Supplementary Notes to

Differential Geometry, Lie Groups and Symmetric Spaces

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Page 17^5 means fifth line from top of page 17 and page 81_6 means the sixth line from below on page 81.

Page 28^{19}. For the applications of Lemma 5.1 in this chapter the local version of the lemma:

$$G(\gamma(t)) = g(t) \qquad \text{for } t \text{ near } t_0$$

(the proof of which does not require partition of unity) is sufficient.

Page 101 (middle). The statement $u_p = cX^*(M)$ actually holds in a stronger form:

$$u_p = m_1! \dots m_n! X^*(M)$$

where m_k is the number of entries in the sequence $(i_1, \dots, i_p)$ which equal k and $M = (m_1, \dots, m_n)$. To see this write X_j for X_j^* and note that

$$(t_{i_1} X_{i_1} + \dots + t_{i_p} X_{i_p})^p = \sum_{|M|=p} t^M S_M\,,$$

where

$$S_M = \sum_\sigma X_{i_{\sigma(1)}} \dots X_{i_{\sigma(p)}} = p! u_p\,.$$

In each term m_k factors equal X_k. One term is $X_1^{m_1} \cdots X_n^{m_n}$ and the others are obtained by shuffling. In the sum $\sum_\sigma$ each term will appear $m_1! \dots m_n!$ times. Now

$$(t_1 X_1 + \cdots t_n X_n)^p = p! \sum_{|M|=p} t^M X(M)\,.$$

Here $p! X(M)$ is the sum of the terms obtained from $X_1^{m_1} \cdots X_n^{m_n}$ by shuffling, each term appearing *exactly once*. Hence

$$p!\, u_p = m_1! \dots m_n!\; p!\; X(M)\,.$$

Page 188 (middle). G_{21}^- should be $-G_{21}^-$

Page 101 195_4. $X'_{\alpha,\beta}$ should be $X'_{\alpha+\beta}$

Page 325. Proposition 8.10. In response to a question by Adam Korányi, this proposition has the following extension:

Assuming Σ irreducible each automorphism of it extends to an automorphism of Δ.

This follows from Theorem 3.29, Ch. X.

Only the cases (ii) and (iii) have to be considered. The cases $\mathfrak{e}_6$ and $\mathfrak{d}_4$ for Σ only occur for the normal form so statement is obvious. This leaves the cases $\mathfrak{a}_\ell(\ell \geq 2)$ and $\mathfrak{d}_\ell(\ell > 4)$ and here we have (by (ii)) to look at one automorphism of Σ which is not induced by $W(\Sigma)$ and show that it is induced by an automorphism of Δ. For the form A1 (Satake diagram, Exercises F, Ch. X) there is nothing to prove since AI is a normal form. Same for the case $\mathfrak{d}_\ell$. For the form AII the extra automorphism is -1 (Exercise B1, Ch. X) so here the statement follows too.

As a corollary we see (considering Theorem 5.4, Ch. IX) that each automorphism of Σ is induced by an automorphism of $\mathfrak{u}$.

Page 349. Exercise 10. As observed to me by A. Onishchik the connectedness assumption for K is unnecessary here; see G. Fels, "A Note on Homogeneous Locally Symmetric Spaces,", Transformation Groups 2 (1997), 269–277.

Page 396_{11}. The following exercise (response to a question by Morris Hirsch) follows from some results in this chapter.

Proposition. *If G is a connected simple Lie group which does not contain a nontrivial compact subgroup then G is the universal covering group of* $\mathbf{SL}(2, \mathbf{R})$.

Proof: Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition. Because of Theorem 5.4 the action of $\mathfrak{k}$ on $\mathfrak{p}$ is irreducible. Now $\mathfrak{k}$ is the direct sum of its center $\mathfrak{c}$ and its semisimple part $[\mathfrak{k}, \mathfrak{k}]$. Since any connected Lie group with Lie algebra $[\mathfrak{k}, \mathfrak{k}]$ is compact the assumption implies $\mathfrak{k} = \mathfrak{c}$. Let $\widetilde{G}$ denote the universal covering group of G. Let $\widetilde{K}$ denote the analytic subgroup corresponding to $\mathfrak{k}$. Then $\widetilde{G}/\widetilde{K} = G/K$ and by Prop. 6.2, $\mathfrak{k}$ is 1-dimensional. From Table V on page 518 we see that $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{R})$. Thus $\widetilde{G}$ = universal covering group of $SL(2, \mathbf{R})$ and $\widetilde{K}$ is isomorphic to $\mathbf{R}$. Thus $K = \mathbf{R}/N$ where N is a discrete subgroup of $\mathbf{R}$. Since K is noncompact by assumption, N reduces to the identity. Since the covering map $j : \widetilde{G} \to G$ has kernel contained in $\widetilde{K}$ (Theorem 1.1, Ch. VI) and j is injective on $\widetilde{K}$ we deduce $G = \widetilde{G}$ as claimed.

