There is a family of spaces $X_k$ which are defined by identifying particular subspaces of $\Lambda^k(S^8 \cup e^{16})$. The cell structure looks like

$$ \begin{array}{cccccc}
\sigma & 2\sigma & 3\sigma & \ldots & k\sigma \\
8k & 2k & 3k & \ldots & k^2 \\
\end{array} $$

This shows $0 \in \langle \sigma, 2\sigma, \ldots, (k-1)\sigma, k\sigma \rangle$.

**Theorem 15.** If the bracket $\langle \sigma, 2\sigma, \ldots, (2^j-2)\sigma, 2^{j-3}\sigma \rangle$ can be formed then $\theta_j$ is in it.

This requires a mild interpretation because $16\sigma = 0$, but it is not hard to see what should be done.

A paraphrase of Theorem 15 is the question of whether the attaching map in the construction $X_k$ can be halved.

Another amusing approach:

**Theorem 16.** If $2\theta_{j-1} = 0 = 2\theta_j$, and $\langle \theta_{j-1}, 2\sigma, \theta_j \rangle = 0$ then $\theta_{j+1}$ exists.
PROOF. Take

\[ \begin{array}{c}
\begin{array}{c}
\circ \quad 2t \\
\downarrow \theta_j \\
\circ \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ \quad 4t \\
\downarrow \theta_{j-1} \\
\circ \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ \quad \eta_{j-1} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ \quad <\theta_j, 2t, \theta_{j-1}> \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ \quad <\theta_j, 2t, \theta_j> \\
\end{array}
\end{array}
\end{array}
\]

This construction is \( \varphi_{i+1,j+1} \) carrying.

CONJECTURE. \( \langle \theta_j, 2t, \theta_{j-1} \rangle = [l, \beta_{j-1}] \) where \( n = 2^i + 2^{i-1} - \varphi(j - 1) \).
(Although \( \langle \theta_k, 2\sigma, \varphi \rangle = 0 \), it is not known whether \( \langle \theta_k, 2t, \sigma \sigma \rangle = 0 \).)

The strongest evidence for the existence of the \( \theta_j \)'s is

**Theorem 17.** There exists a sequence of integers \( n_i \) such that \( 2^i - 2 < n_i \leq 2^{i+1} - 2 \) and elements \( \alpha_i \in \Pi^6_{n_i} \). If \( n_i = 2^{i+1} - 2 \) then the constructed elements are \( \theta_i \).

PROOF. Simply stated, the \( \alpha_i \) are the stable Hopf invariants of the \( \beta_i \) in stem \( 2^{i+1} - \varphi(i) - 1 \). To be more precise consider the \( X_{2^i,e} \) constructing. The attaching map of the cell in dimension \( 2^i + 2 \) can be halved at least through the \( 2^{i+2} - 2 \)-skeleton. If it can be halved through the \( 8 + 2^{i+1} \)-skeleton then we have \( \theta_i \), otherwise there is an obstruction. We call this obstruction \( \alpha_i \).

Another amusing result is the following.

Let \( Y \to S^0 \to K(\mathbb{Z}, 0) \) define \( Y \). The map \( \lambda : P \to S^0 \) lifts to \( Y \).

**Remark.** \( Y/P \) has cohomology which is free over \( Sq^1 \) and \( Sq^2 \). Thus as far as \( bo \) homology is concerned, \( P \) and \( Y \) are equally interesting.

Consider the spectrum \( \text{Im} J \) defined by the fibration

\[ \text{Im} J \to BO[8k, \ldots] \to BO[8k + 4, \ldots]. \]

**Theorem 18.** \( \Pi_i (P \wedge \text{Im} J) = \)

\[
\begin{array}{cccccccc}
i = 0 & 1 & 2 & 3 & 4 & 5 & -2 & -1 & 0 & 1 & 2 & 4 & 5 \pmod{8} \\
\end{array}
\]

where \( \lambda_i \) is the 2-primary order of the image of the \( J \)-homomorphism. (That is, if \( i + 1 \equiv 2^\ell \pmod{2^{\ell+1}} \) then \( \lambda_i = 2^{\ell+1} \).) If \( \lambda : Y \to P \wedge \text{Im} J \), then \( \lambda_i \) are elements in \( \Pi_i Y \) given by the image of \( J \), the \( \mu \)'s of Barratt and Adams, the elements \( \eta_j \) and \( \theta_i \) generate the image of \( \lambda_i \).

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