SOLVABILITY OF INVARIANT DIFFERENTIAL OPERATORS ON HOMONOGEOUS MANIFOLDS

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§1. Introduction and summary.

Over a 100 years ago S. Lie [18] raised the following question:

Given a differential equation, how can knowledge about its invariance group be utilized towards its solution?

Special case. Consider the differential equation

\[(*) \quad \frac{dy}{dx} = \frac{Y(x,y)}{X(x,y)}\]

in the plane. It is called stable under a one-parameter group \( \psi_t \ (t \in \mathbb{R}) \) of diffeomorphisms if each \( \psi_t \) permutes the integral curves (all concepts are here local).

Example.

\[\left(\frac{dy}{dx} - \frac{Y}{x}\right) / \left(1 + \frac{y}{x} \frac{dy}{dx}\right) = x^2 + y^2.\]

The left hand side represents the tangent of the angle between the radius vector and the integral curve. Thus the equation shows that the integral curves intersect each circle \( x^2 + y^2 = r^2 \) under a fixed angle. Hence the group
of rotations around the origin permutes the integral curves, i.e., leaves the equation stable.

For a one-parameter group \( \phi_t \) of transformations in the plane let \( U \) denote the induced vector field

\[
U_p = \left( \frac{d(\phi_t \cdot p)}{dt} \right)_{t=0} = \xi(p)\frac{\partial}{\partial x} + \eta(p)\frac{\partial}{\partial y}, \quad p \in \mathbb{R}^2.
\]

Theorem (Lie). Equation \( (*) \) is stable under \( \phi_t \) if and only if the vector field \( Z = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} \) satisfies

\[
[U, Z] = \lambda Z \quad (\lambda \text{ a function})
\]

In this case \( (X\eta - Y\xi)^{-1} \) is an integrating factor for the equation \( Xdy - Ydx = 0 \).

Thus knowing a stability group for a differential equation provides a way to solve it. For the example above the equation is stable under the group

\[
\phi_t(x,y) = (x \cos t - y \sin t, x \sin t + y \cos t)
\]

for which

\[
U = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}
\]

and the theorem gives the solution

\[
y = x \tan \left( \frac{1}{2}(x^2 + y^2) + C \right)
\]

In spite of extensions of this theorem to higher order partial differential equations these methods have had
a limited influence on the general theory of differential equations. Nevertheless, Lie's question has been of enormous importance in mathematics, in fact leading to the theory of Lie groups.

Aiming for less generality we can pose a few problems in Lie's spirit. Let $G/H$ be a coset space, $H$ being a closed subgroup of a Lie group $G$ and let $D$ be a differential operator on the manifold $G/H$ which is invariant under $G$ in the sense that

$$D(f \circ \tau(g)) = (Df) \circ \tau(g) \quad f \in C^\infty(G/H)$$

for all $g \in G$, the mapping $\tau(g)$ denoting the translation $\tau(g):xH \rightarrow gxH$ of $G/H$ onto itself. Let $\circ$ denote the origin $eH$ of $G/H$. We can then ask the following questions.

A. Does there exist a fundamental solution for $D$, i.e., a distribution $T$ on $G/H$ such that

$$DT = \delta,$$

$\delta$ denoting the delta-distribution at $\circ$ on $G/H$?

B. Is $D$ locally solvable, i.e., is

$$DC^\infty(V) \supseteq C_c^\infty(V)$$

for some neighborhood $V$ of $\circ$?

C. Is $D$ globally solvable, i.e., is

$$DC^\infty(G/H) = C^\infty(G/H)$$?
Let $D(G/H)$ denote the algebra of all $G$-invariant differential operators on $G/H$. For example, if $G$ is the group of translations of $\mathbb{R}^n$ and $H = \{0\}$ then $D(G/H)$ consists of all differential operators with constant coefficients; by results of Ehrenpreis and Malgrange, questions A), B) and C) have positive answers in this case. We shall see however that in general the answers depend on the space $G/H$.

We shall now survey — in chronological order — some of the principal results which have been obtained for the questions above.

First if $X = G/K$ is a symmetric space of the noncompact type (i.e., $G$ noncompact, connected semisimple with finite center and $K$ a maximal compact subgroup) then as proved in [11] each $D \in D(G/K)$ has a fundamental solution, and consequently, by convolution, $D$ is locally solvable. Since in solving the equation $DT = \delta$ we may, by the $K$-invariance of $B$ and $\delta$, assume $T$ to be $K$-invariant, the problem can be handled within Fourier analysis of $K$-invariant functions on $X$. The global solvability discussed later does not seem to be accessible by this method.