Page 418_{10}. add "satisfying $J(\mathfrak{h}) = \mathfrak{h}$".

Page 500^4. "implied" should read "followed from".

Page 517^2. Type IV should be Type III.

Page 521_{12}. v_{sl} should be $v_{5\ell}$.

Page $553^{8,9}$. This statement follows from the theorem on page 596.

Page $579^{1,2}$. $\frac{1}{2}$ should be 2.

Page 580. Solution to Exercise B6. $\operatorname{Aut}(\mathfrak{u})$ acts on $\widetilde{U}$ and on $\widetilde{Z}$. If $s \in \operatorname{Aut}(\mathfrak{u})$ acts trivially on $\widetilde{Z}$ then by Lemma 6.5, Ch. VII it acts trivially on the lattice $\mathfrak{t}(\mathfrak{u})$ so acts trivially on $\mathfrak{t}$. Hence by Prop. 5.3, Ch. IX s is in $\operatorname{Int}(\mathfrak{u})$. Thus $\operatorname{Aut}(\mathfrak{u})/\operatorname{Int}(\mathfrak{u})$ acts faithfully on $\widetilde{Z}$ so by Theorem 5.4, Ch. IX and Lemma 3.30, Ch. X, $\operatorname{Aut}(R)/W(R)$ acts faithfully on $\widetilde{T}(R)/T(R)$.

If $-1 \in W(R)$ then -1 acts trivially on $\widetilde{T}(R)/T(R)$ so $-w \equiv w \mod T(R)$, whence $2\widetilde{T}(R) \subset T(R)$. Conversely, if $2\widetilde{T}(R) \subset T(R)$ then $-w \equiv w \mod T(R)$ so -1 induces the identity map of $\widetilde{T}(R)/T(R)$ so $-1 \in W(R)$.

Page 591^1. As the notation implies the top line applies to $j > i$ and then $\epsilon_{ij}(-1)^{i-1} = (-1)^{j-1}$. If $j < i$ then $[X_i, X_j]$ appears at the $(i-1)$-place so $\epsilon_{ij}(-1)^{i-2}$ is again $(-1)^{j-1}$.

One more proof of Lemma 7.1 is the following: If ω is a bi-invariant p-form then so is the form $J^*\omega$ where J is the map $x \to x^{-1}$ of G to G. But $(J^*\omega)_e = (-1)^p\omega_e$ so $J^*\omega = (-1)^p(\omega)$. Since d commutes with J^* ((6) I, §3) and $d\omega$ is a bi-invariant $(p+1)$ form,

$$d(-1)^p\omega = dJ^*\omega = J^*\, d\omega = (-1)^{p+1}\, d\omega$$

so $d\omega = 0$.

Page 591. Theorem 8.1. Note that in contrast to Chevalley [2], p. 112 this proof does not require the monodromy theorem. However, the following standard result was used in our proof as well as at some other places in the text.

Lemma. *A discrete normal subgroup D of a connected topological group G is contained in the center.*

In fact let $d \in D$ and N a neighborhood of d such that $N \cap D = (d)$ and let V be a neighborhood of e in G such that $VdV^{-1} \subset N$. Since D is normal, $\sigma \in V$ implies $\sigma d \sigma^{-1} \subset N \cap D = d$. Since G is connected, V generates G so $gdg^{-1} = d$ for all $g \in G$.

Theorem. *Let G be a connected, locally arcwise connected topological group. Let $H \subset G$ be a closed locally connected subgroup and H_0 the identity component of H. Then*

(i) G/H is connected and locally arcwise connected.

(ii) The natural map $G/H_0 \to G/H$ is a covering.

(iii) If G/H is simply connected then $H = H_0$.

(iv) If H is discrete, $G \to G/H$ is a covering.

