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Orbital Integrals, Symmetric Fourier Analysis and Eigenspace Representations

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Abstract. These lectures give an informal exposition of the three topics in the title. Although the topics are closely related and share notational conventions the three sections should be readable independently.

The first section describes Harish-Chandra's Plancherel formula for semisimple Lie groups G which is based on the study of the integrals of functions over conjugacy classes in G. The second section deals with the Fourier transform on the symmetric space $X = G/K$ associated with G and selected applications of this transform to differential equations. In the last section we discuss irreducibility properties of representations of G on eigenspaces of invariant differential operators on various homogeneous spaces of G.

§1. Orbital Integrals and Plancherel Formula

1. The Plancherel Formula.

In this section I shall attempt to describe in an informal way the approach to the Plancherel formula on a semisimple Lie group via orbital integral theory.

The following notation will be used: $\mathbf{C}$, $\mathbf{R}$, and $\mathbf{Z}$ denote the complex number, real numbers and integers, $\mathbf{R}^+ = \{t \in \mathbf{R} : t \geq 0\}$ and $\mathbf{Z}^+ = \mathbf{R}^+ \cap \mathbf{Z}$. For a topological space X, $C(X)$ and $C_c(X)$ denote the spaces of continuous functions, the subscript c denoting compact support. If X has a metric d, $B_r(X)$ denotes the ball $\{y \in X : d(x,y) < r\}$ and $S_r(X)$ the sphere $\{y \in X : d(x,y) = r\}$. If X is a manifold we use the notation $\mathcal{E}(X)$ for $C^\infty(X)$ and $\mathcal{D}(X)$ for $C_c^\infty(X)$.

Lie groups will be denoted by capital letters, $A, B, \ldots$ and the corresponding Lie algebras by corresponding German letters $\mathfrak{a}, \mathfrak{b}, \ldots$. The adjoint representations of A and $\mathfrak{a}$ are denoted by Ad_A (or Ad and $ad_{\mathfrak{a}}$ (or ad)).

If A is an abelian group with character group $\widehat{A}$ (and Haar measure dx) the Fourier transform on A is given by (see [**W**])

$$\tilde{f}(\chi) = \int_A f(x)\chi(x^{-1})\,dx \qquad \chi \in \widehat{A}\,. \tag{1}$$

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With the Haar measure $d\chi$ suitably normalized one has the inversion formula

$$f(x) = \int_{\widehat{A}} \tilde{f}(\chi)\chi(x)\,d\chi \tag{2}$$

for f in a suitable dense subspace of $L^1(A)$. We can write (2) in the form

$$\delta = \int_{\widehat{A}} \chi\,d\chi\,, \tag{3}$$

where δ is the delta function of A at e and χ denotes the measure $f \to \tilde{f}(\chi)$ on A.

Next let G be a semisimple Lie group with finite center, $\widehat{G}$ the set of equivalence classes of irreducible unitary representations of G. With $\pi \in \widehat{G}$ operating in the Hilbert space $\mathcal{H}_\pi$ the *Fourier transform* of a function f on G is defined by

$$\tilde{f}(\pi) = \int_G f(x)\pi(x)\,dx\,, \tag{4}$$

dx being the Haar measure on G. Thus the Fourier transform assigns to f a family of operators on different Hilbert spaces $\mathcal{H}_\pi$. For π in the principal series of the complex classical groups, Gelfand and Naimark showed [**GN**] that π always had a character defined almost everywhere on the group. This was completed by Harish-Chandra [**HC1**] where he showed for any G and any $f \in C_c^\infty(G)$, $\pi \in \widehat{G}$ the operator $\tilde{f}(\pi)$ has a trace $\chi_\pi(f)$ and that the functional $f \to \chi_\pi(f)$ is a distribution on G, the *character* of π. The principal step in his proof is showing that if K is a maximal compact subgroup of G and $\delta \in \widehat{K}$ then the restriction $\pi|K$ contains δ at most $N \dim\delta$ times (N being a constant). The objective is then, in analogy with (2), to find a measure $d\pi$ on $\widehat{G}$ such that

$$f(e) = \int_{\widehat{G}} \chi_\pi(f)\,d\pi\,, \qquad f \in C_c^\infty(G)\,. \tag{5}$$

If we use this on the function $f * f^*$ where $f^*(x) = \overline{f(x^{-1})}$ we would get

$$\int_G |f(x)|^2\,dx = \int_{\widehat{G}} \chi_\pi(f * f^*)\,d\pi\,.$$

However $\chi_\pi(f * f^*) = \operatorname{Tr}\left(\tilde{f}(\pi)\tilde{f}(\pi)^*\right) = \|\tilde{f}(\pi)\|^2$ ($\|\ \|$ = Hilbert-Schmidt norm) so we have the Plancherel formula

$$\int_G |f(x)|^2\,dx = \int_{\widehat{G}} \|\tilde{f}(\pi)\|^2\,d\pi\,. \tag{6}$$

The existence of a measure $d\pi$ satisfying (6) (for G locally compact unimodular and $\|\ \|$ a more abstract operator algebra norm) had been proved by Segal [**Se**]. However (5) is a more precise decomposition of $\delta = \delta_e$ into characters:

$$\delta = \int_{\widehat{G}} \chi_\pi\,d\pi\,. \tag{7}$$

2. Compact Groups.

Following Harish-Chandra let us restate the Peter-Weyl theorem in this framework for G compact and simply connected. It can be written

$$f = \sum_{\pi \in \widehat{G}} d_\pi (\chi_\pi * f) \qquad f \in C^\infty(G) \tag{8}$$

where d_π is the degree and $*$ denotes convolution. Let $T \subset G$ denote a maximal torus in G and $\mathfrak{t} \subset \mathfrak{g}$ their Lie algebras, $\mathfrak{t}^c$, $\mathfrak{g}^c$ their complexifications. Consider the weight lattice

$$\Lambda = \left\{ \lambda \in \mathfrak{t}_c^* : 2\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbf{Z} \quad \text{for} \quad \alpha \in \Delta \right\}$$

$\langle , \rangle$ denoting the Killing form and Δ the set of roots of $(\mathfrak{g}^c, \mathfrak{t}^c)$. Under the bijection $\mu \to e^\mu$ $\big(e^\mu(\exp H) = e^{\mu(H)} (H \in \mathfrak{t})\big)$ the lattice Λ is identified with the character group $\widehat{T}$. By the highest weight theory the set $\Lambda(+)$ of dominant weights is identified with $\widehat{G}$ so we have with Δ^+ the set of positive roots,

$$\widehat{G} = \Lambda(+) = \left\{ \lambda \in \Lambda : \langle \lambda, \alpha \rangle \geq 0 \quad \text{for} \quad \alpha \in \Delta^+ \right\} = \Lambda/W = \widehat{T}/W\,,$$

where W denotes the Weyl group.

If π has highest weight λ we have Weyl's formulas

$$d_\pi = \prod_{\alpha \in \Delta^+} \frac{\langle \alpha + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}\,, \qquad \chi_\pi(\exp H) = \frac{\Sigma_s \det s\, e^{s(\lambda+\rho)(H)}}{\Sigma_s \det s\, e^{s\rho(H)}}\,, \tag{9}$$

where s runs through W. Denoting the denominator in the χ_π formula by $D(t)$ $(t = \exp H)$ we have the integral formula

$$|W| \int_G f(g)\, dg = \int_T |D(t)|^2 \int_G f(gtg^{-1})\, dg\, dt\,. \tag{10}$$

We consider now the orbital integral

$$F_f(t) = D(t) \int_G f(gtg^{-1}) dg\,, \qquad t \in T\,.$$

Integrating $f(g)\chi_\lambda(g)$ over G we obtain from (9) and (10) for the Fourier coefficients of F_f

$$\chi_\lambda(f) = c'(F_f)^\sim(\lambda + \rho) \qquad (c' = \text{const.})\,, \tag{11}$$

a formula which relates the Fourier transforms on G and on T. If Ω denotes the differential operator $\Pi_{\alpha \in \Delta^+} H_\alpha$ ($H_\alpha \in \mathfrak{t}^c$ corresponding to α) we obtain from (9)

$$(\Omega F_f)^\sim (\lambda + \rho) = c'' d_\lambda F_f^\sim(\lambda + \rho)\,,$$

where $d_\lambda = d_\pi$ and c'' another constant. This (5) implies the orbital integral formula

$$f(e) = c(\Omega F_f)(e) \qquad (c = \text{const}) \tag{12}$$

The point is now, that since $D(t)$ vanishes at $t = e$ to the same order as the degree of Ω, formula (12) is quite immediate. Going backwards one can then derive the Peter-Weyl decomposition (8).