It was proved in Peetre [19] (cf. also Hörmander [15], p. 170) that a differential operator of constant strength is necessarily locally solvable. Without entering into the precise definition, a differential operator of constant strength is in a certain sense a bounded perturba-
tion of a constant coefficient operator. One might wonder whether an invariant differential operator is not necessarily of constant strength. This can not be so because then it would be locally solvable and the operator
\[ \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} + ix \frac{\partial}{\partial z} \]
which is essentially the non-locally solvable Levi-operator is (cf. [4]) a left invariant differential operator on the Heisenberg group
\[
\begin{pmatrix}
1 & x & y \\
0 & 1 & z \\
0 & 0 & 1
\end{pmatrix}
\]
\(x, y, z \in \mathbb{R}.

Thus we see that \( B \) has in general a negative answer. In view of this, one can modify the problem and ask: For what Lie groups \( L \) are all left invariant differential operators (i.e., all members of \( D(L) = D(L/\{e\}) \)) locally solvable. Using Hörmander's necessary condition for the local solvability of a differential operator [15], p. 157, he, and independently Cèrezo-Rouvière [4], proved that in order that each \( D \in D(L) \) should be locally solvable it is necessary and sufficient that the Lie algebra of \( L \) be either abelian or isomorphic to the algebra of \( n \times n \) matrices where each row except the first is 0 (and thus has an abelian ideal of codimension 1).

The contrast between this negative result for \( L \) and the positive result for the symmetric space \( G/K \)
disappears when we view $L$ in the context of symmetric spaces. A coset space $B/C$ ($B$ a Lie group, $C$ a closed subgroup) is called a symmetric coset space if there exists an involutive automorphism $\sigma$ of $B$ with fixed point set $C$. The spaces $G/K$ considered above have this property.

Now we consider $L$ as a homogeneous space under left and right translations simultaneously, i.e., we let the product group $L \times L$ act on $L$ by

$$(g_1, g_2): g + g_1^{-1}g_2, \quad g \in L.$$ 

The subgroup leaving $e$ fixed is the diagonal $L^\# = L$ so we have the coset space representation

$$L = (L \times L)/L^\#.$$ 

Then the algebra $D(L \times L/L^\#)$ is canonically isomorphic to the algebra $Z(L)$ of bi-invariant differential operators on $L$ and the natural problem becomes: Given $D \in Z(L)$, is it locally (globally) solvable on $L$?

This problem was considered by Räis for simply connected nilpotent Lie groups $L$. He proved that each $D \in Z(L)$ has a fundamental solution and hence is locally solvable. The method is based on harmonic analysis on $L$.

Shortly afterwards [13] I proved local solvability for each $D \in Z(L)$ for the case when $L$ is semisimple. The proof was an easy consequence of a structure theorem of Harish-Chandra for the operators in $Z(L)$ but did not require any harmonic analysis on $L$. 
At about the same time I proved global solvability, 
$DC^\infty(G/K) = C^\infty(G/K)$ for each invariant differential operator 
$D$ on the symmetric space $G/K$. The proof [13] is based on 
a characterization of the image of $C^\infty_c(G/K)$ under the Radon 
transform on $G/K$.

What about global solvability on $L$? A remarkable 
example of Cérèzo-Rouvière [5] shows that for any complex 
semisimple Lie group $L$ there exists an element $w \in Z(L)$ 
which is not globally solvable. This $w$ can be taken as the 
imaginary part of the complex Casimir operator on $L$. But 
$L$ viewed as a real Lie group has a Casimir operator $C$

which they showed, using harmonic analysis on $L$, is globally 
solvable. This global solvability was more recently proved 
by Rauch-Wigner [21], for all non compact real semisimple 
Lie groups $L$, with finite center, using entirely different 
methods. The main step in their proof is verifying that no 
null bicharacteristic of $C$ lies over a compact subset of $G$.

The case of a solvable Lie group $L$ has so far been 
left out. Rais' method for the nilpotent case was extended 
by Duflo-Rais [7] to solvable $L$ and they proved that each 
$D \in Z(L)$ is locally solvable. An alternative proof was 
found by Rouvière [22], modifying in an interesting manner 
methods which Hörmander had introduced for constant 
coefficient operators. His method, being local, does not 
give the supplementary result of [7] that if $L$ is an 
exponential solvable Lie group then $D$ has a fundamental
solution.

The results I have quoted on the Lie groups $L$ and on the symmetric space $G/K$ suggest the following more general question.

Let $B/C$ be a symmetric coset space and $D$ a $B$-invariant differential operator on it. Is $D$ necessarily locally solvable?

§2. Solvability results. Indications of proofs.

We shall now indicate the proofs of the principal results already stated. We emphasize that the proofs of Theorems 1-5 are quite independent and so can be read in any order.

We have already mentioned the fact that a left-invariant differential operator on the Heisenberg group is not necessarily locally solvable. In order to describe the proof of Rais' positive result for nilpotent groups I recall the main results from harmonic analysis on nilpotent Lie groups [17].

