

Integral Geometry and Multitemporal Wave Equations

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In Memory of Fritz John

Revised

Introduction

While the wave equation $(L - \partial^2/\partial t^2)u = 0$ on a Euclidean space has an analog for any Riemannian manifold X the case when X is a symmetric space offers another natural analog where the operator $L - \partial^2/\partial t^2$ is replaced by a system (see (2.7)(2.8)) and the time variable is now multi-dimensional. This was introduced by Semenov-Tjan-Shansky in [STS] and was further developed by Shahshahani [S] and Phillips and Shahshahani [PS] .

In the present paper we study the above system by means of Fourier transform methods described in § 3. In particular we show (Theorem 6.3) that the spectral representations \mathcal{E}^σ are directly related to the Fourier transform on X . This is the basis of the proof of the spectral representation theorem (Theorem 6.4).

In §4 we prove a uniqueness theorem for the system and obtain some new formulas for the solution. In § 5 we prove a characterization of incoming waves in terms of their sources. This is an analog of results by Lax and Phillips for the flat case. This proof follows ideas of Phillips and Shahshahani for similar results but because of the available tools is a bit simpler. I am indebted to Shahshahani for useful discussions and to Schlichtkrull for some concrete suggestions, specified later.

The following notation will be used. \mathbf{R}^n and \mathbf{S}^n denote the Euclidean n -space and n -sphere, respectively. $\mathcal{S}(\mathbf{R}^n)$ denotes the space of rapidly decreasing functions on \mathbf{R}^n . If X is a manifold, $\mathcal{E}(X)$ and $\mathcal{D}(X)$, respectively, constitute the space of smooth functions on X , resp. smooth functions of compact support. Their duals $\mathcal{D}'(X)$ and $\mathcal{E}'(X)$ constitute the space of distributions on X , resp. distributions of compact support. Support is denoted supp . If σ is a diffeomorphism of X , $f \in \mathcal{E}(X)$ and J an operator on $\mathcal{E}(X)$ then we put

$$f^\sigma = f \circ \sigma^{-1}, \quad J^\sigma(f) = (Jf^{\sigma^{-1}})^\sigma$$

for $f \in \mathcal{E}(X)$. If X is a metric space, $B_r(x)$ denotes the open ball of radius r and center x and $S_r(x)$ denotes the corresponding sphere. If G is a group with Haar measure then $*$ denotes the convolution on G ; if $K \subset G$ is a compact subgroup then \times denotes the convolution on

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G/K which is derived from $*$ on G . The translation of $gK \rightarrow hgK$ on G/K is denoted $\tau(h)$. The adjoint representation of a Lie group is denoted by Ad and tA denotes the transpose of a linear transformation A .

1 The Wave Equation in \mathbf{R}^n

Let f be a function on \mathbf{R}^n integrable over each hyperplane. Let $\widehat{f}(\xi)$ denote the integral

$$\widehat{f}(\xi) = \int_{\xi} f(x) dm(x) \quad (1.1)$$

of f over the hyperplane ξ , dm being the Euclidean measure. The map $f \rightarrow \widehat{f}$ is called the *Radon transform*. We can write the hyperplane ξ in the form $\langle x, \omega \rangle = p$ where $\langle \cdot, \cdot \rangle$ is the scalar product, ω a unit vector and $p \in \mathbf{R}$. Defining $\widehat{f}(\omega, p) = \widehat{f}(\xi)$, \widehat{f} becomes a function on $\mathbf{S}^{n-1} \times \mathbf{R}$ satisfying the symmetry condition $\widehat{f}(-\omega, -p) = \widehat{f}(\omega, p)$. If φ is a smooth function on $\mathbf{S}^{n-1} \times \mathbf{R}$ which satisfies this symmetry condition $\varphi(-\omega, -p) = \varphi(\omega, p)$ and in addition is rapidly decreasing in p we define the operator Λ by

$$(\Lambda\varphi)(\omega, p) = \begin{cases} \frac{d^{n-1}}{dp^{n-1}}\varphi(\omega, p) & n \text{ odd} , \\ \mathcal{H}_p \frac{d^{n-1}}{dp^{n-1}}\varphi(\omega, p) & n \text{ even} , \end{cases}$$

where \mathcal{H} is the Hilbert transform

$$(\mathcal{H}F)(t) = \frac{i}{\pi} \int_{\mathbf{R}} \frac{F(p)}{p-t} dp.$$

The function $\Lambda\varphi$ also satisfies the above symmetry condition and thus can be viewed as a function on the space Ξ of hyperplanes.

The Radon transform is then inverted by the formula

$$c_0 f(x) = (-i)^{n-1} \int_{\mathbf{S}^{n-1}} \left\{ (\Lambda\widehat{f})(\omega, p) \right\}_{p=\langle \omega, x \rangle} d\omega, \quad (1.2)$$

where $c_0 = (2\pi)^n / \pi$.

We consider now the Cauchy problem for the wave equation

$$Lu = \frac{\partial^2 u}{\partial t^2} \quad u(x, 0) = f_0(x), \quad u_t(x, 0) = f_1(x) \quad (1.3)$$

f_0 and f_1 being given functions in $\mathcal{D}(\mathbf{R}^n)$.

Lemma 1.1. *If $h \in C^2(\mathbf{R})$ and $\omega \in \mathbf{S}^{n-1}$ then the function*

$$v(x, t) = h(\langle x, \omega \rangle + t) \quad (1.4)$$

satisfies $Lv = \partial^2 / \partial t^2 v$.

The proof is obvious. It is now easy on the basis of (1.2) and (1.4) to write down the solution to the Cauchy problem (1.3). For a different method in the case n odd see Lax-Phillips, [LP], IV, § 2.

Theorem 1.2. *The solution to (1.3) is given by*

$$u(x, t) = \int_{\mathbf{S}^{n-1}} (Sf)(\omega, \langle x, \omega \rangle + t) d\omega, \quad (1.5)$$

where

$$Sf = \begin{cases} c(\partial^{n-1}\widehat{f}_0 + \partial^{n-2}\widehat{f}_1), & n \text{ odd, } > 1. \\ c(\mathcal{H}(\partial^{n-1}\widehat{f}_0 + \partial^{n-2}\widehat{f}_1)), & n \text{ even.} \end{cases} \quad (1.6)$$

Here $\partial = \partial/\partial p$, and the constant c equals

$$c = \frac{1}{2}(2\pi i)^{1-n}.$$

Proof: Because of Lemma 1.1 we just have to check the initial conditions in (1.3).