3. Orbital Integrals.

The orbital integral problem amounts to determining the value of a function on G at e in terms of its integrals over (generic) conjugacy classes. For $G = \mathbf{SL}(n, \mathbf{C})$ this problem was solved by Gelfand and Naimark [**GN**] and used to prove the Plancherel formula (6). Having determined the characters of the principal series by a formula analogous to (9), the method above for G compact illustrates their underlying idea for proving (6) for $\mathbf{SL}(n, \mathbf{C})$. Their method was shortly afterwards generalized by Harish-Chandra to all complex semisimple G [**HC2**].

Recognizing the importance of the orbital integral problem Gelfand and Graev [**GG1**] found in 1953 a new elegant solution for all complex classical G. We sketch this method.

Consider a quadratic form $\omega(x_1, \dots, x_m)$ and the generalized Riesz integral

$$R(\lambda) = \int_{\omega \geq 0} f(x_1, \dots, x_m)\omega(x_1, \dots, x_m)^{\lambda/2}\, dx \qquad \lambda \in \mathbf{C}$$

Assume ω has signature (s, t) with s odd, t even. Then $R(\lambda)$ is meromorphic in $\mathbf{C}$ with simple poles at

$$\lambda = m - 2k \quad (k = 0, 1, 2, \dots)$$

and corresponding residues

$$\operatorname{Res}_{\lambda=m-2k} R(\lambda) = c(L^k f)(0)\,, \tag{13}$$

where $c \neq 0$ is a constant and L the Laplacian.

If $H \subset G$ is a Cartan subgroup with Lie algebra $\mathfrak{h}$ let Δ be determined by

$$\int_G f(g)\, dg = \int_H \Delta(h)^2 \left(\int_{G/H} f(ghg^{-1})\, dg_H \right) dh$$

so

$$\int_G f(g)\, dg = \int_H F_f(h)\Delta(h)\, dh\,, \tag{14}$$

where F_f is the orbital integral

$$F_f(h) = \Delta(h) \int_{G/H} f(ghg^{-1})\, dg_H\,. \tag{15}$$

Let f have support in a neighborhood of e which is a diffeomorphic image of a neighborhood of O in $\mathfrak{g}$ under exp. Let $J(X) = dg/dX$ and ω the Killing form. Then we have by (14)

$$\int_{\mathfrak{g}} f(\exp X)\omega(X, X)^{\lambda/2} J(X)\, dX = \int_{\mathfrak{h}} F_f(\exp H)\Delta(\exp H)\omega(H, H)^{\lambda/2}\, dH\,. \tag{16}$$

In order to use (13) we actually need two such formulas, namely for $\omega \geq 0$ and for $\omega < 0$. Ignoring this complication let us use (16) for $G = \mathbf{SL}(n, \mathbf{C})$, calculating residues for $\lambda = -\dim G$ $(= -2n^2 + 2)$. On the left hand side we get $cf(e)$ $(c \neq 0)$ and since $\dim \mathfrak{h} = 2n - 2 = \dim \mathfrak{g} - 2(n(n-1))$ we get on the right hand side

$$\left(L^{n(n-1)}(F_f \Delta) \right)(e)\,.$$

After some manipulation this gives the orbital integral formula

$$f(e) = (\Omega F_f)(e)\,, \tag{17}$$

where Ω is an explicit invariant differential operator on H. A minor complication is that the necessary parity condition on (s,t) in the residue theorem is not satisfied here. This is remedied by going over to the group $G_1 = \{g \in \mathbf{SL}(n,\mathbf{C}) : \det g \in \mathbf{R}\}$ of dimension $2n^2 - 1$.

In [**HC4**] Harish-Chandra extended (17) to all semisimple G, real or complex. (See also [**HC3**] for $\mathbf{SL}(2,\mathbf{R})$ and Gelfand-Graev [**GG2**] for $\mathbf{SL}(n,\mathbf{R})$). For this theorem H has to be a fundamental Cartan subgroup, that is one for which the compact part has maximal dimension. This method was based on the Fourier transform on $\mathfrak{g}$, relative to the indefinite form ω. In [**HC7**], (1975) he gave yet another method based on fundamental solutions for powers of the Casimir polynomial ω on $\mathfrak{g}$. I shall indicate this method for the case (including G complex) when all the Cartan subgroups are conjugate.

Let $m = [\dim \mathfrak{g}/2]$ and Ξ a specific fundamental solution of $\partial(\omega^m)$. Let $\delta_{\mathfrak{g}}$, $\delta_{\mathfrak{h}}$ denote the delta functions on $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and $\Xi_{\mathfrak{h}}$, $\omega_{\mathfrak{h}}$ the restrictions of Ξ and ω to $\mathfrak{h}$. Then not only is

$$\partial(\omega^m)\Xi = \delta_{\mathfrak{g}} \tag{18}$$

but if $k = [\dim \mathfrak{h}/2]$ we have also

$$\partial(\omega_{\mathfrak{h}}^k)\Xi_{\mathfrak{h}} = c\delta_{\mathfrak{h}} \qquad c \neq 0 \text{ constant}\,. \tag{19}$$

Given a function f on $\mathfrak{g}$ we consider the Lie algebra orbital integral

$$\varphi_f(Z) = \epsilon(Z)\pi(Z)\int_{G/H} f(g\cdot Z)\,dg_H\,, \quad Z \in \mathfrak{h}\,, \tag{20}$$

where π is the product of the positive roots and $\epsilon(Z)$ is locally constant (changes from one Weyl chamber to another). From (18) we have

$$f(0) = \int_{\mathfrak{g}} \left(\Xi\partial(\omega^m)f\right)(X)\,dX \tag{21}$$

and since $dX = \pi(Z)^2 dg_H\, dZ$ we deduce from (20)

$$f(0) = \text{const}\int_{\mathfrak{h}} \Xi_{\mathfrak{h}}(Z)\pi(Z)\phi_{f_m}(Z)\,dZ\,, \tag{22}$$

where $f_m = \partial(\omega^m)f$. However,

$$\phi_{f_m} = \partial(\omega_{\mathfrak{h}}^m)\phi_f \tag{23}$$

and a direct calculation shows that

$$\pi \circ \partial(\omega_{\mathfrak{h}}^m) = \partial(\omega_{\mathfrak{h}}^k) \circ \eta\,, \tag{24}$$

where η is a differential operator on $\mathfrak{h}$ with polynomial coefficients whose local expression η_0 at O is a constant multiple of $\partial(\pi)$. Substituting (23) into (22) we get

$$f(0) = \text{const}\int_{\mathfrak{h}} \Xi_{\mathfrak{h}}(Z)\partial(\omega_{\mathfrak{h}}^k)(\eta\phi_f)(Z)\,dZ$$

so by (24) and the mentioned formula $\eta_0 = \text{const.}\,\partial(\pi)$ we get

$$f(0) = C\,(\partial(\pi)\phi_f)\,(0) \qquad C = \text{const.} \tag{25}$$

Separate arguments are then needed to lift this formula to the group.

For G complex the principal series $\pi_{\widehat{h}}$ is parameterized by characters $\widehat{h}$ of the Cartan subgroup $H \subset G$. Because of the formula indicated for the character $T_{\widehat{h}}$ of $\pi_{\widehat{h}}$ one has in analogy with (11),

$$T_{\widehat{h}}(f) = (\widetilde{F}_f)(\widehat{h}) \tag{26}$$

and since $(\Omega F_f)^{\sim} = \widetilde{\Omega}\widetilde{F}_f$ ($\widetilde{\Omega}$ a certain polynomial on $\widehat{H}$) the limit formula (17) implies the Plancherel formula

$$\delta = \int_{\widehat{H}} T_{\widehat{h}}\widehat{\Omega}(\widehat{h})\,d\widehat{h}\,. \tag{27}$$

The argument of course relies on the smoothness of F_f and this remains valid if G has just one conjugacy class of Cartan subgroups. The argument for (27) can still be carried out (with $\widetilde{\Omega}$ a polynomial).

Dropping the assumption of all Cartan subgroups being conjugate presents major obstacles which were eventually overcome by Harish-Chandra [**HC6**] through the results sketched below.

While (17) remains valid for general G provided H is fundamental its use is complicated by the fact that F_f is no longer smooth, in fact its jumps are related to orbital integrals for other Cartan subgroups (see (32) below for $\mathbf{SL}(2,\mathbf{R})$.