Let $G$ be a simply connected nilpotent Lie group, $\mathfrak{g}$ its Lie algebra; $G$ acts on $\mathfrak{g}$ via the adjoint representation and by duality on $\mathfrak{g}^*$, the dual vector space to $\mathfrak{g}$. To each $f \in \mathfrak{g}^*$ is associated a unitary irreducible representation $\pi_f$ as follows: Let $H$ be a closed connected subgroup with the property that its Lie algebra
\( \mathfrak{h} \) satisfies \( f([\mathfrak{h}, \mathfrak{h}]) = 0 \). Assume \( H \) of maximum dimension with these properties. Then we can define a homomorphism \( \chi_f \) of \( H \) into the circle group \( T \) by

\[
\chi_f(\exp X) = e^{2\pi i f(X)}, \quad X \in \mathfrak{h},
\]

and let \( \pi_f \) denote the representation of \( G \) induced by \( \chi_f \).

It is known that \( \pi_f \) is independent of the choice of \( H \); moreover if \( f_1, f_2 \in \mathfrak{g}^* \), then \( \pi_{f_1} \) and \( \pi_{f_2} \) are unitarily equivalent if and only if \( f_1 \) and \( f_2 \) are in the same orbit of \( G \) acting on \( \mathfrak{g}^* \). Thus the orbit space \( \mathfrak{g}^*/G \) is identified with \( \hat{G} \), the dual space of \( G \). The character \( \chi_f \) of \( \pi_f \) is a distribution on \( G \) (Dixmier [6]) which is given by the formula

\[
(1) \quad \chi_f(\phi) = \int_{G \cdot f} (\phi \circ \exp) \tilde{f}(d\mu) \quad \phi \in C_c^\infty(G)
\]

where \( \tilde{f} \) denotes the Euclidean Fourier transform and \( \mu \) is a \( G \)-invariant measure on the orbit \( G \cdot f \). Finally there is a Plancherel formula: there exists a measure \( \lambda \) on \( \mathfrak{g}^*/G \) such that

\[
(2) \quad \phi(e) = \int_{\mathfrak{g}^*/G} \chi_f(\phi) d\lambda(f_0),
\]

\( f_0 \) denoting the orbit \( G \cdot f \). Using (2) and (1) for \( \phi \circ \exp = \tilde{\psi} \), we get for a Haar measure \( df \) on \( \mathfrak{g}^* \)
For a general Lie algebra \( \mathfrak{g} \) there is a bijection \( s \) of the symmetric algebra \( S(\mathfrak{g}) \) onto the universal enveloping algebra \( U(\mathfrak{g}) \); \( s \) is determined by the property \( s(x^m) = x^m \) for \( m \in \mathbb{Z}^+ \). Also \( s \) commutes with the action of \( G \) so maps the ring of \( G \)-invariants \( I(\mathfrak{g}) \subset S(\mathfrak{g}) \) onto the ring of \( G \)-invariants in \( U(\mathfrak{g}) \), the center \( Z(\mathfrak{g}) \) of \( U(\mathfrak{g}) \). Recall now that in terms of differential operators, \( U(\mathfrak{g}) = D(G) \) and \( Z(\mathfrak{g}) = Z(G) \). The character \( T_f \) is an eigendistribution of each \( Z \in Z(G) \)

\[
(4) \quad ZT_f = \hat{Z}(f)T_f
\]

and the eigenvalue \( \hat{Z}(f) \) is just the polynomial \( s^{-1}(Z) \) evaluated at \( f \).

Let us also recall the following general fact ([2], [3], [20]): Let \( P \geq 0 \) be a positive polynomial on \( \mathbb{R}^N \). Then if \( \text{Re}\ s > 0 \), \( P^s \) defines a distribution \( \phi \mapsto \int_{\mathbb{R}^N} P^s(x)\phi(x)dx \) on \( \mathbb{R}^N \), the mapping \( s \mapsto P^s \) extends to a meromorphic function on \( C \) with values in the space \( S'(\mathbb{R}^N) \) of tempered distribution on \( \mathbb{R}^N \), and the poles occur among the numbers \( -1/N, -2/N, \ldots \) (\( N \) being an integer depending on \( P \)). Moreover

\[
(5) \quad P^0 = 1.
\]
This can be used to construct a fundamental solution for a constant coefficient differential operator $D$ on $\mathbb{R}^n$:

Taking Fourier transform $f \to \hat{f}$ we have $(Df) = Pf$ where $P$ is a polynomial, which we may assume positive, replacing $D$ by $DD^*$ if necessary ($^*$ = adjoint). The tempered distribution

$$T_s: \phi \mapsto \int_{\mathbb{R}^n} P^s(x)\hat{\phi}(x)dx$$

then satisfies $T_1 \ast T_s = T_{s+1}$ which is holomorphic near $s = -1$. Expanding $T_s$ in a Laurent series near this point we get $T_s = \sum_{-\infty}^{\infty} a_k(s+1)^k$ where each $a_k$ is a distribution. The relation above then gives $T_1 \ast a_0 = \delta$ and up to a factor, $a_0$ is a fundamental solution for $D$.

