Problem 1. Let $X$ and $Y$ be pointed CW complexes. Suppose that $X$ is $m$-connected and $Y$ is $n$-connected. How connected are the following spaces?

(a) $X \vee Y$
(b) $X \times Y$
(c) $X \wedge Y$

*Hint:* Recall that a CW complex is $n$-connected iff it is homotopy equivalent to one with no cells in dimensions $\leq n$.

Problem 2. Let $X$ and $Y$ be pointed spaces.

(a) Show that the inclusion $X \vee Y \hookrightarrow X \times Y$ admits a *section* up to pointed homotopy after taking loops.

(b) Deduce that, for $n \geq 2$,
\[ \pi_n(X \vee Y) \cong \pi_n(X) \oplus \pi_n(Y) \oplus \pi_{n+1}(X \times Y, X \vee Y). \]

(c) Assume for simplicity that $X$ and $Y$ are CW complexes. What are the minimal connectivity assumptions on $X$ and $Y$ guaranteeing that the canonical map $\pi_{n+1}(X \times Y, X \vee Y) \rightarrow \pi_{n+1}(X \wedge Y)$ is an isomorphism?

Problem 3. Let $f : X \rightarrow Y$ be a pointed map with homotopy fiber $F$, and consider the exact sequence of pointed sets
\[ \pi_1(Y) \xrightarrow{\partial} \pi_0(F) \rightarrow \pi_0(X). \]
Show that $\pi_1(Y)$ acts on the set $\pi_0(F)$ in such a way that:

- $\partial$ is a $\pi_1(Y)$-equivariant map;
- two elements $a, b \in \pi_0(F)$ have the same image in $\pi_0(X)$ iff there exists $\gamma \in \pi_1(Y)$ such that $\gamma a = b$.

Problem 4. Let $n \geq 0$. A space is called *$n$-truncated* if its homotopy groups are trivial in degrees greater than $n$. If $X$ is any space, an *$n$-truncation* of $X$ is an $n$-connected map $X \rightarrow X_n$ where $X_n$ is $n$-truncated.
(a) Let $X$ be a CW complex. Show that there exists a tower

$$
\begin{array}{c}
X \\
\downarrow \\
\cdots \rightarrow X_{n+1} \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_0
\end{array}
$$

where each map $X \rightarrow X_n$ is an $n$-truncation of $X$.

*Hint:* First construct an $n$-truncation $X \rightarrow X_n$ by attaching cells of dimensions $\geq n+2$. Then, show that the $n$-truncation factors through the $(n+1)$-truncation.

This is called the *Postnikov tower* of $X$; one can show that it is unique in the appropriate sense. Note that the homotopy fibers of $X_n \rightarrow X_{n-1}$ have no nontrivial homotopy groups except in degree $n$ (such spaces are called *Eilenberg–Mac Lane spaces* of degree $n$, see Problem 5 below).

(b) Show that $S^2 = \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$ is a 2-truncation of $S^2$.

**Problem 5.** (Eilenberg–Mac Lane spaces) Let $n \geq 1$ and let $A$ be a group (abelian if $n \geq 2$).

(a) Without using Brown representability, construct a CW complex $K(A,n)$ such that

$$
\pi_i(K(A,n)) \cong \begin{cases} A & \text{if } i = n, \\ 0 & \text{otherwise}, \end{cases}
$$

and show that any two CW complexes with this property are homotopy equivalent.

(b) *Optional.* Show that $[K(A,n), K(B,n)]_* \cong \text{Hom}(A,B)$. In other words, the construction $A \mapsto K(A,n)$ is an equivalence of categories between (abelian) groups and the homotopy category of pointed Eilenberg–Mac Lane spaces of degree $n$.

**Problem 6.**

(a) Compute the homotopy groups of $\mathbb{R}P^n$ and $\mathbb{C}P^n$ in terms of the homotopy groups of spheres.

(b) Let $n \geq 2$. Compute the relative homotopy group $\pi_n(\mathbb{R}P^n, \mathbb{R}P^{n-1})$ and observe that the canonical map $\pi_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \rightarrow \pi_n(\mathbb{R}P^n/\mathbb{R}P^{n-1})$ is not an isomorphism.

**Problem 7.** (Whitehead products) Let $p, q \geq 1$. Define $h: S^{p+q-1} \rightarrow S^p \vee S^q$ by the commutativity of the following diagram:

$$
\begin{array}{ccc}
\partial I^{p+q} & \hookrightarrow & I^{p+q} = I^p \times I^q \\
& \nearrow h & \searrow \\
& & I^p/\partial I^p \times I^q/\partial I^q
\end{array}
$$
It is not difficult to show that the maps \( h \) define a structure of \textit{graded Lie coalgebra} on the graded cogroup \( \{ S^{n+1} \}_{n \geq 0} \) in the pointed homotopy category. Equivalently, the homotopy groups \( \pi_{*+1}(X) \) of a pointed space \( X \) have the structure of a graded Lie algebra, with bracket \( [-,-]: \pi_p(X) \times \pi_q(X) \to \pi_{p+q-1}(X) \) given by:

\[
[\alpha,\beta]: S^{p+q-1} \xrightarrow{h} S^{p} \vee S^{q} \xrightarrow{\alpha \vee \beta} X \vee X \xrightarrow{\nabla} X.
\]

This bracket is called the \textit{Whitehead product}.

(a) What is the mapping cone of \( h \)?

(b) Describe \( [-,-]: \pi_1(X) \times \pi_n(X) \to \pi_n(X) \) in terms of the action of \( \pi_1(X) \) on \( \pi_n(X) \).

(c) Suppose that \( X \) is an \textit{H}-space, i.e., that the fold map \( \nabla: X \vee X \to X \) extends to a map \( \mu: X \times X \to X \) in the pointed homotopy category. Show that all Whitehead products on \( \pi_{*+1}(X) \) are zero.

(d) Show that the suspension map \( \pi_{p+q-1}(X) \to \pi_{p+q}(\Sigma X) \) sends all Whitehead products to zero (\textit{Hint}: loop spaces are \textit{H}-spaces).

\textbf{Problem 8.} Prove the homotopy version of the Poincaré conjecture: every simply connected compact smooth 3-manifold is homotopy equivalent to a 3-sphere. You can use the fact that any compact smooth manifold is homotopy equivalent to a CW complex.