Let $H_1, \ldots, H_r$ be a maximal family of nonconjugate Cartan subgroups, all invariant under a fixed Cartan involution of G. Then (14) is replaced by

$$\int_G f(g)\,dg = \sum_i \frac{1}{|W_i|}\int_{H_i} F_f^i(h)\Delta_i(h)\,dh\,, \tag{28}$$

where F_f^i is the orbital integral relative to the group H_i and W_i is the corresponding Weyl group. The Jacobians $|\Delta_i|$ coincide on the complexifications of the H_i. If θ is a conjugacy invariant distribution which equals F on the regular set of G then

$$\theta(f) = \sum_i \frac{1}{|W_i|}\int_{H_i} F_f^i(h)\Delta_i(h)F(h)\,dh\,. \tag{29}$$

Let H be one of the H_i, A its vector part (with Lie algebra $\mathfrak{a}$) and $P = MAN$ a parabolic subgroup for which MA is the centralizer of A in G and $M \cap A = \{e\}$. With an arbitrary character $a \to e^{i\lambda(\log a)}$ of A ($\lambda \in \mathfrak{a}^*$), an arbitrary discrete series representation σ of M we obtain a representation $\pi_{\sigma,\lambda}$ of G induced by the representation $man \to \sigma(m)e^{i\lambda(\log a)}$ of P. For $\lambda \in \mathfrak{a}^*$ regular $\pi_{\sigma,\lambda}$ is irreducible. Let $\theta_{\sigma,\lambda}$ denote its character. Denoting by $\mathcal{S}(G)$ the L^2 Schwartz space of G, let $\mathcal{S}_H(G)$ denote the subspace of functions orthogonal to the matrix coefficients of $\pi_{\sigma,\lambda}$ coming from the other Cartan subgroups $H_i \neq H$. Then ([**HC6**])

$$\mathcal{S}(G) = \bigoplus_i \mathcal{S}_{H_i}(G) \tag{30}$$

and for each individual $\mathcal{S}_H(G)$ one has

$$f(e) = \int_{\widehat{M}_d \times \mathfrak{a}^*} \theta_{\sigma,\lambda}(f)\, d\mu_H(\sigma,\lambda)\,, \quad f \in \mathcal{S}_H(G) \tag{31}$$

where $\widehat{M}_d$ is the discrete series of M and μ_H a certain explicitly determined positive measure. Combining (30) and (31) we get Harish-Chandra's formula (5) in an explicit form.

Example. $G = \mathbf{SL}(2, \mathbf{R})$. This case was settled already in 1952 by Harish-Chandra. The Lie algebra $\mathfrak{sl}(2, \mathbf{R})$ is given by

$$\{X = x_1X_1 + x_2X_2 + x_3X_3 : x_1, x_2, x_3 \in \mathbf{R}\}$$

where

$$X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It has the two non-conjugate Cartan subalgebras

$$\mathfrak{b} = \mathbf{R}X_1\,, \qquad \mathfrak{a} = \mathbf{R}X_2$$

invariant under the Cartan involution $X \to -{}^tX$. We write ψ_f^B and ψ_f^A for the corresponding φ_f in (20). The Cartan subgroups H_1, H_2 are now

$$B = \left\{k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right\}, \qquad A = \left\{\epsilon a_t : a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad \epsilon = \pm 1\right\}.$$

The adjoint action of G on $\mathfrak{sl}(2, \mathbf{R})$ is $X \to gXg^{-1}$. Under the mapping $X \to (x_1, x_2, x_3)$ the orbits

$$G \cdot \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} (\theta > 0) \quad G \cdot \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}, (\theta < 0)\,, \quad G \cdot \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$$

are respectively

$$\begin{aligned} H_+ &: \text{hyperboloid } x_1^2 - x_2^2 - x_3^2 = \theta^2 && (x_1 > 0), \\ H_- &: \text{hyperboloid } x_1^2 - x_2^2 - x_3^2 = \theta^2 && (x_1 < 0), \\ H &: \text{hyperboloid } x_1^2 - x_2^2 - x_3^2 = -t^2\,, \end{aligned}$$

which are indicated on the figure. Let $C_\pm$ denote the upper and lower light cone, respectively, and $C = C_+ \cup C_-$.

Then, as the geometry suggests,

$$\lim_{\theta \to 0+} \psi_f^B(\theta) = \int_{C_+} f$$

$$\lim_{\theta \to \theta-} \psi_f^B(\theta) = -\int_{C_-} f$$

so for the jump of ψ_f^B we have

$$\psi_f^B(0^+) - \psi_f^B(0^-) = \int_C f = \psi_f^A(0)\,. \tag{32}$$

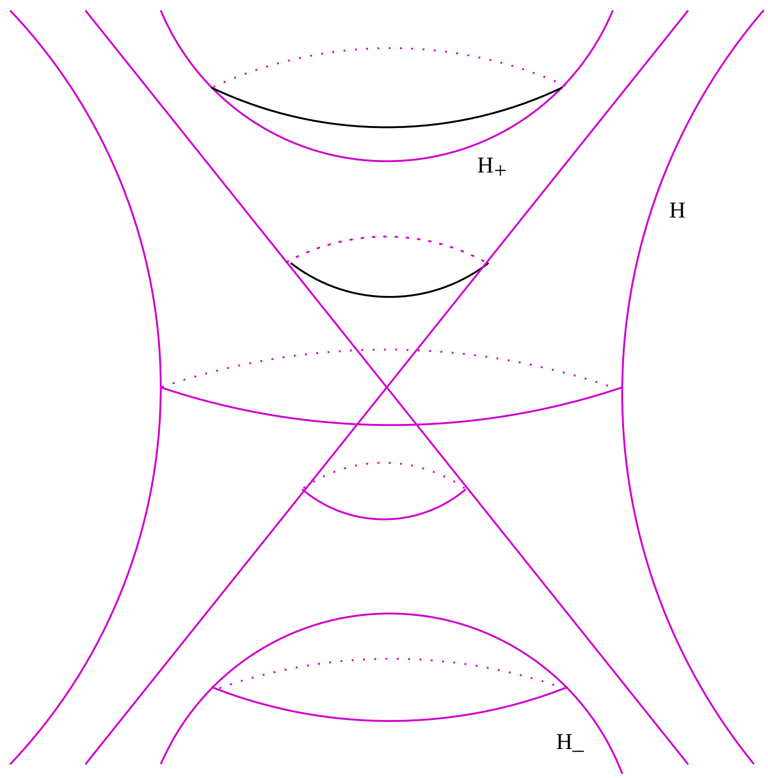

FIGURE 1

On the other hand, $d\psi_f^B/d\theta$ is continuous at $0 = 0$ and since the Cartan subgroup B is fundamental,

$$\frac{d\psi_f^B}{d\theta}(0) = \text{const.} \quad f(0)\,. \tag{33}$$

Formulas similar to (32) and (33) hold for the group orbital integrals (where $x^g = gxg^{-1}$)

$$F_f^B(k_\theta) = \left(e^{i\theta} - e^{-i\theta}\right) \int_G f(k_\theta^g)\, dg\,, \qquad (\theta \notin \pi\mathbf{Z})\,.$$
$$F_f^A(\epsilon a_t) = \left|e^t - e^{-t}\right| \int_{G/A} f(\epsilon a_t^g)\, dg_A\,, \quad (t \neq 0)\,.$$

Next we write down the characters of the representations $\pi_{\sigma,\lambda}$ above.