We can now prove the following result (cf. [20]).

**Theorem 1.** Each bi-invariant differential operator $Z$ on a nilpotent Lie group $G$ is locally solvable.

We may assume $G$ simply connected and use the notation above. Replacing $Z$ by $Z\bar{Z}$ we may assume the polynomial $P = \hat{Z}(f)$ in (4) to be positive. Let $\xi$ denote the distribution $\phi \to (Z^*\phi)(e)$ ($^*$ = adjoint). Then by (2) - (4)

$$\xi(\phi) = \int_{\mathbb{Q}^*} \hat{Z}(f)(\phi \circ \exp)^{-1}(f)df.$$

In analogy with $T_s$ above we define $\xi_s$ for $\text{Re } s > 0$ by
\[ \xi_s(\phi) = \int_{q_s} \hat{Z}(f)^s(\phi \circ \exp)^{-1}(f)\,df. \]

By the result about \( P \) quoted, \( s + \xi_s \) extends to a meromorphic distribution-valued function on \( C \) and by (5) and (6),

\[ \xi_0 = \lim_{s \to 0} \xi_s = \delta, \quad \xi_1 = \xi. \]

Moreover, by (1) and (3),

\[ \xi_s(\phi) = \int_{q_s/G} \hat{Z}(f)^s T_\phi(\phi)\,d\lambda(f_0), \]

whence \( Z\xi_s = \xi_{s+1} \), i.e., \( \xi \ast \xi_s = \xi_{s+1} \). Again the term \( b_0 \) in the Laurent series \( \xi_s = \sum_{k=-\infty}^{\infty} b_k(s+1)^k \) gives a fundamental solution for \( Z \).

Duflo and Rais [7] extended this argument to solvable groups giving

**Theorem 2.** A bi-invariant differential operator \( D \) on a solvable Lie group is locally solvable.

A completely different proof was found by Rouvière [22]. We sketch his argument below.

Let \( X_1, \ldots, X_n \) be a basis of any Lie algebra \( \mathfrak{g} \) and let \( \mathfrak{d}_\mathfrak{g} \) denote the derived algebra \( [\mathfrak{g}, \mathfrak{g}] \). For any \( j, 1 \leq j \leq n \) let \( \partial_j \) denote the endomorphism of \( \mathfrak{U}(\mathfrak{g}) \) given by
Lemma 1. Assume \( X_j \notin Dq \). Then \( \partial_j \) is a derivation of \( U(q) \) and \( \partial_j Z(q) \subseteq Z(q) \).

In fact the restriction of \( \partial_j \) to \( q + \mathbb{R} \) vanishes on \( \mathbb{R} \) and extends to a derivation \( \partial \) of the tensor algebra \( T(q) \). Since \( \partial([q,q]) = 0 \) and \( \partial(X \otimes Y - Y \otimes X) = 0 \), \( \partial \) induces a derivation of \( U(q) \). Then (8) follows for \( \partial \) immediately by induction, so \( \partial = \partial_j \). Finally, if \( X \in q \), \( Z \in Z(q) \) then

\[
0 = \partial_j [X,Z] = [\partial_j X, Z] + [X, \partial_j Z] = 0 + [X, \partial_j Z]
\]

so

\[
\partial_j Z \in Z(q).
\]

Remark. The proof in [22] is quite different and proceeds by showing that in a suitable neighborhood of \( e \) in \( G \),

(9) \[
\partial_j P = [P, f_j] \quad P \in U(q)
\]

where \( f_j \) is a suitably chosen function which vanishes at \( e \). This will be used below.

Next we have the general identity for the \( L^2 \) norm,

\[
\|P'u\|^2 = -(P'u, fPu) + (P'u, f(P')u) - (P'u, (P'')u) + ([P', P'u], fu)
\]

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on an open set \( U \subseteq G \), \( u \in C^\infty_c(U) \), \( f \in C^\infty(U) \), \( P \) any differential operator on \( U \) with adjoint \( P^* \), where \( P' = [P,f] \), \( P'' = [P',f] \). Using this on \( P \in Z(H) \), \( f = f_j \) in (9) we get easily by Lemma 1 and (9)

\[
\|a_j Pu\|^2 \leq \|Pu\| (2C\|a_j Pu\| + \|a_j^2 Pu\|)
\]

where \( C = \sup |f_j| \). This implies by induction on the order \( m \) of \( P \)

(10) \[ \|a_j Pu\| \leq 2mC\|Pu\| \]

**Lemma 2.** Let \( r \) be an integer \( > 1 \) and \( P \in Z(H) \), \( P \notin Z(D^r H) \). Then there exists an operator \( Q \in Z(D^r H) \cap Z(D^{r-1} H) \), \( Q \notin 0 \) and a neighborhood \( U \) of \( e \) in \( G \) such that

(11) \[ \|Qu\| \leq \|Pu\| \]

for all \( u \in C^\infty_c(U) \).