Proof: (i) The natural map $\pi : G \to G/H$ is continuous so G/H is connected. If a is in an open subset $V \subset G/H$ take $\widetilde{a} \in \pi^{-1}(a)$ and a connected arcwise connected neighborhood W of $\widetilde{a}$ contained in $\pi^{-1}(V)$. Then $\pi(W)$ is an arcwise connected neighborhood of a contained in V.

For (ii) let $\pi_0 : G \to G/H_0$, $\pi : G \to G/H$ and $\sigma : G/H_0 \to G/H$ be the natural maps. If $U \subset G/H$ is open then $\sigma^{-1}(U) = \pi_0(\pi^{-1}(U))$ is open so σ is continuous. Also if $V \subset G/H_0$ is open, $\sigma(V) = \pi(\pi_0^{-1}(V))$ is open so σ is a continuous open mapping. H is locally connected so H_0 is open in H. Thus there exists an open subset $V \subset G$ such that $V \cap H = H_0$. Choose a connected neighborhood U of e in G such that $U^{-1}U \subset V$ and $U^{-1}U \cap H \subset H_0$. Then UH is a neighborhood of the origin in G/H.

Consider the inverse image

$$\sigma^{-1}(UH) = \{gH_0 : \sigma(gH_0) \in UH\} = \{UhH_0 \in G/H_0 : h \in H\}$$

the latter equality holding since $gH \in UH$ implies $g = uh$. Each UhH_0 is open in G/H_0 and is connected and their union is $\sigma^{-1}(UH)$. If

$$Uh_1H_0 \cap Uh_2H_0 \neq \emptyset \tag{1}$$

then since H_0 is normal in H,

$$UH_0h_1 \cap UH_0h_2 \neq \emptyset\,,$$

and hence $U^{-1}UH_0 \cap H_0h_2h_1^{-1} \neq \emptyset$. Again since H_0 is normal in H,

$$U^{-1}U \cap H_0h_2^{-1}h_1 \neq \emptyset\,.$$

Thus $U^{-1}U$ contains an element $h \in H$ so since $U^{-1}U \cap H \subset H_0$, $h \in H_0$ so $h_2^{-1}h_1 \in H_0$. Thus (1) implies

$$Uh_1H_0 = Uh_2H_0\,. \tag{2}$$

We claim now that for each $h \in H$ the map $\sigma : UhH_0 \to UH$ is a bijection. In fact if $\sigma(u_1hH_0) = \sigma(u_2hH_0)$ then $u_1h = u_2hh^*$ for some $h^* \in H$ so $u_2^{-1}u_1 \in H$ so by $U^{-1}U \cap H \subset H_0$, $u_1 = u_2h_0$ for some $h_0 \in H_0$, whence the injectivity of σ. The surjectivity is obvious. The sets UhH_0 being the components of $\sigma^{-1}(UH)$, UH is evenly covered. By translation we see that σ is a covering. Now (iii) and (iv) follow from (ii).

Page 597^{10}. $d_n = z_n$ should be $d_n = z_n + d^*$.

Page 597^{15}. $v_i \in V$ should be $v_i \in D$.

Errata

Page and line in { above / below	Instead of:	Read:
249_2	automorphism	inner automorphism
250_5	g_+	g_{+1}
264^2	$\wedge$	Δ
291_{14}	lemma	theorem
314^7	linear	affine
373^1, 375^1	Hermetion	Hermitian
430_7	$u - 1$	$u - I$
432^8	definining	defining
$433_{3,5}$	2	$2I$
485_8	representatives	representatives ξ', ξ, η'
498^{17}	$\equiv 0$	$= 0$
587^4	$\zeta, \zeta' \in B_r$	$\zeta \in B_r$
592_3	$=$	$\in$
603^{12}	remarquablz	remarquable
603^{16}	géométrique	géométrie
603_3	Scouten	Schouten
604^{11}	halbeinfachen	halbeinfacher
605^{11}	les	leurs
606^{16}	enveloppants	enveloppantes
611_{17}	geschlossenen	geschlossener
612^9	Gétt	Gött
616^{14}	dei Gruppi e	dei gruppi finiti e
616_{10}	Integrabilitetsfaktors	Integrabilitätsfaktors
623^{16}	Lie complexes	Lie semi-simple complexes
625^{24}	metrische homogene	metrisch homogenen
633_{13}	$\mathfrak{h} et$	$\mathfrak{h}_{\mathfrak{p}_0}$