For $H_1 = B$ there is no vector part so we just have the discrete series of G. It is parameterized by $k \in \mathbf{Z} - \{0\}$ and the characters are given by

$$\theta_k(a_t) = (-1)^{k-1}\theta_k(-a_t) = \frac{e^{-|kt|}}{|e^t - e^{-t}|}\,, \qquad \theta_k(k_\theta) = -(\operatorname{sign} k)\frac{e^{ik\theta}}{e^{i\theta} - e^{-i\theta}}\,.$$

On the other hand the characters $\theta_{\pm,s}$ of the representations of G induced by the representations

$$\epsilon a_t \to e^{st} \qquad \epsilon a_t \to (\operatorname{sign}\epsilon)e^{st}$$

of AN are given by

$$\begin{aligned} \theta_{+,s}(a_t) = \theta_{+,s}(-a_t) &= \frac{e^{st}+e^{-st}}{|e^t - e^{-t}|}\,, \qquad \theta_{+,s}(k_\theta) = 0 \\ \theta_{-,s}(a_t) = -\theta_{-,s}(-a_t) &= \frac{e^{st}+e^{-st}}{|e^t - e^{-t}|}\,, \qquad \theta_{-,s}(k_\theta) = 0 \end{aligned}$$

Since the Weyl groups W_B and W_A have orders 1 and 2 respectively we conclude from (28)

$$\begin{aligned} \int_G f(g)\,dg = \frac{1}{2}\int_{A_0} \left|e^t - e^{-t}\right| \left(F_f^A(a_t) + F_f^A(-a_t)\right)\,dt \\ \frac{1}{2\pi}\int_0^{2\pi} \left(e^{-i\theta} - e^{i\theta}\right) F_f^B(k_\theta)\,d\theta\,. \end{aligned}$$

Applying (29) to $T = \theta_k$ we obtain

$$\begin{aligned} \theta_k(f) = \frac{1}{2}\int_{A_0} e^{-|kt|}\left(F_f^A(a_t) + (-1)^{k-1}F_f^A(-a_t)\right)\,dt \\ + (\operatorname{sign} k)\frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta}F_f^B(k_\theta)\,d\theta \end{aligned} \tag{34}$$

and for $T = \theta_{\pm,s}$, since $F_f^A(\pm a_t) = F_f^A(\pm a_{-t})$,

$$\theta_{\pm,s}(f) = \frac{1}{2}\int_{A_0} \left[F_f^A(a_t) \mp F_f^A(-a_t)\right] e^{st}\,dt\,. \tag{35}$$

Formulas (34) and (35) relate the characters to the orbital integrals, the latter formula being analogous to (26). Finally, one uses (33) combined with (34) to get the decomposition of δ into characters. Here the jumps of F_f^B for $\theta = 0, \pi$ come into play. The full details of the calculation can be found in [**HC3**] and in [**K**], Ch. XI § 3 and in [**L**].

The final result is as follows.

Theorem 1.1 (Harish-Chandra). *Let* $G = \mathbf{SL}(2,\mathbf{R})$. *Then we have for* $f \in C_c^\infty(G)$,

$$\begin{aligned} 8\pi f(e) = \sum_{m\neq 0} m\theta_m(f) + \int_0^\infty \frac{\lambda}{2}\tanh\left(\frac{\pi\lambda}{2}\right)\theta_{+,i\lambda}(f)\,d\lambda \\ + \int_0^\infty \frac{\lambda}{2}\coth\left(\frac{\pi\lambda}{2}\right)\theta_{-,i\lambda}(f)\,d\lambda\,. \end{aligned}$$

All the characters represent irreducible representations except $\theta_{-,0}$ *which is a sum of two irreducible characters.*

The argument can also be used to show that each conjugacy invariant eigendistribution of the Casimir operator on G is a linear combination of irreducible characters ([**S**], [**H5**]).

For another proof of the theorem in which Rossmann's formula [**Ro1**] for the orbital integral plays a prominent role see a beautiful account by Vergne [**Ve3**]. See also [**Ve1**] and [**Ve2**]. Yet another approach is in [**SW**], [**He2**] and [**HW**]. See also Varadarajan's article [**V**] for a lucid description of Harish-Chandra's original proof.

For further work on the orbital integral problem for semisimple G see e.g. Herb [**He1**], Bouaziz [**Bo**] and Shelstad [**Sl**]. Viewing G as $G \times G$/diagonal, the orbital problem seems of considerable interest for a semisimple symmetric space G/H although it remains to be seen whether it will play a role in harmonic analysis as in the group case. For G/H Lorentzian of constant curvature the inversion problem is solved in [**H1**], and for G/H of rank one by Orloff [**Or**].

In [**Ha**] Harinck investigates these orbital integrals for $G^{\mathbf{C}}/G$ and obtains the Plancherel formula for the corresponding spherical transform.

§2. Analysis on Riemannian Symmetric Spaces

1. The Fourier Transform.

The Fourier transform on a semisimple Lie group G

$$\tilde{f}(\pi) = \int_G f(x)\pi(x)dx \qquad \pi \in \widehat{G} \tag{1}$$

has a nice Plancherel formula

$$\int_G |\, f(x) \,|^2 \; dx = \int_{\widehat{G}} \| \, \tilde{f}(\pi) \, \|^2 \; d\pi \, ,$$

$\| \quad \|$ being the Hilbert-Schmidt norm. An explicit description of the measure $d\pi$ in terms of the structure of G was given by Harish-Chandra. There is also a characterization of the range $\mathcal{D}(G)^{\sim}$ through efforts of many people, primarily Arthur (see [**A**] and references there.)

Classical Fourier analysis in $\mathbf{R}^n$, which originated in the study of the heat equation, certainly has one of its principal applications in the theory of partial differential equations. For the semisimple G the Fourier transform is actually a rather unwieldy gadget in that it associates to a function f on G an object $\tilde{f}$ which is a family of operators on different Hilbert spaces. Thus it is not immediately suited for applications to differential equations.

Nevertheless, there has been certain amount of activity in studying invariant differential equations on G. It was proved in [**H7**] that if D is a bi-invariant differential operator on G then D is locally solvable, i.e. there exists an open neighborhood V of the identity e in G such that for each $f \in \mathcal{D}(V)$ there is a $u \in E(V)$ satisfying

$$Du = f$$

Such a result had been proved by Raïs [**R**] for nilpotent groups, was extended to solvable groups by Rouvière [**Ro**] and Duflo and Raïs [**DR**] and by Duflo [**D**] for arbitrary Lie groups. The Fourier transform however was never involved in the proofs.

On the symmetric space $X = G/K$ (G semisimple with finite center, $K \subset G$ maximal compact) one can however define a Fourier transform [**H4**] which is scalar-valued and concrete enough to be directly applicable to differential equations.

Consider the Iwasawa decompositions of our semisimple G

$$G = NAK \qquad \mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k} \tag{2}$$

with N, A and K nilpotent, abelian and compact, respectively, $\mathfrak{n}$, $\mathfrak{a}$ and $\mathfrak{k}$ their respective Lie algebras. Let θ be the Cartan involution of G with fixed group K. Let $\sum$ denote the set of roots for$(\mathfrak{g}, \mathfrak{a})$; for $\alpha \in \sum$ let $\mathfrak{g}_\alpha$ denote the corresponding root space and m_α its dimension. Then $\mathfrak{n} = \oplus_{\alpha \in \sum^+} \mathfrak{g}_\alpha$ where $\sum^+$ is the set of positive roots. Let $\mathfrak{a}^+$ denote the Weyl chamber where the $\alpha \in \sum^+$ are positive. Let $\mathfrak{a}^*$ (resp. $\mathfrak{a}_c^*$) denote the space of $\mathbf{R}$-linear maps $\mathfrak{a} \to \mathbf{R}$ (resp. $\mathfrak{a}_c^* \to \mathbf{C}$) and $\rho \in \mathfrak{a}^*$ given by $2\rho = \mathrm{Tr}(adH \mid \mathfrak{n})$, $H \in \mathfrak{a}$. Let M denote the centralizer if A in K and put $B = K/M$. We define $A(g) \in \mathfrak{a}$ in terms of (2) by $g = n \exp A(g) k$ and the vector-valued inner product

$$A : X \times B \to \mathfrak{a}$$

by

$$A(gK, kM) = A(k^{-1}g)\,.$$

The *Fourier transform* $f \to \tilde{f}$ on X is then defined [**H4**] by

$$\tilde{f}(\lambda, b) = \int_X f(x) e^{(-i\lambda+\rho)(A(x,b))}\, dx \tag{3}$$

for all $\lambda \in \mathfrak{a}_c^*$, $b \in B$ for which the integral converges. Note that in contrast to (1) $\tilde{f}(\lambda, b)$ is scalar-valued.

The basic results for this Fourier transform are ([**H4**], [**H6**], 1970):

a) Inversion. *If $f \in \mathcal{D}(x)$ then*

$$f(x) = \frac{1}{w} \int_{\mathfrak{a}^* \times B} \tilde{f}(\lambda, b) e^{(i\lambda+\rho)(A(x,b))}\, d\mu(\lambda, b)\,,$$

where w is the order of the Weyl group $W = W(\mathfrak{g}, \mathfrak{a})$ and $d\mu = |c(\lambda)|^{-2}\, d\lambda\, db$.

Here $d\lambda$, db are suitably normalized invariant measures on $\mathfrak{a}^*$ and B, respectively, and $c(\lambda)$ the Harish-Chandra c-function which is given by the following integral over $\overline{N} = \theta N$

$$c(\lambda) = \int_{\overline{N}} e^{(i\lambda+\rho)(A(\bar{n}))}\, d\bar{n}$$

and can be expressed in terms of Γ functions ([**GK**]).

b) Plancherel Theorem. *The map $f \to \tilde{f}$ extends to an isometry of $L^2(X)$ onto $L^2(\mathfrak{a}_+^* \times B, d\mu)$ ($\mathfrak{a}_+^*$ being the positive Weyl chamber in $\mathfrak{a}^*$).*

If f is K-invariant, i.e. $f(k \cdot x) \equiv f(x)$ these results reduce to the principal results in Harish-Chandra's theory of spherical functions [**HC5**]. However, for applications to analysis on X that invariance condition is of course too restrictive.