This is proved by induction on \( r \). The case \( r = 1 \) comes from (10) and Lemma 1. In fact, let \( j = \dim Q - \dim DQ \). Then \( j \geq 1 \) and we take for \( Q \) a nonzero derivative of \( P \) of maximal order in \( X_1, \ldots, X_j \). Then \( Q \in Z(H) \) and \( Q \in U(DQ) \) so \( Q \in Z(H) \cap Z(DQ) \). For the passage from \( r \) to \( r+1 \) one uses the case \( r = 1 \) on the analytic group \( H \subseteq G \) with Lie algebra \( \mathfrak{h} = D^r Q \). This yields (11) after some analysis on the factor group \( G/H \).

Now if the group \( G \) is solvable let \( r \) be such
that \( D^r \alpha = \{0\} \) so \( Z(D^r \alpha) = c \). Then Lemma 2 gives an inequality

\[ \| u \| \leq C \| Pu \| \]

which as well known \([14]\) implies local solvability, i.e., Theorem 2.

Next we consider the semisimple case. Here we have by \([10]\),

Theorem 3. Each bi-invariant differential operator \( D \) on a semisimple Lie group \( G \) is locally solvable.

For this let \( U \) be a connected open neighborhood of \( 0 \) in \( \mathfrak{g} \) such that \( \exp \) is a diffeomorphism of \( U \) onto an open neighborhood \( U_0 \) of \( e \) in \( G \). In addition we assume, as we can (cf. \([9]\)) that \( U \) is completely invariant, that is, for each compact subset \( C \subset U \), the closure \( \text{Cl}(\text{Ad}(G) \cdot C) \subset U \). Let \( \eta \) denote the Jacobian of the exponential mapping, i.e.,

\[ \eta(X) = \det \left( \frac{1-e^{-\text{ad} X}}{\text{ad} X} \right) . \]

Then according to Harish-Chandra there exists a specific \( \text{Ad}(G) \)-invariant differential operator \( p_D \) on \( \mathfrak{g} \) with constant coefficients such that

\[ (12) \quad (Df) \circ \exp = \eta^{-1/2} p_D(\eta^{1/2} f \circ \exp) \]

whenever \( f \) is a \( C^\infty \) function on \( U_0 \), locally invariant under inner automorphisms. (This formula is proved in \([9]\)
by computing the radial parts of the operators $D$ and $p_D$; for an analogous global formula for the symmetric space $G/K$ (with $G$ complex) see [11]).

Denoting by $A$ the mapping

$$A : f \mapsto \eta^{1/2}(f \circ \exp) \quad f \in \mathcal{C}^\infty(U_\mathfrak{g})$$

we have by (12),

(13) \quad ADf = p_DAf.

If $G$ is compact, the $\text{Ad}(G)$-invariant distributions on $U_\mathfrak{g}$ form in a natural way the dual to the space of Ad($G$)-invariant functions in $\mathcal{C}^\infty(U_\mathfrak{g})$. The adjoint of $A$ is given by $A^*S = (\eta^{1/2}S)^\exp$, the image of the distribution $\eta^{1/2}S$ under $\exp$, so by (13)

(14) \quad D(\eta^{1/2}S)^\exp = (\eta^{1/2}(p_D^*)^*S)^\exp$

if $S$ is an $\text{Ad}(G)$-invariant distribution on $U_\mathfrak{g}$. Harish-Chandra proved ([8], [9], p. 477), that (14) is true even if $G$ is noncompact; since the process of averaging over $G$ is not available, entirely different methods are required. Now $(p_D^*)^*$ has a fundamental solution $S$ on $\mathfrak{g}$ which we may take $\text{Ad}(G)$-invariant (cf. the construction of $w_0$ before Theorem 1). Using (14) we get a fundamental solution for $D$ on $U_\mathfrak{g}$, $DT = \delta$, and Theorem 3 follows by convolution.

Global solvability does not in general hold in Theorem 3 as shown by the following example of Cerèzo-Rouvière [5]. Let $G$ be a complex semisimple Lie group,
K a maximal compact subgroup and \( \mathfrak{g}, \mathfrak{k} \) their respective Lie algebras. Then \( \mathfrak{g} = \mathfrak{k} + J\mathfrak{k} \) where \( J \) is the complex structure of \( \mathfrak{g} \). Let \( (T_i) \) be a basis of \( \mathfrak{k} \), orthonormal for the negative of the Killing form, and put

\[
\omega = \sum_i (JT_i)T_i.
\]