- (i) If $n > 1$ is odd then $\omega \rightarrow (\partial^{n-1}\widehat{f}_0)(\omega, \langle x, \omega \rangle)$ is an even function on \mathbf{S}^{n-1} but $\omega \rightarrow (\partial^{n-2}\widehat{f}_1)(\omega, \langle x, \omega \rangle)$ is odd. Thus $u(x, 0) = f_0(x)$ by (1.2). Applying $\partial/\partial t$ to the right hand side of (1.5) and putting $t = 0$ gives $u_t(x, 0) = f_1(x)$, this time because the function $\omega \rightarrow (\partial^n \widehat{f}_0)(\omega, \langle x, \omega \rangle)$ is odd and $\omega \rightarrow (\partial^{n-1}\widehat{f}_1)(\omega, \langle x, \omega \rangle)$ is even.
- (ii) n even. Here the proof is the same if one remarks that \mathcal{H} interchanges even and odd functions on \mathbf{R} .

Definition For the initial data $f = \{f_0, f_1\}$ we shall refer to the function Sf in (1.6) on $\mathbf{S}^{n-1} \times \mathbf{R}$ as the *source*.

Note that for n odd, Lax and Phillips refer to $-\partial^{(n+1)/2}\widehat{f}_0 + \partial^{(n-1)/2}\widehat{f}_1$ as the *translation representation* ([LP], IV, Theorem 2.2). They also call the wave $u(x, t)$ *outgoing* if $u(x, t) = 0$ in the *forward cone* $|x| < t$ and *incoming* if $u(x, t) = 0$ in the *backward cone* $|x| < -t$.

Corollary 1.3. *The solution $u(x, t)$ to (1.3) is*

- (i) *outgoing if and only if $(Sf)(\omega, s) = 0$ for $s > 0$.*
- (ii) *incoming if and only if $(Sf)(\omega, s) = 0$ for $s < 0$.*

Proof: (adapted from [LP] IV, § 2 for n odd).

For (i) suppose $(Sf)(\omega, s) = 0$ for $s > 0$. Then if $|x| < t$ we have $\langle x, \omega \rangle + t > -|x| + t > 0$ so by (1.5) u is outgoing. Conversely, suppose $u(x, t) = 0$ for $|x| < t$. Let $t_0 > 0$ be arbitrary

and $\varphi \in \mathcal{D}(t_0, \infty)$. Then if $|x| < t_0$ we have

$$\begin{aligned} 0 &= \int_{\mathbf{R}} u(x, t) \varphi(t) dt = \int_{\mathbf{S}^{n-1}} d\omega \int_{\mathbf{R}} (Sf)(\omega, \langle x, \omega \rangle + t) \varphi(t) dt \\ &= \int_{\mathbf{S}^{n-1}} d\omega \int_{\mathbf{R}} (Sf)(\omega, p) \varphi(p - \langle x, \omega \rangle) dp = 0. \end{aligned}$$

Taking $\partial^{|k|}/\partial x_{i_1}, \dots, \partial x_{i_k}$ at $x = 0$ we deduce

$$\int_{\mathbf{R}} \left(\int_{\mathbf{S}^{n-1}} (Sf)(\omega, p) \omega_{i_1} \dots \omega_{i_k} d\omega \right) (\partial^{|k|} \varphi)(p) dp = 0$$

for each k and each $\varphi \in \mathcal{D}(t_0, \infty)$. Integrating by parts in the \mathbf{R} -integral we deduce that the function

$$p \rightarrow \int_{\mathbf{S}^{n-1}} (Sf)(\omega, p) \omega_{i_1} \dots \omega_{i_k} d\omega \quad (1.7)$$

has its k th derivative $\equiv 0$ for $p > t_0$. Having compact support if n is odd and being $O(\log |p|)$ if n is even we deduce that the function (1.7) vanishes identically for $p > t_0$. Now varying k we see that $(Sf)(\omega, p)$ vanishes for $p > t_0$. Since $t_0 > 0$ was arbitrary this proves (i). Part (ii) is proved in the same way.

Remark The solution to (1.3) can also be written in the well-known form

$$u(x, t) = (f_0 * T'_t)(x) + (f_1 * T_t)(x) \quad T'_t = dT_t/dt \quad (1.8)$$

where $*$ denotes convolution on \mathbf{R}^n , T_t and T'_t in $\mathcal{E}'(\mathbf{R}^n)$ are given by their Fourier transforms,

$$\frac{\sin(|\lambda|t)}{|\lambda|} = \int_{\mathbf{R}^n} e^{-i(\lambda, x)} dT_t(x), \quad \cos(|\lambda|t) = \int_{\mathbf{R}^n} e^{-i(\lambda, x)} dT'_t(x). \quad (1.9)$$

Since

$$LT_t = \frac{d^2}{dt^2} T_t, \quad LT'_t = \frac{d^2}{dt^2} T'_t \quad (1.10)$$

$$(T_t)_{t=0} = 0, \quad (T'_t)_{t=0} = \delta_0, \quad (1.11)$$

one refers to the pair $\{T_t, T'_t\}$ as the *fundamental solution* to the wave equation.

For n odd, (1.5) implies immediately the classical Huygens' principle.

Corollary 1.4. *Let $n > 1$ be odd. Assume f_0 and f_1 have support in $B_R(0)$. Then u has support in the conical shell*

$$|t| - R \leq |x| \leq |t| + R. \quad (1.12)$$

In fact, for n odd (1.5) implies

$$u(0, t) = 0 \quad \text{for } |t| \geq R. \quad (1.13)$$

If $y \in \mathbf{R}^n$ the translated initial data f_0^{-y}, f_1^{-y} give the solution $x \rightarrow u(x + y, t)$. Thus by (1.13) since $\text{supp}(f_i^{-y}) \subset B_{R+|y|}(0)$,

$$u(y, t) = 0 \quad \text{for } t \geq R + |y|.$$

On the other hand, since T_t in (1.8) has support in $B_{|t|}(0)$, (1.8) implies that u has support in the region $|x| \leq |t| + R$. This proves (1.12).