2. Applications.

As is very familiar to analysts the characterization of the Fourier transform space $\mathcal{D}(\mathbf{R}^n)$ as the space of entire functions of exponential type (the Paley-Wiener theorem) is an important tool in partial differential equation theory. The analog for X is the following [**H7**].

c) Paley-Wiener type theorem. *The range $\mathcal{D}(X)^{\sim}$ is identical to the space*

$$\left\{ \varphi \in \mathcal{E}(\mathfrak{a}^* \times B) : \begin{array}{ll} (i) & \lambda \to \varphi(\lambda, b) \text{ is holomorphic in } \mathfrak{a}_c^* \\ & \text{of exponential type uniformly in } b. \\ (ii) & \int_B \varphi(\lambda, b) e^{(i\lambda+\rho)(A(x,b))}\, db \text{ is } W\text{-invariant in } \lambda. \end{array} \right\}$$

The first application is the following existence theorem for members D of the algebra $\mathbf{D}(G/K)$ of differential operators on $X = G/K$ which are invariant under the action of G.

Theorem 2.1. *Each nonzero $D \in \mathbf{D}(G/K)$ is surjective on $\mathcal{E}(X)$, i.e.*

$$D\mathcal{E}(X) = \mathcal{E}(X)\,. \tag{4}$$

The first step is to get a fundamental solution for D, i.e. a distribution J such that $DJ = \delta$. This can be done by means of c) in the K-invariant case which then can be extended to distributions.

Once the existence of J has been established one can by fairly general functional analysis methods reduce the problem to the proof of the following implication, V being the closure of a ball $B_R(o)$ in X (R arbitrary):

$$f \in \mathcal{D}(X)\,, \qquad \operatorname{supp}(Df)^{\sim} \subset V \Longrightarrow \operatorname{supp}(f) \subset V \tag{5}$$

This is an easy consequence of c). In fact the functions $x \to e^{\nu(A(x,b))}$ are eigenfunctions of each $D \in \mathbf{D}(G/K)$ and

$$(Df)^{\sim}(\lambda, b) = p(\lambda)\tilde{f}(\lambda), b)$$

where $p(\lambda)$ is a polynomial. The conditions $f \in \mathcal{D}(X)$, $\operatorname{supp}(Df) \subset V$ imply that $p(\lambda)\tilde{f}(\lambda, b)$ is entire and of exponential type $\leq R$. From complex variable theory this implies that $\tilde{f}(\lambda, b)$ has exponential type $\leq R$. Hence $\operatorname{supp}(f) \subset V$, verifying (5).

The Paley-Wiener theorem in c) asserts that $f \to \tilde{f}$ is a bijection between two function spaces. However, although $\mathcal{D}(X)$ has a natural topology we have not introduced any natural topology on the other side. Using the induced topology of $\mathcal{E}(X \times \mathfrak{a}^*)$ would not make $f \to \tilde{f}$ a homeomorphism.

This can be remedied by specializing the theorem to the subspace of $\mathcal{D}(X)$ consisting of functions of a given K-type. For $\delta \in \widehat{K}$ acting on V_δ with V_δ^M the fixed space under $\delta(M)$ consider the subspace

$$\mathcal{D}_{\check{\delta}}(X) \subset \mathcal{D}(X) \qquad (\check{\delta} = \text{ contragredient to } \delta) \tag{6}$$

consisting of $f \in \mathcal{D}(X)$ of K-type $\check{\delta}$. Then

$$\tilde{f}(\lambda, kM) = \operatorname{Tr}\left(\delta(k)\tilde{f}(\lambda)\right)$$

where $\tilde{f}(\lambda)$ is the vector-valued Fourier transform

$$\tilde{f}(\lambda) = d(\delta) \int_X f(x) \Phi_{\bar{\lambda},\delta}(x)^*\, dx \tag{7}$$

with $\Phi_{\lambda,\delta}$ the generalized spherical function (or Eisenstein integral)

$$\Phi_{\lambda,\delta}(x) = \int_K e^{(i\lambda+\rho)(A(x,kM))} \delta(k)\, dk\,. \tag{8}$$

In (7) $*$ refers to adjoint and $\bar{\lambda}$ is the complex conjugate of the $\mathbf{R}$-linear function λ. For $a \in A$, $\Phi_{\lambda,\delta}(a)$ is a linear transformation of V_δ^M into itself. If we expand $\Phi_{\bar{\lambda},\delta}(a)^*$ near $a = e$ using the simple roots $\alpha_1, \dots, \alpha_l$ we have

$$\Phi_{\bar{\lambda},\delta}(a)^* = \sum_n P_{(n)}(\lambda)\alpha_1(\log a)^{n_1} \cdots \alpha_\ell(\log a)^{n_\ell}$$

and each $P_{(n)}(\lambda)$ is divisible by a certain polynomial matrix $Q^{\check{\delta}}(\lambda)$ and we have the following result ([**H6**], II).

Theorem 2.2. *The Eisenstein integral satisfies the following functional equations*

$$Q^{\check{\delta}}(\lambda)^{-1}\Phi_{\bar{\lambda},\delta}(a)^* \qquad \textit{is } W\textit{-invariant in } \lambda\,. \tag{9}$$

From this one can deduce a stronger topological version of c). Let $H(\mathfrak{a}_c^*, \mathrm{Hom}(V_\delta, V_\delta^M))$ denote the space of holomorphic functions on $\mathfrak{a}_c^*$ of exponential type with values in $\mathrm{Hom}(V_\delta, V_\delta^M)$, $\mathcal{I}^\delta(\mathfrak{a}^*)$ the subspace of W-invariants and $\mathcal{H}^\delta(\mathfrak{a}^*)$ the subspace $Q^{\check{\delta}}\mathcal{I}^\delta(\mathfrak{a}^*)$ with the induced topology.

Theorem 2.3. *The Fourier transform $f \to \tilde{f}$ given by* (7) *is a homeomorphism of $\mathcal{D}_{\check{\delta}}(X)$ onto $Q^{\check{\delta}}\mathcal{I}^\delta(\mathfrak{a}^*)$.*

Corollary 2.4. *The K-finite joint eigenfunctions of $\mathbf{D}(G/K)$ are precisely the integrals*

$$f(x) = \int_B e^{(i\lambda+\rho)(A(x,b))}F(b)\,db \tag{10}$$

where F is a K-finite continuous function on B.

Sketch of proof of the corollary ([**H6**], II). Fix $\delta \in \widehat{K}$ and let f be a joint eigenfunction of $\mathbf{D}(G/K)$. Consider the Harish-Chandra isomorphism $\Gamma : \mathbf{D}(G/K) \to I(\mathfrak{a})$ (W-invariants in $S(\mathfrak{a})$) then

$$Df = \Gamma(D)(i\lambda)f \tag{11}$$

for some $\lambda \in \mathfrak{a}_c^*$. Let $\mathcal{E}_\delta(B)$ be the space of K-finite continuous functions on B of type $\delta, \mathcal{E}_\lambda(X)$ the space of all $f \in \mathcal{E}(X)$ satisfying (11) and $\mathcal{E}_{\lambda,\delta}(X)$ the space of K-finite elements in $\mathcal{E}_\lambda(X)$ of type δ. The Poisson transform

$$\mathcal{P}_\lambda : F(b) \to f(x) = \int_B e^{(i\lambda+\rho)(A(x,b))}F(b)\,db \tag{12}$$

maps $\mathcal{E}(B)$ into $\mathcal{E}_\lambda(X)$ and $\mathcal{E}_\delta(B)$ into $\mathcal{E}_{\lambda,\delta}(X)$. For at least one $s \in W, \mathcal{P}_{s\lambda}$ is injective (see Theorem 3.3 below) so since $\mathcal{E}_{s\lambda,\delta}(X) = \mathcal{E}_{\lambda,\delta}(X)$ for $s \in W$ we may assume $\mathcal{P}_\lambda$ injective. Then

$$\dim \mathcal{E}_{\lambda,\delta}(X) \geq \dim \mathcal{E}_\delta(B) = \dim V_\delta \dim V_\delta^M\,. \tag{13}$$