Then if \( T \in \mathfrak{k} \), \( [T, T_i] = \sum J c_{ij} T_j \), where \( (c_{ij}) \) is skew, so

\[
[T, \omega] = \sum_i [T, JT_i]T_i + (JT_i)[T, T_i] \]

and

\[
= \sum_{i,j} c_{ij} (JT_j)T_i + \sum_{i,j} c_{ij} (JT_i)T_j
\]

\[
= \sum_{i,j} c_{ij} (JT_j)T_i - \sum_{i,j} c_{ij} (JT_i)T_j
\]

\[
= \sum_{i,j} c_{ij} (JT_j)T_i - \sum_{i,j} c_{ij} (JT_i)T_j = 0.
\]

Similarly \( [JT, \omega] = 0 \) since \( \omega \) can also be written \( \omega = \sum T_i (JT_i) \). Thus \( \omega \) is a bi-invariant differential operator on \( G \). Since it annihilates all \( C^\infty \) functions on \( G \) which are right invariant under \( K \) it is not globally solvable.

Nevertheless we have by [5] for complex \( G \) and by [21] for real \( G \), Theorem 4. Let \( G \) be a connected noncompact semisimple Lie group with finite center. Then the Casimir operator \( C \) is globally solvable.

This follows from general results of Hörmander [16] once the following three facts have been established [21].
(I) $C$ is one-to-one on $C_0^\infty(G)$.

(II) For any compact set $\Gamma \subset G$ there exists another compact set $\Gamma' \subset G$ such that $\Gamma \subset \text{Int} \Gamma'$ and if $u \in E'(G)$ (distributions on $G$ with compact support) with $\text{supp}(Cu) \subset \Gamma$ then $\text{supp} u \subset \Gamma'$.

(III) No null characteristic of $C$ lies over a compact subset of $G$.

The Holmgren uniqueness theorem implies for the analytic operator $C$ that if a hyper-surface $S$ in $G$ is non-characteristic at $x_0$ then a solution $u$ to $Cu = 0$ which vanishes on one side of $S$ must vanish in a neighborhood of $x_0$ in $G$. This implies (I) and (II) without difficulty because it is easy to determine plenty of non-characteristic surfaces for $C$.

In order to describe the proof of (III) let us recall a few facts concerning null bicharacteristics.

Let $M$ be a manifold, $T^*M$ its cotangent bundle. If $(x_1, \ldots, x_n)$ is a local coordinate system on $M$ then

$$(m, \omega_m) \rightarrow (x_1(m), \ldots, x_n(m), \omega_m(\frac{\partial}{\partial x_1}), \ldots, \omega_m(\frac{\partial}{\partial x_n}))$$

is a local coordinate system $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ on $T^*(M)$. The two-form

$$\Omega = \sum_{i=1}^n d\xi_i \wedge dx_i$$

is called the canonical 2-form on $T^*M$. It has the property
that if \( \phi \) is a diffeomorphism of \( \mathcal{M} \) then the lifted
diffeomorphism of \( T^*\mathcal{M} \) preserves \( \Omega \) (cf. e.g. [1], p. 97).
Given a 1-form \( \omega \) on \( T^*\mathcal{M} \) let \( X_\omega \) denote the vector field
on \( T^*\mathcal{M} \) given by
\[
\Omega(\cdot, X_\omega) = \omega(\cdot),
\]
If \( \omega = \sum a_i dx_i + \sum b_j d\xi_j \), then
\[
2X_\omega = \sum b_j \frac{\partial}{\partial x_i} - \sum a_i \frac{\partial}{\partial \xi_j}.
\]
If \( \omega = df \) where \( f \in C^\infty(T^*\mathcal{M}) \) then \( X_{df} \) is called a
Hamiltonian vector field; the function \( f \) is easily seen to
be constant on the integral curves of \( X_{df} \). In fact, if
\((x_1(t), \ldots, \xi_n(t))\) is an integral curve of \( X_{df} \) then by
(16),
\[
\frac{dx_k}{dt} = \frac{1}{2} \frac{\partial f}{\partial x_k}, \quad \frac{d\xi_k}{dt} = -\frac{1}{2} \frac{\partial f}{\partial \xi_k}
\]
so
\[
\frac{df}{dt}(x_1(t), \ldots, \xi_n(t)) = 0.
\]
In particular, let \( c: T^*\mathcal{G} \rightarrow \mathbb{R} \) be the principal
symbol of \( c \), i.e., \( c(\xi, \xi) = \sum_{i,j} \varepsilon_{ij}^I \xi_i^I \xi_j^I \) if \( \varepsilon_{ij}^I \) is the
metric tensor determined by the Killing form. The
bicharacteristics of \( c \) are by definition the integral
curves in \( T^*\mathcal{G} - (0) \) to the Hamiltonian vector field \( X_{dc} \).
If the constant value of \( c \) on an integral curve of \( X_{dc} \)
is 0, this curve is called a null bicharacteristic.