Remark The shell (1.12) is the union

$$\bigcup_{|y| \leq R} C_y$$

where C_y is the light cone $\{(x, t) : |x - y| = |t|\}$.

For $n > 1$ odd one has also the following limit theorem of Friedlander.

Theorem 1.5. *For n odd the solution u to (1.3) satisfies*

$$\lim_{|t| \rightarrow \infty} t^{\frac{n-1}{2}} u_t((t+p)\omega, t) = (-\partial^{(n+1)/2} \widehat{f}_0 + \partial^{(n-1)/2} \widehat{f}_1)(\omega, p).$$

For a proof see Lax-Phillips [LP] p.108.

2 Multitemporal Wave Equations on Symmetric Spaces

Since the Laplacian in (1.3) generalizes to Riemannian manifolds, the analog of the Cauchy problem (1.3) exists for any Riemannian manifold X . In the case when X is a Riemannian symmetric space there is another natural analog of (1.3), defined by Semenov-Tjan-Shansky in [STS]; it is a system of equations and the ‘‘time variable’’ becomes multidimensional.

In order to describe the problem we need to recall some notions from symmetric space theory. Let $X = G/K$ be a symmetric space of the noncompact type, that is G is a connected noncompact semi-simple Lie group with finite center and K a maximal compact subgroup. Let \mathfrak{g} and \mathfrak{k} denote the corresponding Lie algebras, θ the Cartan involution of \mathfrak{g} with fixed point set \mathfrak{k} . Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where \mathfrak{p} , the -1 eigenspace of θ is the orthogonal complement of \mathfrak{k} in \mathfrak{g} , relative to the Killing form B of \mathfrak{g} . For $X, Y \in \mathfrak{p}$ we put $\langle X, Y \rangle = B(X, Y)$ and $|X| = \langle X, X \rangle^{1/2}$. We fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$, \mathfrak{a}^* its dual, and let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the system of roots of \mathfrak{g} relative to \mathfrak{a} . Let Σ_0 denote the set of indivisible roots. The Weyl group W of Σ acts on \mathfrak{a} and \mathfrak{a}^* by

$$(\sigma \cdot \lambda)(H) = \lambda(\sigma^{-1}H) = \lambda^\sigma(H), \quad \sigma \in W, \lambda \in \mathfrak{a}^*, H \in \mathfrak{a}.$$

This action extends to automorphisms of the symmetric algebras $S(\mathfrak{a})$ and $S(\mathfrak{a}^*)$.

Fix a Weyl chamber \mathfrak{a}^+ in \mathfrak{a} and let \mathfrak{a}_+^* correspond to \mathfrak{a}^+ under the map $\lambda \rightarrow H_\lambda$ where $\langle H, H_\lambda \rangle = \lambda(H)$ ($H \in \mathfrak{a}$). Let Σ^+ be the corresponding set of positive roots, let 2ρ denote their sum with multiplicity, and put $\Sigma_0^+ = \Sigma^+ \cap \Sigma_0$. Let

$$G = NAK, \quad \mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k} \quad (2.1)$$

be the Iwasawa decompositions corresponding to this ordering, \mathfrak{n} the (nilpotent) Lie algebra of N . Let M be the centralizer of A in K and put $B = K/M, \overline{N} = \theta N$. If $H \in \mathfrak{a}$ let $\partial(H)$ denote the corresponding directional derivative. The map $H \rightarrow \partial(H)$ extends to an isomorphism of $S(\mathfrak{a})$ (respectively $S^C(\mathfrak{a}) = \mathbf{C} \otimes S(\mathfrak{a})$) onto the algebra of all differential operators on \mathfrak{a} with constant real (resp. complex) coefficients. The map $\lambda \rightarrow H_\lambda$ extends to an isomorphism of $S^C(\mathfrak{a}^*)$ onto $S^C(\mathfrak{a})$.

Let $I(\mathfrak{a})$ denote the algebra of W -invariants in $S(\mathfrak{a})$ and $I_+(\mathfrak{a})$ the set of $q \in I(\mathfrak{a})$ without constant term. A polynomial $p \in S(\mathfrak{a}^*)$ is said to be W -harmonic if $\partial(q)p = 0$ for all $q \in I_+(\mathfrak{a})$. Let $H(\mathfrak{a}^*)$ be the space of W -harmonic polynomials and let $H(\mathfrak{a}) \subset S(\mathfrak{a})$ correspond to $H(\mathfrak{a}^*)$ under the extension of $\lambda \rightarrow H_\lambda$. It is well known that

$$S(\mathfrak{a}) = S(\mathfrak{a})I_+(\mathfrak{a}) \oplus H(\mathfrak{a}) \quad (2.2)$$

and

$$\dim H(\mathfrak{a}) = w \quad (\text{the order of } W). \quad (2.3)$$

For the adjoint action of K on \mathfrak{p} we have analogous notions, $S(\mathfrak{p}), S(\mathfrak{p}^*), I(\mathfrak{p}), I_+(\mathfrak{p}), H(\mathfrak{p})$, and $H(\mathfrak{p}^*)$. The analog to (2.2) is also valid ([H6], III, Theorem 1.2).

Let $\mathbf{D}(G)$ denote the algebra of left-invariant differential operators on G and $\mathbf{D}_K(G)$ the subalgebra of $D \in \mathbf{D}(G)$ which are right invariant under K . If $X \in \mathfrak{g}$ let $\tilde{X} \in \mathbf{D}(G)$ denote the corresponding left-invariant vector field. The *symmetrization* ([H6], II, § 4)

$$Y_1 \cdots Y_p \rightarrow \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \tilde{Y}_{\sigma(1)} \cdots \tilde{Y}_{\sigma(p)}$$

(\mathfrak{S}_p = the symmetric group) is a linear bijection of $S(\mathfrak{g})$ onto $\mathbf{D}(G)$. Let H^* denote the image of $H(\mathfrak{p})$ under this map.