The corollary will be concluded by proving the converse inequality. For this let $h \in \mathcal{E}_{\lambda,\delta}(X)$. We view it as a distribution on X, define its Fourier transform $\tilde{h}$ as a linear form on $\mathcal{H}^\delta(\mathfrak{a}^*)$ by

$$\tilde{h}(\tilde{f}) = h(f) = \int_X h(x)f(x)\,dx \qquad f \in \mathcal{D}_{\check{\delta}}(X)\,.$$

Then the map $\psi \in \mathcal{I}^\delta(\mathfrak{a}^*) \to \tilde{h}(Q^{\check{\delta}}\psi))$ is a continuous linear functional so by Theorem 3.1 (for $\delta = I$) this map is given by the Fourier transform of a K-invariant distribution j on X, i.e.

$$\tilde{j}(\tilde{\varphi}) = \tilde{h}(Q^{\check{\delta}}\tilde{\varphi}) = j(\varphi) \tag{14}$$

for all K-invariant φ in $\mathcal{D}(X, \mathrm{Hom}(V_\delta, V_{\check{\delta}}^M)$. Putting $p_D(\mu) = \Gamma(D)(-i\mu)$ formula (11) implies easily

$$p_D\tilde{h} = p_D(-\lambda)\tilde{h}\,.$$

Combining this with (14) one proves $p_D\tilde{j} = p_D(-\lambda)\tilde{j}$ which in turn implies $Dj = \Gamma(D)(i\lambda)j$. But j is also K-invariant so $j = \varphi_\lambda A$ where φ_λ is the zonal spherical function ((8) for δ, I) and $A \in \mathrm{Hom}(V_\delta, V_{\check{\delta}}^M)$. This proves the converse of (13) and the corollary.

An alternative proof was later found by Ban and Schlichtkrull [**BS**].

3. Multitemporal Wave Equations.

We now explain another application of Theorem 2.3, namely to the system of differential equations

$$Du = \partial(\Gamma(D))u \qquad D \in \mathbf{D}(G/K)\,. \tag{15}$$

Here $u \in \mathcal{E}(X \times \mathfrak{a})$, D operates on the first argument, $\partial(p)$ is the constant coefficient operator on $\mathfrak{a}$ corresponding to $p \in S(\mathfrak{a})$.

The following result is easily established.

Proposition. *Let $f \in C^2(A)$ and put*

$$u(x, H) = f(\exp(A(x, b) + H))e^{-\rho(H)}\,, \qquad b \in B\,.$$

Then u is a solution to (15).

Next we impose initial conditions on the system (15). We choose a real homogeneous basis $p_1 = 1, p_2, \ldots, p_w$ of the W-harmonic polynomial on $\mathfrak{a}$. Given $f_1, \ldots, f_w \in \mathcal{D}(X)$ we impose the initial conditions

$$(\partial(p_i)u)(x, 0) = f_i(x) \qquad 1 \leq i \leq w \tag{16}$$

on the solution to (15). The system (15)–(16) was first considered by [**STS**] and then by [**Sh**] and [**PS**].

If G/K has rank 1 the Laplacian L_X on X generates $\mathbf{D}(G/K)$ and $\Gamma(L_X) = L_A - |\rho|^2$ where L_A is the Laplacian on the 1-dimensional space $\mathfrak{a}$. Also $p_1 = 1$, $p_2 = H$ so the system (15)–(16) reduce to the shifted wave equation

$$(L_X + |\rho|^2)u = \frac{\partial^2 u}{\partial t^2}\,, \qquad u(x, 0) = f_1(x)\,, \qquad u_t(x, 0) = f_2(x) \tag{17}$$

On the quotient field $C(S(\mathfrak{a}))$ we consider the bilinear form

$$(a, b) = \sum_{\sigma \in W} a^\sigma b^\sigma$$

which has values in $C(I(\mathfrak{a}))$. We determine $q^j \in C(S(\mathfrak{a}))$ by $(q^j, p_i) = \delta_{ij}$. If π denotes the product of the positive indivisible roots it is known from [**HC5**] I, §3 that $\pi q^j \in S(\mathfrak{a})$. The matrix $A = (A_{ij})$ where

$$A_{ij} = \big(\pi q^j, \theta(\pi q^i)\big) \qquad 1 \leq i\,,\ j \leq w \tag{18}$$

has entries in $I(\mathfrak{a})$ so we can consider the matrix $\mathcal{A} = (\mathcal{A}_{ij})$ with entries in $\mathbf{D}(G/K)$ where $\Gamma(\mathcal{A}_{ij}) = A_{ij}$. Given $u, v \in \mathcal{E}(X \times \mathfrak{a})$ define the column vectors μ and ν by

$$\mu_i(x, H) = (\partial(p_i)u)(x, H) \qquad 1 \leq i \leq w \tag{19}$$

$$\nu_i(x, H) = (\partial(p_i)v)(x, H) \qquad 1 \leq i \leq w \tag{20}$$

The *energy* is defined by

$$E(u, v; H) = \int_X ({}^t\mu \mathcal{A} \bar{\nu})(x, H)\, dx \tag{21}$$

whenever the integral converges. For the special case (17) this reduces to the usual energy

$$E(u, u; 0) = c \int_X \left(-f_1(L_X + |\rho|^2)\bar{f}_1 + |f_2|^2\right)\, dx\,.$$

As proved by Shahshahani [**Sh**], if u is a solution to (15)–(16) then $E(u, u; H)$ is independent of H. This disagrees with Proposition 3, § 1.1 in [**STS**] where the same statement is made with a different definition of the energy, namely with θ missing in (18).

For simplicity put $E(u, v; 0) = E(u, v)$. Also let F denote the row vector

$$F(x) = (f_1(x), \ldots, f_w(x))$$

and for each $\sigma \in W$, $\lambda \in \mathfrak{a}_c^*$, $b \in B$ let μ^σ be the column vector with components

$$\mu_j^\sigma(x, H; \lambda, b) = \partial(p_j)_H\big(e^{i\lambda(H) + (i\sigma\lambda + \rho)(A(x,b))}\big)$$

We consider then the linear map

$$\mathcal{E}^\sigma : F(x) \to E(F, \mu^\sigma)(\lambda, b) = \int_X ({}^t F \mathcal{A} \bar{\mu}^\sigma)(x, 0; \lambda, b)\, dx$$

which maps $\mathcal{D}(X) \times \ldots \times \mathcal{D}(X)$ (w times) into a function space on $\mathfrak{a}_c^* \times B$.

Theorem 2.6. *For each $\sigma \in W$ the map $\mathcal{E}^\sigma$ is an injective norm-preserving map of $\mathcal{D}(X) \times \ldots \times \mathcal{D}(X)$ onto a dense subspace of $L^2(\mathfrak{a}^* \times B, d\lambda db/\left|\pi(\lambda)c(\lambda)\right|^2)$. Thus*

$$\int_{\mathfrak{a}^* \times B} \mathcal{E}^\sigma(F)\overline{\mathcal{E}^\sigma(F)}(\lambda, b) \frac{d\lambda\, db}{\left|\pi(\lambda)c(\lambda)\right|^2} = E(F, F)\,.$$

For $\sigma = e$ a rather complicated proof of this was given in [**Sh**]. Our more general result is based on Theorem 2.3 and the following new identity which relates $\mathcal{E}^\sigma$ to the Fourier transform $f \to \tilde{f}$.

THEOREM 2.7. *For each* $\sigma \in W$,

$$\mathcal{E}^\sigma(F)(\lambda, b) = \pi(\lambda)^2 \sum_{k=1}^{w} q^k(i\lambda)\tilde{f}_k(\sigma\lambda, b)\,.$$

For $H \in \mathfrak{a}$ let U_H denote the operator

$$F(x) \to {}^t\mu(x, H)\,.$$

Then the translation invariance of (15) implies that U_{H_o} maps ${}^t\mu(x, H)$ to ${}^t\mu(x, H_0 + H)$. The mapping $\mathcal{E}^\sigma$ in Theorem 2.7 is then easily shown to have the following property.

THEOREM 2.8. *For* $H \in \mathfrak{a}$ *let* $e(H)$ *denote the endomorphism*

$$e(H) : \varphi(\lambda, b) \to e^{i\lambda(H)}\varphi(\lambda, b) \quad \textit{of} \quad L^2\left(\mathfrak{a}^* \times B, d\lambda\, db/\left|\pi(\lambda)c(\lambda)\right|^2\right)$$

Then

$$\mathcal{E}^\sigma \circ U_{H_o} = e(H_o)\mathcal{E}^\sigma\,, \qquad H_o \in \mathfrak{a}\,.$$

This means that the wave motion $u(x, H) \to u(x, H + H_o)$ corresponds under $\mathcal{E}^\sigma$ to the simple map $e(H_o)$.