By the invariance property of \( \Omega \) stated above it
is clear that $\Omega$, as a 2-form on $T^*G$, is bi-invariant. Since $C$ is bi-invariant on $G$, $c$ is a bi-invariant function on $T^*G$ so $X_{dc}$ is a bi-invariant vector field on $T^*G$. Using in addition the fact that $X_{dc}$ is Hamiltonian the authors prove that its integral curves have the form

$$\gamma(t) = (a \exp tX, \omega_a \exp tX)$$

where $a \in G$, $X \in \mathfrak{g}$ and $\omega$ a left invariant 1-form on $G$.

Now suppose $\gamma$ is a null bicharacteristic for $C$ lying over a compact subset of $G$. Then the same is true of the curve

$$\gamma_0(t) = (\exp tX, \omega \exp tX)$$

so $c(\gamma_0(t)) \equiv 0$ and $\exp tX$ lies in a maximal compact subgroup $K$ of $G$. But then $B(X,X) < 0$ implies easily that $c$ is strictly negative on $\gamma_0$, which is a contradiction.

Finally, we have by [11] and [13],

**Theorem 5.** Let $X = G/K$ be a symmetric space of the non-compact type, $D$ a $G$-invariant differential operator on $X$. Then

(i) $D$ has a fundamental solution.

(ii) $D$ is globally solvable.

This theorem can be proved either by means of the Fourier transform on $X$ or by means of the Radon transform on $X$. After (i) is proved, (ii) follows on the basis of general functional analysis methods provided one can prove for each ball $B \subset X$: 
Here "supp" stands for "support".

We now sketch how (1) and (17) can be proved by means of the Radon transform on $X$. Let $G = KAN$ be an Iwasawa decomposition of $G$ where $A$ is abelian and $N$ nilpotent. A horocycle in $X$ is by definition an orbit of a point in $X$ under a subgroup of $G$ conjugate to $N$. The group $G$ permutes the horocycles transitively; more precisely, if $\xi_0$ denotes the horocycle $N \cdot o$ ($o$ = origin in $X$) then each horocycle $\xi$ can be written $\xi = ka \cdot \xi_0$ where $a \in A$ is unique and $k \in K$ is unique up to a right multiplication by an element $m$ in the centralizer $M$ of $A$ in $K$. Thus the space $\mathcal{E}$ of all horocycles is naturally identified with $(K/M) \times A$. Each horocycle $\xi$ is a submanifold of $X$ so inherits a volume element $d\sigma$ from $X$; the Radon transform of a function $f$ on $X$ is defined by

$$\hat{f}(\xi) = \int_{\xi} f(x) d\sigma(x), \quad (\xi \in \mathcal{E})$$

whenever this integral exists. There exists a $G$-invariant differential operator $\hat{D}$ on the manifold $\mathcal{E}$ such that

$$(\hat{D}f)^\wedge = \hat{D}\hat{f} \quad f \in C_0(X);$$

moreover $\hat{D}$ has the form

$$(\hat{D}f)(ka \cdot \xi_0) = P_a(\hat{f}(ka \cdot \xi_0)) \quad k \in K, a \in A),$$

where $P$ is a translation-invariant differential operator on the Euclidean space $A$ (cf. [10], §4-5). Thus the Radon
transform converts the operator \( D \) into a constant coefficient differential operator. This principle gives a fundamental solution for \( D \) by a method similar to the one used for Theorem 3 (cf. [11]).

In order to use this principle to deduce (17) the following lemma is decisive.

**Lemma 3.** Let \( f \in C_c^\infty(X) \) and \( B \subset X \) a closed ball. Assume that \( \hat{f}(\xi) = 0 \) whenever the horocycle \( \xi \) is disjoint from \( B \). Then \( f \) vanishes identically outside \( B \).

To verify (17) we may assume \( B \) centered at \( o \). Let \( R \) denote its radius. Now \( \text{supp}(Df) \subset B \) implies by (18) and (19) and \( d(o,ka \cdot \xi_o) \geq d(o,a \cdot o) \) that

\[
P_a(\hat{f}(ka \cdot \xi_o)) = 0 \quad \text{if} \quad d(o,a \cdot o) > R,
\]

\( d \) denoting distance. But the function \( a \rightarrow \hat{f}(ka \cdot \xi_o) \) has compact support so by the Lions-Titchmarsh convexity theorem we conclude that \( \hat{f}(ka \cdot \xi_o) = 0 \) if \( d(o,a \cdot o) > R \). But then Lemma 3 implies \( \text{supp}(f) \subset B^R(\varphi) \) so (17) is verified, proving Theorem 5.

Plausible as Lemma 3 looks, its proof (cf. [13]) is too technical to describe here. Instead, I sketch a proof of the analogue of Lemma 3 for \( \mathbb{R}^n \). Here the Radon transform \( \hat{F} \) of a function \( F(x) \) is defined by

\[
\hat{F}(\xi) = \int_{\xi} F(x) d\sigma(x),
\]

\( \xi \) being an arbitrary hyperplane and \( d\sigma \) being the surface element on it.
Lemma 4. Let $F \in C^\infty(\mathbb{R}^n)$ satisfy the conditions:

(i) For each integer $k \geq 0$, $F(x)|x|^k$ is bounded.