According to (2.1) we have the direct decomposition

$$\mathbf{D}(G) = \mathbf{D}(A) \oplus (\tilde{\mathfrak{n}}\mathbf{D}(G) + \mathbf{D}(G)\tilde{\mathfrak{k}}). \quad (2.4)$$

Let $D \rightarrow q^D$ denote the corresponding projection of $\mathbf{D}(G)$ onto $\mathbf{D}(A) = S(\mathfrak{a})$. Then the map

$$\gamma : D \rightarrow e^{-\rho} q^D \circ e^\rho$$

is a homomorphism of $\mathbf{D}_K(G)$ onto $I(\mathfrak{a})$ with kernel $\mathbf{D}_K(G) \cap \mathbf{D}(G)\tilde{\mathfrak{k}}$. ([HC] § 3 or [H6], III, Theorem 5.17). On the other hand, G/K being reductive, the algebra $\mathbf{D}_K(G)/\mathbf{D}_K(G) \cap \mathbf{D}(G)\tilde{\mathfrak{k}}$ is naturally isomorphic to the algebra $\mathbf{D}(G/K)$ of G -invariant differential operators on G/K , [H1]). This gives a surjective isomorphism

$$\Gamma : \mathbf{D}(G/K) \rightarrow I(\mathfrak{a}). \quad (2.5)$$

We now fix a real homogeneous basis

$$p_1 = 1, p_2, \dots, p_w \quad (2.6)$$

of $H(\mathfrak{a})$. For fixed $f_i \in \mathcal{D}(X)$ ($1 \leq i \leq w$) we consider now the ‘‘Cauchy problem’’ for $u \in \mathcal{E}(X \times \mathfrak{a})$,

$$Du = \partial(\Gamma(D))u \quad D \in \mathbf{D}(G/K) \quad (2.7)$$

with initial conditions

$$(\partial(p_i)u)(x, 0) = f_i(x), \quad 1 \leq i \leq w. \quad (2.8)$$

In (2.7) D operates on the first variable and $\partial(\Gamma(D))$ operates on the second variable. As remarked by Schlichtkrull the system (2.8) is equivalent to

$$(\partial(p)u)(x, 0) = F(p) \quad p \in H(\mathfrak{a})$$

where

$$F \in \text{Hom}_{\mathbf{R}}(H(\mathfrak{a}), \mathcal{D}(X)), \quad (F(p_i) = f_i).$$

The system (2.7)(2.8) was first considered by Semenov-Tjan-Shansky [STS] and further studied by Shahshahani and Phillips and Shahshahani ([S] and [PS]). Here we shall present some additional results, some generalizations of older results and some new and simpler proofs.

3 The Fourier Transform

To study the system (2.7)(2.8) we use the Fourier transform theory on X developed in [H2]-[H5] and [H7]. We now indicate the results needed. Let us normalize the invariant measures on $G, N, A, K, M, \mathfrak{a}, \mathfrak{a}^*, X = G/K, B = K/M$ as in [H7], II, § 3.1. According to (2.1) each $g \in G$ is decomposed $g = n \exp A(g)k$ ($A(g) \in \mathfrak{a}$) and we can define the ‘‘inner product’’

$$A(gK, kM) = A(k^{-1}g), \quad g \in G, k \in K. \quad (3.1)$$

Then for each $b \in B$ the function $x \rightarrow e^{(i\lambda+\rho)(A(x,b))}$ is an eigenfunction of each $D \in \mathbf{D}(G/K)$ and

$$D_x (e^{(i\lambda+\rho)(A(x,b))}) = \Gamma(D)(i\lambda)e^{(i\lambda+\rho)(A(x,b))}. \quad (3.2)$$

For a function f on X we define the *Fourier transform* \tilde{f} by

$$\tilde{f}(\lambda, b) = \int_X f(x) e^{-(i\lambda+\rho)(A(x,b))} dx \quad (3.3)$$

for all $b \in B$, and $\lambda \in \mathfrak{a}_c^* = \mathbf{C} \otimes \mathfrak{a}^*$ for which the integral exists. For $f \in \mathcal{D}(X)$ we have the *inversion formula*

$$f(x) = \frac{1}{w} \int_{\mathfrak{a}^* \times B} \tilde{f}(\lambda, b) e^{(i\lambda+\rho)(A(x,b))} d\mu(\lambda, b), \quad (3.4)$$

where $d\mu(\lambda, b) = |\mathbf{c}(\lambda)|^{-2} d\lambda db$, $\mathbf{c}(\lambda)$ being Harish-Chandra’s \mathbf{c} -function. The *Plancherel theorem* asserts that $f \rightarrow \tilde{f}$ extends to an isometry of $L^2(X)$ onto $L^2(\mathfrak{a}_c^* \times B, \mu)$. The *Paley-Wiener theorem* describes the range $\mathcal{D}(X)^\sim$ as a certain subspace of the space $\mathcal{H}(\mathfrak{a}_c^* \times B)$ of

smooth functions $\varphi(\lambda, b)$ holomorphic in $\lambda \in \mathfrak{a}_C^*$ of exponential type (uniform in b). That subspace is specified by the identities

$$\int_B \varphi(\lambda, b) e^{(i\lambda + \rho)(A(x, b))} db \quad \text{is } W\text{-invariant in } \lambda. \quad (3.5)$$

We now relate this theory to representations of K . Let \widehat{K}_M be the set of unitary irreducible representations δ of K with fixed vector under $\delta(M)$. Let V_δ be the representation space of δ and V_δ^M the space of fixed vectors under $\delta(M)$. We choose an orthonormal basis $v_1, \dots, v_{d(\delta)}$ of V_δ such that $v_1, \dots, v_{\ell(\delta)}$ span V_δ^M . With H^* defined before let $E_\delta = \text{Hom}_K(V_\delta, H^*)$ denote the space of linear maps $V_\delta \rightarrow H^*$ intertwining $\delta(k)$ and $Ad(k)$ ($k \in K$). By [KR], $\dim E_\delta = \ell(\delta)$. Let $\check{\delta}$ be the contragredient to δ operating on the dual space $V'_\delta = V_\delta$. For $v \in V_\delta$ let $v' \in V'_\delta$ be determined by $v'(u) = \langle u, v \rangle$ for $u \in V_\delta$. For $\lambda \in \mathfrak{a}_C^*$ consider the linear map

$$Q^\delta(\lambda) : E_\delta \rightarrow V_{\check{\delta}}^M \quad (3.6)$$

given by

$$(Q^\delta(\lambda)(e))(v) = q^{e(v)}(\rho - i\lambda) \quad e \in E_\delta, v \in V_\delta, \quad (3.7)$$

q^D being defined after (3.4). If $\epsilon_1, \dots, \epsilon_{\ell(\delta)}$ is any basis of E_δ we define

$$Q^\delta(\lambda)_{ij} = q^{\epsilon_j(v_i)}(\rho - i\lambda), \quad 1 \leq i, j \leq \ell(\delta).$$