§3. Eigenspace Representations

1. The Symmetric Space Case.

Spherical harmonics are by definition the eigenfunctions on the unit sphere $\mathbf{S}^{n-1}$ of the Laplacian $L = L_{\mathbf{S}^{n-1}}$. The name comes from the fact that these eigenfunctions are precisely the restrictions to $\mathbf{S}^{n-1}$ of homogeneous harmonic polynomials on $\mathbf{R}^n$. Given $c \in \mathbf{C}^n$ the eigenspace

$$E_c = \left\{f \in \mathcal{E}\left(\mathbf{S}^{n-1}\right) : L_{\mathbf{S}^{n-1}}f = cf\right\} \tag{1}$$

is invariant under each rotation of $\mathbf{S}^{n-1}$. This gives a representation of $\mathbf{O}(n)$ on E_c. The space E_c is $\neq 0$ if and only if

$$c = -k(k + n - 2) \qquad k \in \mathbf{Z}^+$$

and $\mathbf{O}(n)$ acts irreducibly on E_c. Also, since $\mathbf{S}^{n-1}$ is two-point homogeneous under $\mathbf{O}(n)$ the only differential operators on $\mathbf{S}^{n-1}$ which are invariant under $\mathbf{O}(n)$ are the polynomials in $L_{\mathbf{S}^{n-1}}$ ([**H1**]).

This example motivates the definition of a fairly general class of representation which I called *eigenspace representations* in [**H6**]. Given a Lie group L and a closed subgroup H let $\mathbf{D}(L/H)$ denote the algebra of differential operators on L/H which are invariant under L. Given a homomorphism $\chi : \mathbf{D}(L/H) \to \mathbf{C}$ consider the joint eigenspace

$$E_\chi = \{f \in \mathcal{E}(L/H) : Df = \chi(D)f \quad \text{for} \quad D \in \mathbf{D}(L/H)\} \tag{2}$$

with the topology induced by that of $\mathcal{E}(L/H)$. Let T_χ denote the natural representation of L on E_χ, i.e. $\left(T_\chi(\ell)f\right)(xH) = f(\ell^{-1}xH)$.

If $\mathbf{D}(L/H)$ is not commutative it would be natural to replace it by a commutative subalgebra in order to have a rich supply of joint eigenfunctions. It might also be natural to pass from eigenfunctions in (2) to eigendistributions. Further natural generalization is obtained by replacing functions in (2) by sections of vector bundles.

Coming back to T_χ above the following problem arises naturally:

For which χ is T_χ irreducible and what are the representations of L so obtained?

Note that in our setup there is no Hilbert space in sight; in particular there is no particular emphasis on unitary representations.

Example. Consider $\mathbf{R}^n$ as the quotient space $\mathbf{M}(n)/\mathbf{O}(n)$ where $\mathbf{M}(n)$ is the group of isometries of $\mathbf{R}^n$. Here $\mathbf{D}\left(\mathbf{M}(n)/\mathbf{O}(n)\right)$ consists of the polynomials in the Laplacian $L = L_{\mathbf{R}^n}$. Here our problem has a simple solution ([**H8**]). Given $\lambda \in \mathbf{C}$ consider the eigenspace

$$\mathcal{E}_\lambda(\mathbf{R}^n) = \left\{ f \in \mathcal{E}(\mathbf{R}^n) : Lf = -\lambda^2 f \right\}, \tag{3}$$

and let T_λ denote the corresponding eigenspace representation.

Theorem 3.1. *T_λ is irreducible if and only if $\lambda \neq 0$.*

Proof (Sketch). If $\lambda = 0$ the space of harmonic polynomials of degree $\leq k$ form a closed invariant subspace for each k. For the converse let $\lambda \neq 0$ and consider the Poisson transform

$$\mathcal{P}_\lambda : F \in \mathcal{E}\left(\mathbf{S}^{n-1}\right) \to f \in \mathcal{E}_\lambda\left(\mathbf{R}^n\right) \tag{4}$$

given by

$$f(x) = \int_{\mathbf{S}^{n-1}} e^{i\lambda(x,\omega)} F(\omega)\, d\omega$$

This mapping commutes with the $\mathbf{O}(n)$ action and is injective for $\lambda \neq 0$. Using PDE techniques one can prove ([**H6**] '70) that there exists a sphere $S = S_r(0)$ (r depending on λ) such that the restriction map

$$f \in \mathcal{E}_\lambda\left(\mathbf{R}^n\right) \to f|S \tag{5}$$

is injective. Let the subscript δ refer to the spaces of $\mathbf{O}(n)$-finite functions of type δ. Then from the injectivity of the maps (4) and (5) we deduce

$$\dim\Big(\mathcal{E}(S^{n-1})_\delta\Big) \leq \dim\Big(\mathcal{E}_\lambda(\mathbf{R}^n)_\delta\Big) \leq \dim\Big(\mathcal{E}(S)_\delta\Big). \tag{6}$$

The extremes having the same dimension equality holds in (6) so

$$\mathcal{E}_\lambda\left(\mathbf{R}^n\right)_\delta = \mathcal{P}_\lambda\left(\mathcal{E}(\mathbf{S}^{n-1})_\delta\right). \tag{7}$$

Consider the Hilbert space

$$\mathcal{H}_\lambda = \left\{ f(x) = \int_{\mathbf{S}^{n-1}} e^{i\lambda(x,\omega)} F(\omega)\, d\omega : F \in L^2(S^{n-1}) \right\}$$

the norm of f taken as the L^2 norm of F. Expanding $f \in \mathcal{E}_\lambda(\mathbf{R}^n)$ according to its δ-components, $f = \Sigma_\delta f_\delta$ with $f_\delta \in \mathcal{E}_\lambda(\mathbf{R}^n)_\delta$ we see that $\mathcal{H}_\lambda$ is dense in $\mathcal{E}_\lambda(\mathbf{R}^n)$. The action of $\mathbf{M}(n)$ on $\mathcal{H}_\lambda$ is easy to analyze and irreducibility follows quickly. Using the density of $\mathcal{H}_\lambda$ in $\mathcal{E}_\lambda(\mathbf{R}^n)$ it is easy to deduce the irreducibility of T_λ.

Consider now the case of a symmetric space G/K of the noncompact type. We adopt the notation from § 2 and let Σ_0 denote the set of indivisible roots. With Γ as in § 2 consider for each $\lambda \in \mathfrak{a}_c^*$ the joint eigenspace

$$\mathcal{E}_\lambda(X) = \{f \in \mathcal{E}(X) : Df = \Gamma(D)(i\lambda)f \quad \text{for} \quad D \in \mathbf{D}(G/K)\} \ .$$

Each joint eigenspace is of this form for some $\lambda \in \mathfrak{a}_c^*$ and $\mathcal{E}_{s\lambda}(X) = \mathcal{E}_\lambda(X)$ for each $s \in W$. Let T_λ denote the eigenspace representation of G on $\mathcal{E}_\lambda(X)$. When is it irreducible?

Consider the product

$$\Gamma_X(\lambda) = \prod_{\alpha\in\Sigma_0} \Gamma\left(\tfrac{1}{2}\left(\tfrac{1}{2}m_\alpha + 1 + \langle i\lambda, \alpha_0\rangle\right)\right)\Gamma\left(\tfrac{1}{2}\left(\tfrac{1}{2}m_\alpha + m_{2\alpha} + \langle i\lambda, \alpha_0\rangle\right)\right)$$

"the Gamma function of X" where α_0 stands for $\alpha/\langle\alpha,\alpha\rangle$. This $\Gamma_X(\lambda)$ is the denominator in the Gindikin-Karpelevich formula for $c(\lambda)c(-\lambda)$. The counterpart to Theorem 3.1 for the symmetric space G/K is the following result ([**H6**]).

Theorem 3.2. *Let $\lambda \in \mathfrak{a}_c^*$. Then T_λ is irreducible if and only if*

$$\frac{1}{\Gamma_X(\lambda)} \neq 0\,.$$

The proof involves a study of the Poisson transform

$$(\mathcal{P}_\lambda F)(x) = \int_B e^{(i\lambda+\rho)(A(x,b))}F(b)\,db\,.$$

Here the principal property is the following result ([**H6**], I and II).

