5. (a) Let $A$ be a chain complex (of abelian groups). Assume that it is acyclic; i.e. $H(A) = 0$. Prove that it is contractible (i.e. chain-homotopy-equivalent to the trivial chain complex) if and only if for every $n$ the inclusion $Z_n A \hookrightarrow A_n$ is a split monomorphism of abelian groups.

(b) Give an example of an acyclic chain complex that is not contractible.

6. (a) Propose a construction of the product and the coproduct of two spaces in the homotopy category, and check that your proposal serves the purpose.

(b) Let $\text{VS}$ be the category of finite-dimensional vector spaces over a fixed field (with vector space homomorphisms for morphisms). Show that the assignment $V \mapsto V^*$ sending $\text{ob} \text{VS} \rightarrow \text{ob} \text{VS}$ can be made part of a functor, contrary to expectation!

7. (a) Let $S$ and $T$ be sets and $A$ an abelian group. Establish a bijection between the set of maps of sets from $S \times T$ to $A$ and the set of bilinear maps $Z(S) \times Z(T) \rightarrow A$.

(b) For positive integers $m, n$, let $\mathbb{Z}/m, \mathbb{Z}/n$ denote the cyclic groups of order $m, n$. Construct a surjective bilinear map $\mu : \mathbb{Z}/m \times \mathbb{Z}/n \rightarrow \mathbb{Z}/\gcd\{m, n\}$. Show that any bilinear map $\mathbb{Z}/m \times \mathbb{Z}/n \rightarrow A$ factors uniquely as $f \circ \mu$ where $f : \mathbb{Z}/\gcd\{m, n\} \rightarrow A$ is a homomorphism.

8. (a) Let $0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0$ be a short exact sequence. Show that the following three sets are in bijection with one another.

(i) The set of homomorphisms $\sigma : C \rightarrow B$ such that $p\sigma = 1_C$.

(ii) The set of homomorphisms $\pi : B \rightarrow A$ such that $\pi i = 1_A$.

(iii) The set of homomorphisms $\alpha : A \oplus C \rightarrow B$ such that $\alpha(a, 0) = ia$ for all $a \in A$ and $p\alpha(a, c) = c$ for all $(a, c) \in A \oplus C$.

Moreover, show that any homomorphism as in (iii) is an isomorphism.

Any one of these data is a splitting of the short exact sequence, and the sequence is then said to be split.

(b) Suppose that

$$
\cdots \rightarrow A_n \xrightarrow{i_n} B_n \xrightarrow{p_n} C_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{i_{n-1}} \cdots
$$

$$
\cdots \rightarrow A'_n \xrightarrow{i'_n} B'_n \xrightarrow{p'_n} C'_n \xrightarrow{\alpha'_n} A'_{n-1} \xrightarrow{i'_{n-1}} \cdots
$$

is a “ladder”: a map of long exact sequences. So both rows are exact and each square commutes. Suppose also that every third vertical map is an isomorphism, as
indicated. Prove that these data determine a long exact sequence

\[ \cdots \rightarrow A_n \rightarrow A'_n \oplus B_n \rightarrow B'_n \rightarrow A_{n-1} \rightarrow \cdots \]

9. (a) ("3 × 3 lemma.") Let

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & A' & B' \\
\downarrow & \downarrow & \downarrow \\
0 & A & B \\
\downarrow & \downarrow & \downarrow \\
0 & A'' & B'' \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]

be a commutative diagram of abelian groups. Assume that all three columns are exact, that all but one of the rows is exact, and that the compositions in the remaining row are trivial. Prove that the remaining row is also exact. (Hint: view each row as a chain complex . . . .)

(b) ("Long exact homology sequence of a triple.") Let \((C, B, A)\) be a "triple," so \(C\) is a space, \(B\) is a subspace of \(C\), and \(A\) is a subspace of \(B\). Show that there are natural transformations \(\partial : H_n(C, B) \rightarrow H_{n-1}(B, A)\) such that

\[ \cdots \rightarrow H_n(B, A) \xrightarrow{i_*} H_n(C, A) \xrightarrow{j_*} H_n(C, B) \xrightarrow{\partial} H_{n-1}(B, A) \rightarrow \cdots \]

is exact, where \(i : (B, A) \rightarrow (C, A)\) and \(j : (C, A) \rightarrow (C, B)\) are the inclusions of pairs. (9 (a) might be useful.)