The matrix $Q^\delta(\lambda)_{ij}$ whose entries are polynomials in λ are related to the P_{ij}^δ in [K] § 4 by a simple variable shift. If $C_\delta : V_{\check{\delta}}^M \rightarrow E_\delta$ is the linear transformation determined by $C_\delta v'_j = \epsilon_j$ then

$$Q^\delta(\lambda) C_\delta v'_j = \sum_{i=1}^{\ell(\delta)} Q^\delta(\lambda)_{ij} v'_i. \quad (3.8)$$

If (3.3) is specialized to $f \in \mathcal{D}_{\check{\delta}}(X)$, the space of K -finite f of type $\check{\delta}$, then ([H5], § 7)

$$\tilde{f}(\lambda, kM) = \text{Tr}_{V_\delta}(\tilde{f}(\lambda)\delta(k)), \quad \text{Tr} = \text{Trace}, \quad (3.9)$$

where $\tilde{f}(\lambda)$ is the δ -spherical transform

$$\tilde{f}(\lambda) = d(\delta) \int_X f(x) \left(\int_K e^{(-i\lambda + \rho)(A(x, kM))} \delta(k^{-1}) dk \right) dx. \quad (3.10)$$

Then $\tilde{f}(\lambda)$ maps V_δ into $V_{\check{\delta}}^M$ and has the invariance property:

$$Q_{\check{\delta}}^\delta(\lambda)^{-1} \tilde{f}(\lambda) \quad \text{is } W\text{-invariant in } \lambda. \quad (3.11)$$

Also as f runs through $\mathcal{D}_{\check{\delta}}(X)$, the functions (3.11) fill up the space $\mathcal{J}^\delta(\mathfrak{a}^*)$ of W -invariants in the space $\mathcal{H}(\mathfrak{a}_C^*, \text{Hom}(V_\delta, V_\delta^M))$ of $\text{Hom}(V_\delta, E_{\check{\delta}})$ -valued holomorphic functions of exponential type on \mathfrak{a}_C^* ([H5] § 7).

4 Properties of Solutions of the Multitemporal Wave Equations

It is known from [STS] and [S] that the system (2.7)(2.8) has a solution which for each $H \in \mathfrak{a}$ has compact support in the first variable. We now prove a uniqueness statement for such solutions.

Proposition 4.1. *Suppose u' and u'' are two solutions to (2.7)(2.8) and that for each $H \in \mathfrak{a}$, the functions $x \rightarrow u'(x, H)$ and $x \rightarrow u''(x, H)$ have compact support. Then $u' = u''$.*

Proof: Let $u = u' - u''$. Then u satisfies (2.7)(2.8) with $f_j \equiv 0$ for all j . Consider the Fourier transform $\tilde{u}(\lambda, b; H)$ of u in the x -variable. Then by (3.2)(3.3) if $D \in \mathbf{D}(G/K)$ and D^* its adjoint,

$$\partial(\Gamma(D))_H(\tilde{u}(\lambda, b; H)) = \Gamma(D^*)(-i\lambda)\tilde{u}(\lambda, b; H) = \Gamma(D)(i\lambda)\tilde{u}(\lambda, b; H) \quad (4.1)$$

by [H6], II, Cor.5.3 and Lemma 5.21. Also

$$\left\{ \partial(p_i)_H(u(x, H)) \right\}_{H=0} \equiv 0. \quad (4.2)$$

Because of (2.2)(2.3) each $p \in S(\mathfrak{a})$ can be written $p = \sum_{j=1}^w q_j p_j$ ($q_j \in I(\mathfrak{a})$). Then using (2.7) (4.1) we get

$$\partial(p)_H(\tilde{u}(\lambda, b; H)) = \sum_j \partial(p_j)\partial(q_j)_H(\tilde{u}(\lambda, b; H)) = \sum_j \partial(p_j)_H(q_j(i\lambda)\tilde{u}(\lambda, b; H))$$

Using the inversion formula for $u(x, H)$ and (4.2) we deduce

$$\left\{ \partial(p)_H(u(x, H)) \right\}_{H=0} = 0, \quad p \in S(\mathfrak{a}). \quad (4.3)$$

On the other hand, if $\lambda \in \mathfrak{a}^*$ is regular (4.1) implies by a result of Steinberg and Harish-Chandra ([H6], III, Theorem 3.13) that for constants C_s

$$\tilde{u}(\lambda, b; H) = \sum_{s \in W} C_s(\lambda, b) e^{is\lambda(H)}.$$

Thus the equation $\left\{ \partial(p)_H(\tilde{u}(\lambda, b; H)) \right\}_{H=0} = 0$ for all p implies

$$\sum_{s \in W} p(is\lambda) C_s(\lambda, b) = 0, \quad p \in S(\mathfrak{a}).$$

Choosing p such that $p(is\lambda) = 0$ for all $s \neq s_0$ in W and $p(is_0\lambda) = 1$ we get $C_{s_0}(\lambda) = 0$. Thus $\tilde{u}(\lambda, b; H) = 0$ for all $\lambda \in \mathfrak{a}^*$ regular and $b \in B$ so by the inversion formula, $u(x, H) \equiv 0$, as desired.