Theorem 3.3. *Let $\lambda \in \mathfrak{a}_c^*$. Then*

$$\mathcal{P}_\lambda \text{ is injective} \Longleftrightarrow \frac{1}{\Gamma_X^+(\lambda)} \neq 0\,.$$

Here $\Gamma_X^+(\lambda)$ equals $\Gamma_X(\lambda)$ except that the product ranges only over $\Sigma_0^+ = \Sigma_0 \cap \Sigma^+$. We call λ *simple* if $\mathcal{P}_\lambda$ is injective so Theorem 3.2 states that T_λ is irreducible if both λ and $-\lambda$ are simple. Here is a sketch of the proof, using Corollary 2.4.

Suppose first both λ and $-\lambda$ are simple. Let $V \subset \mathcal{E}_\lambda(X)$ be a closed invariant subspace. Then clearly $\varphi_\lambda \in V$. Since $-\lambda$ is simple the functions

$$b \to \sum_j a_j e^{(-i\lambda+\rho)(A(g_j\cdot 0,b))} \qquad a_j \in \mathbf{C}\,, \quad g_j \in G$$

form a dense subspace of $L^2(B)$. On the other hand φ_λ has the following symmetry property ([**H6**], I, p. 116):

$$\varphi_\lambda(g^{-1}\cdot x) = \int_B e^{(i\lambda+\rho)(A(x,b))}e^{(-i\lambda+\rho)(A(g\cdot 0,b))}\,db\,,$$

so we can conclude that V contains the space $\mathcal{H}_\lambda$ of functions

$$f(x) = \int_B e^{(i\lambda+\rho)(A(x,b))}F(b)\,db\,, \qquad F \in L^2(B)\,.$$

On the other hand the proof of Corollary 2.4 showed that

$$\mathcal{E}_{\lambda,\delta}(X) = \mathcal{P}_\lambda(\mathcal{E}(B)_\delta)$$

so we conclude that V contains each $\mathcal{E}_{\lambda,\delta}(X)$. However, each $f \in \mathcal{E}(X)$ can be expanded in a Fourier series according to K,

$$f = \sum_{\delta \in \widehat{K}} f_\delta$$

and then $f_\delta \in \mathcal{E}_{\lambda,\delta}(X)$. Consequently V is dense in $\mathcal{E}_\lambda(X)$ so $V = \mathcal{E}_\lambda(X)$, proving the irreducibility. The converse, T_λ irreducible $\Rightarrow \lambda$ and $-\lambda$ simple involves similar ideas.

For semisimple symmetric spaces G/H (H fixed point group of any involution) the eigenspace representations have been relatively little studied. However for G/H isotropic the irreducibility question was completely answered by Schlichtkrull (by classification) and the composition series determined for each case ([**Sc**]). For the real hyperbolic spaces related results had been obtained by Rossmann ([**Ro2**]).

2. The Principal Series.

Next I shall discuss the principal series from the point of view of eigenspace representations.

Given $\delta \in \widehat{M}$ operating on the vector space V_δ and $\lambda \in \mathfrak{a}_c^*$ consider the space $\Gamma_{\delta,\lambda}$ of smooth functions $f : G \to V_\delta$ satisfying

$$f(gman) \equiv \delta(m^{-1})e^{(i\lambda-\rho)(\log a)}f(g)\,. \tag{8}$$

Let $\tau_{\delta,\lambda}$ denote the representation of G on $\Gamma_{\delta,\lambda}$ given by

$$(\tau_{\delta,\lambda}(g_1)f)(g_2) = f(g_1^{-1}g_2) \tag{9}$$

This family of representations is called the *principal series*.

Let us first consider the case when G is complex. In this case MA is a Cartan subgroup H of G and $\dim V_\delta = 1$. Let $\mathbf{D}(H)$ (respectively, $\mathbf{D}(G/N)$) denote the algebras of left invariant differential operators on H (respectively G-invariant differential operators on G/N). Given $U \in \mathbf{D}(H)$ we can define the differential operator D_U on G/N by

$$(D_U f)(gN) = \{U_\hbar(f(ghN))\}_{\hbar=e}\,. \tag{10}$$

The operator D_U is well defined since $\hbar N h^{-1} \subset N$ and it is clearly a G-invariant differential operator on G/N. We now have the following result relating these algebras ([**H3**], [**H6**], I).

THEOREM 3.4. *The mapping $U \to D_U$ is an isomorphism of $\mathbf{D}(H)$ onto $\mathbf{D}(G/N)$. In particular, $\mathbf{D}(G/N)$ is commutative.*

As a consequence the joint eigenspaces are precisely the spaces

$$E_\omega = \{f \in \mathcal{E}(G/N) : f(ghN) \equiv \omega(\hbar)f(gN)\} \tag{11}$$

as ω runs through the C^∞ characters of H. Comparing with (8) we therefore conclude

COROLLARY 3.5 (G COMPLEX). *The principal series representations of G are precisely the eigenspace representations for G/N.*

In order to treat real G in the same spirit it is convenient to use the familiar connection between induced representations and vector bundles. Consider the representation $\delta \otimes 1$ of MN which defines a vector bundle

$$G \times_{MN} V_\delta .$$

The sections of this bundle are the maps $F : G \to V_\delta$ satisfying

$$F(gmn) \equiv \delta(m^{-1})F(g) . \tag{12}$$

Let $\mathbf{D}(A)$ (respectively, $\mathbf{D}(G/MN)$) denote the algebras of left invariant differential operators on A (respectively G-invariant differential operators on G/MN). Given $U \in \mathbf{D}(A)$ we define the G-invariant differential operator D_U on G/MN by

$$(D_U f)(gMN) = \{U_a(f(gaMN))\}_{a=e} .$$

In analogy with Theorem 3.4 we have ([**H3**]):

PROPOSITION 3.6. *The mapping $U \to D_U$ is an isomorphism of $\mathbf{D}(A)$ onto $\mathbf{D}(G/MN)$.*

The operator D_U also operates on the sections (12) of the bundle $G \times_{MN} V_\delta$ by

$$(D_U F)(gmn) = \{U_a\left(F(gamn)\right)\}_{a=e} .$$

Then in fact

$$(D_U F)(gmn) = \delta(m^{-1})(D_U F)(g)$$

and the functions (8) are the joint eigenfunctions of $\mathbf{D}(G/MN)$. This proves

THEOREM 3.7. *Let $\delta \in \widehat{M}$. Then the principal series representations $\tau_{\lambda,\delta}(\lambda \in \mathfrak{a}_c^*)$ are the eigenspace representations for the algebra $\mathbf{D}(G/MN)$ acting on the sections of the bundle $G \times_{MN} V_\delta$.*

3. The Discrete Series.

In the case when rank $G = \operatorname{rank} K, G$ has a discrete series ([**HC6**]). This has been displayed in several models but Hotta's realization of the discrete series (or most of it) fits best in the above framework ([**Ho**]). Schmid's earlier model ([**Sm**]) is set up in similar spirit.

Consider the irreducible representation δ_λ of K on V_λ with lowest weight $\lambda + 2\rho_k$ ($2\rho_k =$ sum of the positive compact roots) relative to a compact Cartan subalgebra $\mathfrak{h} \subset \mathfrak{k}$. Consider the corresponding vector bundle $G \times_K V_\lambda$. Then the Casimir operator Ω is an invariant differential operator on this bundle. Let Γ_{δ_λ} denote the space of corresponding smooth square integrable sections and put

$$\mathcal{E}_\lambda = \{f \in \Gamma_{\delta_\lambda} : \Omega f = \left(\langle \lambda + \rho, \lambda + \rho\rangle - \langle \rho, \rho\rangle\right) f\} .$$

Assume $\langle \lambda + \rho, \alpha\rangle < 0$ for all roots $\alpha > 0$. Then there exists a constant $a > 0$ such that if $|\langle \lambda + \rho, \beta\rangle| > a$ for all noncompact roots β then $\mathcal{E}_\lambda$ realizes the discrete series representation whose character is on a maximal turns T of K is given by

$$\varepsilon \frac{\sum_{s \in W_G} (\det s) e^{s(\lambda+\rho)}}{\prod_{\alpha \in P}\left(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}\right)} .$$

Here $W_G = \widetilde{T}/T$ where $\widetilde{T}$ is the normalizer of T in G, P is the set of positive roots for (G, T) and ε is a power of (-1).

Thus the discrete series arises as eigenspace representations for the algebra generates by Ω.

4. G/H with G nilpotent.

Here the eigenspace representations have been analyzed rather completely by Stetkaer and Jacobsen. This goes beyond the Kirillov theory in that the results are not restricted to unitary representations. They have also extended this to some solvable groups.

References to this work and other results on eigenspace representations can be found in [**H9**], especially Chapter VI.

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