Problem 1. Let \( n \geq 1 \) and let \( T\mathbb{C}P^n \) be the tangent bundle of \( \mathbb{C}P^n \). Recall from Problem Set 4 that \( T\mathbb{C}P^n \cong \gamma \vee \gamma' \), where \( \gamma \) is the tautological line bundle on \( \mathbb{C}P^n \), \( \gamma' \) is its dual, and \( \gamma' \) is defined by the short exact sequence

\[
0 \to \gamma \to \mathbb{C}^{n+1} \to \gamma' \to 0.
\]

(a) Show that \( c_1(\gamma) \in H^2(\mathbb{C}P^n) \) is a generator of the cohomology ring \( H^*(\mathbb{C}P^n) \).

(b) Define an isomorphism \( T\mathbb{C}P^n \oplus \mathbb{C} \cong (\gamma')^{n+1} \), and deduce that

\[
c_i(T\mathbb{C}P^n) = (-1)^i \binom{n+1}{i} c_1(\gamma)^i.
\]

The same arguments show that \( w_1(\gamma) \) is the generator of \( H^*(\mathbb{R}P^n, \mathbb{Z}/2) \) and that

\[
w_i(T\mathbb{R}P^n) = \binom{n+1}{i} w_1(\gamma)^i.
\]

Problem 2. Recall from Problem Set 4 that \( c_1(L \otimes M) = c_1(L) + c_1(M) \) for complex line bundles \( L \) and \( M \). Combining this with the Cartan formula and the splitting principle, it is possible to express the Chern classes of any tensor product \( V \otimes W \) of complex vector bundles in terms of the Chern classes of \( V \) and \( W \).

(a) Do this explicitly when \( V \) and \( W \) have rank at most 2.

(b) Compute the Chern classes of the tangent bundle of \( \text{Gr}_2(\mathbb{C}^4) \) in terms of the Chern classes of the tautological bundle.

Problem 3.

(a) Let \( X \) be a CW complex and \( V \) a vector bundle over \( X \) with a nonvanishing global section. Show that the zero section \( X \to XV \) is nullhomotopic. Assuming \( V \) is \( R \)-orientable, deduce that \( e(V) = 0 \) in \( H^*(X, R) \).

(b) Show that every continuous vector field on \( \mathbb{C}P^n, \mathbb{R}P^{2n}, \) or \( S^{2n} \) has a zero.

Problem 4. (Fundamental classes) Throughout this problem, cohomology is taken with coefficients in \( \mathbb{Z}/2 \), for simplicity. Let \( M \) be smooth manifold and \( i: N \hookrightarrow M \) a compact submanifold of codimension \( c \) with normal bundle \( \nu \). Recall that \( \nu \) is defined by the short exact sequence

\[
0 \to TN \xrightarrow{T_i} i^*(TM) \to \nu \to 0.
\]
The tubular neighborhood theorem states that there exists an open neighborhood $U$ of $N$ in $M$ and a diffeomorphism $\phi: \nu \to U$ such that $\bar{U}$ is compact and the triangle

\[
\begin{array}{ccc}
\nu & \xrightarrow{\phi} & U \\
\uparrow zero & & \\
N & \xrightarrow{\phi} & U
\end{array}
\]

commutes. In particular, we can identify $\bar{U}/\partial U$ with the Thom space $N^\nu$. The map

$M_+ \to M/(M \setminus U) \cong \bar{U}/\partial U \cong N^\nu$

is called the Pontryagin–Thom collapse; up to homotopy, it does not depend on the choices of $U$ and $\phi$. On mod 2 cohomology, the Pontryagin–Thom collapse induces

$$i_! : H^*(N) \cong \tilde{H}^{*+c}(N^\nu) \to \tilde{H}^{*+c}(M_+) \cong H^{*+c}(M),$$

where the first isomorphism is the Thom isomorphism. This map is variously called the transfer map, or the Gysin map, or the Umkehr map. Define the fundamental class of $N$ by

$$[N] = i_!(1) \in H^c(M).$$

(a) Assume that $M$ is compact connected of dimension $d$ and let $*$ be a point in $M$. Show that $[*]$ is the generator of $H^d(M) = \mathbb{Z}/2$.

(b) Let $N_1$ and $N_2$ be compact submanifolds of $M$ intersecting transversely. This means that, for every $x \in N_1 \cap N_2$, $T_xN_1$ and $T_xN_2$ span $T_xM$. By the implicit function theorem, $N_1 \cap N_2$ is a submanifold of $M$ with $T_x(N_1 \cap N_2) = T_xN_1 \cap T_xN_2$. Show that

$$[N_1 \cap N_2] = [N_1] \cup [N_2] \in H^*(M).$$

Problem 5.

(a) Let $M$ be a closed smooth $n$-manifold and $\nu$ an $R$-orientable vector bundle of rank $k$ on $M$. Show that the Euler class $e(\nu) \in H^k(M, R)$ is an obstruction to the existence of an embedding $M \hookrightarrow \mathbb{R}^{n+k}$ with normal bundle $\nu$ (i.e., if $e(\nu) \neq 0$, such an embedding does not exist).

Hint: Use the tubular neighborhood theorem.

(b) Suppose that $\mathbb{R}P^n$ embeds in $\mathbb{R}^{n+1}$. Show that $n + 1$ is a power of 2. (In fact, $n = 0$ or $n = 1$, but this is trickier to prove.)

Problem 6.

(a) The Stiefel–Whitney class $w_i$, viewed as a characteristic class for complex vector bundles of rank $n$, must be a polynomial in the Chern classes $c_1, \ldots, c_n$ modulo 2. What is this polynomial?

Hint: You already know the answer for the top Stiefel–Whitney class: $w_{2n} = e = c_n$.

(b) Define a new characteristic class $r_i$ for real vector bundles of rank $n$ by $r_i(V) = c_i(V \otimes_R \mathbb{C})$. Then $r_i$ modulo 2 must be a polynomial in the Stiefel–Whitney classes $w_1, \ldots, w_n$. What is this polynomial?

The characteristic class $p_i = (-1)^i r_{2i}$ is called the $i$th Pontryagin class.