Consider now the quotient fields $C(S(\mathfrak{a}))$ and $C(I(\mathfrak{a}))$, and the bilinear form $(,)$ on $C(S(\mathfrak{a})) \times C(S(\mathfrak{a}))$ with values in $C(I(\mathfrak{a}))$ given by

$$(a, b) = \sum_{\sigma \in W} a^\sigma b^\sigma. \quad (4.4)$$

This form being nondegenerate we can determine $q^j \in C(S(\mathfrak{a}))$ such that

$$(q^j, p_i) = \delta_{ij}. \quad (4.5)$$

Theorem 4.2. *Given $f_1, \dots, f_w \in \mathcal{E}(X)$ the system*

$$Du = \partial(\Gamma(D))u \quad D \in \mathbf{D}(G/K) \quad (4.6)$$

with initial data

$$(\partial(p_j)u)(x, 0) = f_j(x) \quad (1 \leq j \leq w) \quad (4.7)$$

has a solution given by the convolution

$$u(x, H) = \sum_{j=1}^w (f_j \times S_H^j)(x) \quad (4.8)$$

where for each j , $H \in \mathfrak{a}$, S_H^j is in $\mathcal{E}'(X)$ and is for all b given by

$$\sum_{\sigma \in W} q^j(i\sigma\lambda) e^{(i\sigma\lambda)(H)} = \int_X e^{(-i\lambda+\rho)(A(x,b))} dS_H^j(x). \quad (4.9)$$

Remark Because of Prop. 4.1 we shall refer to (4.8) as *the* solution to (2.7)(2.8).

We need some known results about p_i, q^j from [HC] § 3 for which simplified proofs (due to Steinberg) can be found in [GV] § 5.5. Let $\pi \in S(\mathfrak{a})$ be given by $\pi \in \prod_{\alpha \in \Sigma_0^+} H_\alpha$.

Lemma 4.3. *We have*

$$(i) \quad S(\mathfrak{a}) = \bigoplus_{i=1}^w I(\mathfrak{a})p_i$$

$$(ii) \quad \pi q^j \in S(\mathfrak{a})$$

$$(iii) \quad \frac{1}{\pi} S(\mathfrak{a}) = \{x \in C(S(\mathfrak{a})) : (x, y) \in I(\mathfrak{a}) \text{ for all } y \in S(\mathfrak{a})\}.$$

If s_1, \dots, s_w runs through W (4.5) implies that the matrices

$$e_{ij} = p_j(s_i\lambda) \quad \text{and} \quad f_{k\ell} = q^k(s_\ell\lambda)$$

are inverses. The determinant of e_{ij} being homogeneous in λ it follows from Cramer's rule that each q^j is homogeneous in λ . Because of (ii) above, $q^j = h^j/\pi$ where $h^j \in S(\mathfrak{a})$ so the

left hand side of (4.9) equals

$$\frac{1}{\pi(i\lambda)} \sum_{\sigma} h^j(i\sigma\lambda) \epsilon(\sigma) e^{i\sigma\lambda(H)} \quad (4.10)$$

where $\pi(\sigma\lambda) = \epsilon(\sigma)\pi(\lambda)$ and $\epsilon(\sigma) = \pm 1$. The sum in (4.10) is skew and is thus divisible by $\pi(i\lambda)$ so (4.10) is holomorphic in λ . Also if $\lambda = \xi + i\eta$ ($\xi, \eta \in \mathfrak{a}^*$) we have

$$|e^{i\sigma\lambda(H)}| \leq e^{|\eta||\sigma|}$$

so (4.10) is an entire function of exponential type of polynomial growth for λ real. By the Paley-Wiener theorem extended to $\mathcal{E}'(X)$ ([H4], Theorem 8.5, or [EHO] or [H7], III, Cor. 5.9) there exist unique $S_H^j \in \mathcal{E}'(X)$ satisfying (4.9) and

$$\text{supp}(S_H^j) \subset B_{|H|}(o). \quad (4.11)$$

Next we prove that S_H^j depends smoothly on $H \in \mathfrak{a}$, that is, for each $f \in \mathcal{D}(X)$, $S_H^j(f)$ is smooth in H . Since S_H^j is K -invariant, $(f \times S_H^j)^\sim = \tilde{f}(S_H^j)^\sim$ ([H7], III) so (3.4) and (4.9) imply

$$\int_X f(x) dS_H^j(x) = \int_{\mathfrak{a}^* \times B} \tilde{f}(\lambda, b) \sum_{\sigma} q^j(-i\sigma\lambda) e^{-i\sigma\lambda(H)} d\mu(\lambda, b)$$

so the smoothness in H follows.

Defining $u(x, H)$ by (4.8) we shall now verify (4.6)-(4.7). Since $D(f \times S) = f \times DS$ for $f \in \mathcal{E}(X)$, $S \in \mathcal{E}'(X)$ we consider DS_H^j and shall prove

$$DS_H^j = \partial(\Gamma(D))_H(S_H^j). \quad (4.12)$$

For this we observe that by (4.9) and (2.3)

$$(DS_H^j)^\sim(\lambda, b) = \Gamma(D^*)(-i\lambda)(S_H^j)^\sim(\lambda, b) = \Gamma(D)(i\lambda)(S_H^j)^\sim(\lambda, b).$$

On the other hand, applying $\partial(\Gamma(D))$ to (4.9) we see by the W -invariance of $\Gamma(D)$ that

$$(\partial(\Gamma(D))_H(S_H^j))^\sim(\lambda, b) = \Gamma(D)(i\lambda)(S_H^j)^\sim(\lambda, b).$$

These formulas imply (4.12), so (4.6) follows. Secondly, applying $\partial(p_k)_H$ to (4.9) we get

$$\sum_{\sigma} q^j(i\sigma\lambda) p_k(i\sigma\lambda) e^{i\sigma\lambda(H)} = (\partial(p_k)_H(S_H^j))^\sim(\lambda, b).$$

Putting here $H = 0$ and using (4.5) we obtain

$$\left\{ (\partial(p_k)_H(S_H^j))^\sim(\lambda, b) \right\}_{H=0} = \delta_{jk} \quad (4.13)$$

for all λ, b . By the injectivity of the Fourier transform this means

$$\left\{ \partial(p_k)S_H^j \right\}_{H=0} = \delta_{jk} \delta_o, \quad (4.14)$$

where δ_o the delta distribution of X at o . Now (4.7) follows immediately.

Remark Relations (4.12)-(4.14) are the analogs to (1.10)-(1.11) and one can thus consider the family

$$\left\{ S_H^1, \dots, S_H^w \right\}$$

as the *fundamental solution* to (4.6)-(4.7).

