Due October 5, 2016, in class. Problems 9 and 10 to follow.

Homework is an important part of this class. I hope you gain from the struggle. Collaboration can be effective, but be sure that you grapple with each problem on your own as well. If you do work with others, you must indicate with whom on your solution sheet. Scores will be posted on the Stellar website.

5. (a) Let $A$ be a chain complex (of abelian groups). Assume that it is acyclic; i.e. $H(A) = 0$. Prove that it is contractible (i.e. chain-homotopy-equivalent to the trivial chain complex) if and only if for every $n$ the inclusion $Z_n A \hookrightarrow A_n$ is a split monomorphism of abelian groups.

(b) Give an example of an acyclic chain complex that is not contractible.

6. (a) Propose a construction of the product and the coproduct of two spaces in the homotopy category, and check that your proposal serves the purpose.

(b) Let $\text{VS}$ be the category of finite-dimensional vector spaces over a fixed field (with vector space homomorphisms for morphisms). Show that the assignment $V \mapsto V^*$ sending $\text{ob\,VS} \to \text{ob\,VS}$ cannot be made part of a functor.

7. (a) Let $S$ and $T$ be sets and $A$ an abelian group. Establish a bijection between the set of maps of sets from $S \times T$ to $A$ and the set of bilinear maps $Z^S \times Z^T \to A$.

(b) For positive integers $m, n$, let $\mathbb{Z}/m, \mathbb{Z}/n$ denote the cyclic groups of order $m, n$. Construct a surjective bilinear map $\mu: \mathbb{Z}/m \times \mathbb{Z}/n \to \mathbb{Z}/\gcd\{m, n\}$. Show that any bilinear map $\mathbb{Z}/m \times \mathbb{Z}/n \to A$ factors uniquely as $f \circ \mu$ where $f: \mathbb{Z}/\gcd\{m, n\} \to A$ is a homomorphism.

8. (a) Let $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$ be a short exact sequence. Show that the following three sets are in bijection with one another.

(i) The set of homomorphisms $\sigma: C \to B$ such that $p \sigma = 1_C$.

(ii) The set of homomorphisms $\pi: B \to A$ such that $\pi i = 1_A$.

(iii) The set of homomorphisms $\alpha: A \oplus C \to B$ such that $\alpha(a, 0) = i a$ for all $a \in A$ and $p \alpha(a, c) = c$ for all $(a, c) \in A \oplus C$.

Moreover, show that any homomorphism as in (iii) is an isomorphism. Any one of these data is a splitting of the short exact sequence, and the sequence is then said to be split.

(b) Suppose that

$$
\cdots \longrightarrow A_n \longrightarrow B_n \longrightarrow C_n \longrightarrow A_{n-1} \longrightarrow \cdots
$$

$$
\cdots \longrightarrow A'_n \longrightarrow B'_n \longrightarrow C'_n \longrightarrow A'_{n-1} \longrightarrow \cdots
$$

is a “ladder”: a map of long exact sequences. So both rows are exact and each square commutes. Suppose also that every third vertical map is an isomorphism, as
indicated. Prove that these data determine a long exact sequence

\[ \cdots \rightarrow A_n \rightarrow A'_n \oplus B_n \rightarrow B'_n \rightarrow A_{n-1} \rightarrow \cdots \]