(ii) There exists a constant $A$ such that $\hat{F}(\xi) = 0$ for $d(0,\xi) > A$, $d$ denoting distance.

Then $F$ vanishes identically outside the ball $|x| \leq A$.

Proof (cf. [12]). Suppose first $F$ is a radial function, i.e., $F(x) = \phi(|x|)$ where $\phi \in C^\infty(\mathbb{R})$ is even. Then there exists an even function $\hat{\phi} \in C^\infty(\mathbb{R})$ such that $\hat{\phi}(d(0,\xi)) = \hat{F}(\xi)$. Now (20) takes the form

$$
(21) \quad \hat{\phi}(p) = \int_{\mathbb{R}^{n-1}} \phi((p^2 + |y|^2)^{1/2})dy = A_{n-1} \int_0^\infty \phi(p^2 + t^2)t^{n-2}dt,
$$

where $A_{n-1}$ is the area of the unit sphere in $\mathbb{R}^{n-1}$. By a simple change of variables (21) is transformed into Abel's integral equation and is inverted by

$$
\phi(s^{-1})s^{-n} = c \left( \frac{d}{ds} \right)^{n-1} \int_0^s (s^2 - u^2)^{1/2}u^{n-3}u^{-2}\phi(u^{-1})du,
$$

where $c$ is a constant. Now by (ii), $\hat{\phi}(u^{-1}) = 0$ for $0 < u \leq A^{-1}$ so $\phi(s^{-1}) = 0$ for $0 < s \leq A^{-1}$, proving the lemma for the case when $F$ is radial.

Consider now the general case. Fix $x \in \mathbb{R}^n$ and consider the function
\[ g_x(y) = \int_{O(n)} F(x+k \cdot y) dk, \quad y \in \mathbb{R}^n, \]

where \( dk \) is the normalized Haar-measure on the orthogonal group \( O(n) \); \( g_x(y) \) is the average of \( F \) on the sphere with center \( x \) and radius \( |y| \). We have

\[ \hat{g}_x(\xi) = \int_{O(n)} \hat{F}(x+k \cdot \xi) dk, \]

where \( x+k \cdot \xi \) is the hyperplane \( k \cdot \xi \) translated by \( x \). Clearly

\[ d(0, x+k \cdot \xi) \geq d(0, \xi) - |x| \]

so if \( d(0, \xi) > |x| + A \) we have \( \hat{g}_x(\xi) = 0 \). But \( g_x \) is a radial function so by the first part of the proof,

\[ g_x(y) = 0 \quad \text{for} \quad |y| \geq |x| + A. \]

This means that if \( S \) is a sphere with surface element \( d\omega \) then

\[ \int_S F(s) d\omega(s) = 0 \]

if \( S \) encloses the solid ball \( B^A(0) \) (with center \( 0 \) and radius \( A \)), Lemma 4 will therefore follow from another lemma.

**Lemma 5.** Let \( F \in C^w(\mathbb{R}^n) \) such that for each \( k \geq 0 \), \( F(x)|x|^k \) is bounded. Assume (22) whenever \( S \) encloses the solid ball \( B^A(0) \). Then \( F \) vanishes identically outside \( B^A(0) \).
The idea for proving this is to perturb $S$ in (22) a bit and differentiate with respect to the parameter of perturbation, thereby obtaining additional relations. If $S = S^R(x)$ and $B^R(x)$ the corresponding ball, then by (22),

$$
\int_{B^R(x)} F(y) dy = \int_{R^n} F(y) dy
$$

so

$$
\frac{\partial}{\partial x_1} \int_{B^R(x)} F(y) dy = 0,
$$

that is

(23) \quad \int_{B^R(0)} \frac{\partial}{\partial y_1} F(y) dy = 0.

Using the divergence theorem on the vector field

$$
V(y) = F(y)e_1
$$

we obtain from (23)

$$
\int_{S^R(0)} F(x+s) \frac{s_1}{R} \omega(s) = 0.
$$

Combining this with (22) we deduce

$$
\int_{S^R(x)} F(s)s_1 \omega(s) = 0.
$$

By iteration
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\[ \int_{S^R(x)} F(s)p(s)d\omega(s) = 0 \]

if \( p(s) \) is an arbitrary polynomial, so the lemma follows.

Remark. The function \( F(z) = \frac{1}{z^k} \) (\( z \in \mathbb{C} \)) smoothed out near the origin satisfies (by Cauchy's theorem) assumption (ii) in Lemma 4. Thus condition (i) cannot be weakened. Similar remark applies to Lemma 5.

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