Example G/K of rank one. Let H_0 be the vector in \mathfrak{a}^+ of length 1. Then with $p_1 = 1, p_2 = H_0$ we have $q^1 = 1/2, q^2 = 1/2H_0$. Also $\Gamma(L) = H_0^2 - |\rho|^2$. Writing $v(x, t) = u(x, tH_0)$ equations (4.6) and (4.7) become

$$(L + |\rho|^2)v = \frac{\partial^2}{\partial t^2}v, \quad v(x, 0) = f_0(x), v_t(x, 0) = f_1(x)$$

Writing S_t^j instead of $S_{tH_0}^j$ equation (4.9) becomes (since $\lambda(H_0) = \pm|\lambda|$)

$$\begin{aligned} \cos |\lambda| t &= \int_X e^{(-i\lambda + \rho)(A(x, b))} dS_t^1(x), \\ \frac{\sin |\lambda| t}{|\lambda|} &= \int_X e^{(-i\lambda + \rho)(A(x, b))} dS_t^2(x) \end{aligned}$$

in exact analogy with (1.9).

From Theorem 4.2 we can now deduce the following result stating informally that the speed of propagation is ≤ 1 . The result is established in [PS], Theorem 3.4, in a rather complicated fashion (see also [STS], Lemma 2).

Corollary 4.4. *If $\text{supp}(f_j) \subset B_R(x_0)$ for $1 \leq j \leq w$ then for each $H \in \mathfrak{a}$,*

$$u(x, H) = 0 \quad \text{for } x \notin B_{R+|H|}(x_0). \quad (4.15)$$

By the group invariance we can take $x_0 = o$. Then the result is an immediate consequence of (4.8), (4.11) and the following simple lemma.

Lemma 4.5. *Let $f \in \mathcal{E}(X), S \in \mathcal{E}'(X)$. Then*

$$\text{supp}(f) \subset \overline{B_r(o)}, \text{supp}(S) \subset \overline{B_s(o)} \Rightarrow \text{supp}(f \times S) \subset \overline{B_{r+s}(o)}. \quad (4.16)$$

Proof: By approximation we may assume S is a continuous function. If \tilde{S} denotes the lift of S to G ($\tilde{S}(g) = S(g \cdot o)$) then

$$(f \times S)(g \cdot o) = \int_G f(gh^{-1} \cdot o) \tilde{S}(h) dh.$$

If this is $\neq 0$ then for some h $d(gh^{-1} \cdot o, 0) \leq r$ and $d(h \cdot o, 0) \leq s$ whence,

$$d(o, g \cdot o) = d(g^{-1} \cdot o, o) \leq d(g^{-1} \cdot o, h^{-1} \cdot o) + d(h^{-1} \cdot o, o) \leq r + s,$$

proving (4.16).

We can now reformulate the property in Cor.4.4 as follows.

Proposition 4.6. *Let $f_j \in \mathcal{D}(X)$ ($1 \leq j \leq w$) and $u(x, H)$ the corresponding solution to (4.6), (4.7). Then if*

$$\overline{B_\rho(x_0)} \cap \bigcup_{j=1}^w \text{supp}(f_j) = \emptyset \quad (4.17)$$

we have for each $H \in \mathfrak{a}$,

$$u(x, H) = 0 \quad \text{for } x \in B_{\rho-|H|}(x_0). \quad (4.18)$$

Proof: The compact set $C = \bigcup_{j=1}^w \text{supp}(f_j)$ can be covered by finitely many balls $B_\epsilon(x_i)$ (ϵ fixed) such that

$$\overline{B_\rho(x_0)} \cap \overline{B_\epsilon(x_i)} = \emptyset \quad \text{for each } i. \quad (4.19)$$

Choose a corresponding partition of unity i.e., $\varphi_i \in \mathcal{D}(B_\epsilon(x_i))$, $\varphi_i \geq 0$, $\sum_i \varphi_i \leq 1$ with equality in a neighborhood of C . Then $f_j = \sum_i f_j^i$ ($1 \leq j \leq w$) where $f_j^i = \varphi_i f_j$ has support in $B_\epsilon(x_i)$. Let $u^i(x, H)$ be the solution to (4.6)(4.7) corresponding to f_j^i ($1 \leq j \leq w$). By Cor. 4.4,

$$u^i(x, H) = 0 \quad \text{for } x \notin B_{\epsilon+|H|}(x_i). \quad (4.20)$$

However, $B_{\epsilon+|H|}(x_i)$ is disjoint from $\overline{B_{\rho-|H|}(x_0)}$; in fact a common point y would satisfy $d(x_i, y) \leq \epsilon + |H|$, $d(x_0, y) \leq \rho - |H|$ whence $d(x_i, x_0) \leq \epsilon + \rho$, contradicting (4.19). Thus by (4.20), $u^i(x, H) = 0$ for $x \in B_{\rho-|H|}(x_0)$ so (4.18) follows since $u = \sum_i u^i$.

We shall now derive some other formulas for the solution. If $S \in \mathcal{E}'(X)$ is K -invariant and $f \in \mathcal{D}(X)$ we have

$$(f \times S)^\sim = \tilde{f} \tilde{S}$$

so by (4.8) and the inversion formula for the Fourier transform,

$$u(x, H) = \frac{1}{w} \int_{\mathfrak{a}^*} \sum_{j, \sigma} q^j(i\sigma\lambda) e^{i\sigma\lambda(H)} (\mathbf{c}(\lambda)\mathbf{c}(-\lambda))^{-1} \int_B \tilde{f}_j(\lambda, b) e^{(i\lambda+\rho)(A(x,b))} db d\lambda.$$

