Problem 1. Let $G$ be a topological group and $\phi: G \to G$ an inner automorphism of $G$, i.e., $\phi$ is conjugation by some element of $G$. Show that $B\phi: BG \to BG$ is homotopic to the identity.

Problem 2. Let $n \geq 2$ and let $d$ be a nonzero integer. Let $f_d: S^n \to S^n$ be a map of degree $d$ (i.e., a map inducing multiplication by $d$ on $H_n(S^n)$), and let $F_d$ be the homotopy fiber of $f_d$.

(a) Using the Serre spectral sequence, compute the homology of $F_d$.

(b) Determine the structure of the Serre spectral sequence, starting from $E^2$, for the homotopy fiber sequence $\Omega S^n \to F_d \to S^n$. Ignore signs.

Hint: Recall that $H_*(\Omega S^n) = \{ \mathbb{Z} \text{ if } i \text{ is a multiple of } n-1, 0 \text{ otherwise.} \}$

(c) Assume $n \geq 3$. Determine the structure of the Serre spectral sequence, starting from $E^2$, for the homotopy fiber sequence $\Omega S^n \to \Omega S^n \to F_d$. Determine, up to signs, the group extensions needed to assemble $H_*(\Omega S^n)$ from the $E^\infty$ page.

Problem 3. (a) Pretend that you don’t know $K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$. Use the Serre spectral sequence for $S^1 \to \ast \to K(\mathbb{Z}, 2)$ to compute the cohomology ring $H^*(K(\mathbb{Z}, 2))$.

(b) Use the Serre spectral sequence for $K(\mathbb{Z}, 2) \to \ast \to K(\mathbb{Z}, 3)$ to compute the ring $H^*(K(\mathbb{Z}, 3))/H^{\geq 14}(K(\mathbb{Z}, 3))$.

Problem 4. Let $n \geq 1$ be odd. Show that the cohomology ring $H^*(\Omega S^{n+1})$ has the form $\Lambda_\mathbb{Z}^k(\gamma_1) \otimes \Gamma_\mathbb{Z}(\gamma_2)$ for some cohomology classes $\gamma_1$ in degree $n$ and $\gamma_2$ in degree $2n$. Explicitly, $H^*(\Omega S^{n+1}) \cong \mathbb{Z}[\gamma_1, \gamma_2k | k \geq 1]/(\gamma_1^2, \gamma_2^{k!} - k! \gamma_2)$.

Problem 5. Let $L$ and $M$ be complex line bundles over a CW complex $X$. Show that $c_1(L \otimes M) = c_1(L) + c_1(M)$ in $H^2(X)$.

Hint: The universal pair of line bundles lives over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$.

Problem 6. Let $E$ be a finite-dimensional (real or complex) vector space and let $n \geq 0$.

(a) Let $\gamma_n(E) = \{(V, x) \in \text{Gr}_n(E) \times E | x \in V \}$. Show that the projection $\gamma_n(E) \to \text{Gr}_n(E)$ is a vector bundle. This is called the tautological bundle over $\text{Gr}_n(E)$.

(b) Let $\text{Aff}_n(E)$ be the set of $n$-dimensional affine subspaces of $E$, i.e., translates of $n$-dimensional linear subspaces. Put a topology on this set and show that the “translate to the origin” map $\text{Aff}_n(E) \to \text{Gr}_n(E)$ is a vector bundle.

(c) Explain why the tangent bundle of $\text{Gr}_n(E)$ is $\text{Hom}(\gamma_n(E), \text{Aff}_n(E))$. 