10. This exercise generalizes our computation of the homology of spheres, and introduces several important constructions.

The cone on a space \(X\) is the quotient space \(CX = X \times I/X \times \{0\}\), where \(I\) is the unit interval \([0, 1]\). The cone is a pointed space, with basepoint \(*\) given by the "cone point," i.e. the image of \(X \times \{0\}\). (By convention, the cone on the empty space \(\emptyset\) is a single point, the cone point.) Regard \(X\) as the subspace of \(CX\) of all points of the form \((x, 1)\).

Define the suspension of a space \(X\) to be \(SX = CX/X\). Make \(SX\) a pointed space by declaring the image of \(X \subseteq CX\) to be the basepoint in \(SX\). (By convention, the quotient \(W/\emptyset\) is the disjoint union of \(W\) with a single point, which is declared to be the basepoint. So \(S\emptyset = */\emptyset\) is the discrete two-point space, with the new point as basepoint.)

The quotient map induces a map of pairs \(f : (CX, X) \rightarrow (SX, *)\).
(a) Show that \( CX \) is contractible.

For any \( a, b \in I \) with \( a \leq b \), let \( C^b_a X \) denote the image of \( X \times [a, b] \) in \( CX \). Thus \( C^0_0 X = CX \), \( C^0_0 X = * \), and \( C^1_1 X = X \).

Let \( p : CX \to CX \) send \((x, t)\) to \((x, 3t)\) for \( t \leq 1/3 \) and to \((x, 1)\) if \( t \geq 1/3 \).

(b) Show that \( p \) defines a homotopy equivalence of pairs \((C^2/3_0 X, C^2/3_1 X) \to (CX, X)\).

(c) Show that the evident map \( e : (C^2/3_0 X, C^2/3_1 X) \to (SX, C^1_1 X/X) \) is an excision.

(d) Show that \( p \) defines a homotopy equivalence of pairs \((SX, C^1_1 X/X) \to (SX, *)\).

(e) Conclude from the commutativity of

\[
\begin{array}{ccc}
(C^2/3_0 X, C^2/3_1 X) & \xrightarrow{e} & (SX, C^1_1 X/X) \\
\downarrow & & \downarrow \\
(CX, X) & \xrightarrow{f} & (SX, *)
\end{array}
\]

that \( f \) induces an isomorphism in homology.

(f) Use the long exact sequence for the homology of the pair \((CX, X)\) to construct an isomorphism \( H_{n-1}(X) \to H_n(SX) \) for \( n > 1 \).

To deal smoothly with \( n \leq 1 \), it is convenient to define the augmented singular chain complex \( \tilde{S}_*(X) \) as follows. In non-negative dimensions it is isomorphic to \( S_*(X) \). We define \( \tilde{S}_-1(X) = \mathbb{Z} \), \( d : \tilde{S}_0(X) \to \tilde{S}_-1(X) \) to be the augmentation, and \( \tilde{S}_n(X) = 0 \) for \( n < -1 \). Define the augmented singular homology of \( X \) to be \( \tilde{H}_*(X) = H(\tilde{S}_*(X)) \).

(g) Compute \( \tilde{H}_*(X) \) in terms of \( H_*(X) \), using the natural short exact sequence of chain complexes \( 0 \to \mathbb{Z}[-1] \to \tilde{S}_*(X) \to S_*(X) \to 0 \) where \( \mathbb{Z}[-1] \) denotes the chain complex which is \( \mathbb{Z} \) in dimension \(-1\) and \( 0 \) otherwise.

(h) Show that there is a natural isomorphism \( \tilde{H}_{n-1}(X) \to \overline{H}_n(SX) \), for any \( n \).