Since $(\mathbf{c}(\lambda)\mathbf{c}(-\lambda))^{-1}$ as well as the integral over B are W -invariant in λ (cf.(5) §3) the terms in \sum_σ all have the same integral over \mathfrak{a}^* . Thus we can replace the average $\frac{1}{w} \sum_\sigma$ by a single term. For each $\sigma \in W$ we thus obtain (as in [S], Lemma 8),

$$u(x, H) = \int_{\mathfrak{a}^* \times B} \sum_k j_k^\sigma(\lambda) \tilde{f}_k(\lambda, b) e^{i\lambda(A(x,b)+\sigma H)} e^{\rho(A(x,b))} d\lambda db, \quad (4.21)$$

where

$$j_k(\lambda) = q^k(i\lambda)/\mathbf{c}(\lambda)\mathbf{c}(-\lambda) \quad (4.22)$$

and as usual, $j_k^\sigma(\lambda) = j_k(\sigma^{-1}\lambda)$. Consider now the Euclidean Fourier transform

$$F^*(\lambda) = \int_A e^{-i\lambda(\log a)} F(a) da$$

and let J_k denote the operator on $\mathcal{S}(A)$ determined by

$$(J_k F)^*(\lambda) = j_k(\lambda) F^*(\lambda). \quad (4.23)$$

Since $q^k(\lambda)\pi(\lambda)$ is a polynomial and $(\pi(\lambda)\mathbf{c}(\lambda)\mathbf{c}(-\lambda))^{-1}$ has all its derivatives bounded by a polynomial, J_k does indeed map $\mathcal{S}(A)$ into itself. Also, by a simple computation,

$$(J_k^\sigma F)^*(\lambda) = j_k^\sigma(\lambda)F^*(\lambda).$$

Consider now the Radon transform

$$\widehat{f}(\xi) = \int_{\xi} f(x) dm(x),$$

where ξ is a horocycle $gNg^{-1} \cdot x$ and dm the measure on ξ derived from dn . More explicitly, if $\xi_0 = N \cdot o$,

$$\widehat{f}(g \cdot \xi_0) = \int_N f(gn \cdot o) dn.$$

Since each ξ equals $ka \cdot \xi_0$ where $kM \in K/M$ and $a \in A$ are unique we can write $\widehat{f}(\xi)$ in the form $\widehat{f}(kM, a)$. Then J_k operates on functions on the horocycle space Ξ . Also by [H7] p. 276,

$$\widetilde{f}(\lambda, b) = \int_A \widehat{f}(b, a) e^{(-i\lambda + \rho)(\log a)} da = (e^\rho \widehat{f})^*(\lambda, b). \quad (4.24)$$

Thus by (4.21)

$$u(x, H) = \int_{\mathfrak{a}^* \times B} \sum_k (J_k^\sigma(e^\rho \widehat{f}_k))^*(\lambda, b) e^{i\lambda(A(x, b) + \sigma H)} e^{\rho(A(x, b))} d\lambda db$$

so by the inversion formula for the Euclidean Fourier transform,

$$u(x, H) = \int_B \sum_k J_k^\sigma(e^\rho \widehat{f}_k)(b, \exp(A(x, b) + \sigma H)) e^{\rho(A(x, b))} db. \quad (4.25)$$

Here e^ρ is the function on Ξ given by $e^\rho(kM, a) = e^{\rho(\log a)}$. As in [H7], II, §3 we introduce the operators

$$\Lambda_{k, \sigma} = e^{-\rho} J_k^\sigma \circ e^\rho \quad (1 \leq k \leq w)$$

on the space $\mathcal{S}_\rho(\Xi) = \{\varphi \in \mathcal{E}(\Xi) : e^\rho \varphi \in \mathcal{S}(\Xi)\}$, where $\mathcal{S}(\Xi) = \mathcal{S}(K/M \times A)$. Then the space $\mathcal{S}_\rho(\Xi)$ and the operators $\Lambda_{k, \sigma}$ are G -invariant ([H7], III, Lemma 3.7.)

Theorem 4.7. *Fix $\sigma \in W$. The solution to (4.16)-(4.17) can be written*

$$u(g \cdot o, H) = e^{\rho(\sigma H)} \int_{K/M} \sum_j (\Lambda_{j, \sigma} \widehat{f}_j)(gk \exp \sigma H \cdot \xi_0) dk_M. \quad (4.26)$$

Proof: If $g = e$ the right hand side becomes

$$e^{\rho(\sigma H)} \int_B \sum_j (e^{-\rho} J_j^\sigma(e^\rho \widehat{f}_j))(b, \exp \sigma H) db = \int_B \sum_j (J_j^\sigma(e^\rho \widehat{f}_j))(b, \exp \sigma H) db,$$

which agrees with the right hand side of (4.25) for $x = o$. By the G -invariance of (4.6)-(4.7) in the x -variable the initial data $f_j^{\tau(g^{-1})}$ generate the solution $u^{\tau(g^{-1})}$. Since the Radon transform as well as the operators $\Lambda_{j,\sigma}$ commute with the G -action we deduce from the above

$$\begin{aligned} u(g \cdot o, H) = u^{\tau(g^{-1})}(o, H) &= e^{\rho(\sigma H)} \int_{K/M} \sum_j (\Lambda_{j,\sigma}(f_j^{\tau(g^{-1})})^\widehat{)}(k \exp \sigma H \cdot \xi_0) dk_M \\ &= e^{\rho(\sigma H)} \int_{K/M} \sum_j (\Lambda_{j,\sigma} \widehat{f}_j)(gk \exp \sigma H \cdot \xi_0) dk_M \end{aligned}$$

as claimed.

Definition Given $\sigma \in W$ the function

$$(S_\sigma f)(\xi) = \sum_j (\Lambda_{j,\sigma} \widehat{f}_j)(\xi), \quad \xi \in \Xi$$

will be called the σ -source of the solution u .

From (4.25) it is easy to see that

$$u(x, H) = e^{\rho(\sigma H)} \int_B (S_\sigma f)(b, \exp(A(x, b) + \sigma H)) e^{2\rho(A(x, b))} db \quad (4.27)$$

and this formula is the symmetric space analog to (1.5).

From (4.27) it is easy to deduce the analog of Huygens' principle in Cor. 1.4. This is done in the same way in [S], §4, Cor. 2. See also Prop. 18 in [STS].

Corollary 4.8. *Assume G has all its Cartan subgroups conjugate. Then if $\text{supp}(f_j) \subset B_R(o)$ ($1 \leq j \leq w$) the solution $u(x, H)$ has support in the region*

$$|H| - R \leq d(x, o) \leq |H| + R. \quad (4.28)$$

In fact, the assumption on G is equivalent to all $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ having even multiplicity ([H3], II, §3). This in turn implies by the Gindikin-Karpelevic formula for $\mathfrak{c}(\lambda)$ that $\mathfrak{c}(\lambda)^{-1}$ and even $(\pi(\lambda)\mathfrak{c}(\lambda))^{-1}$ is a polynomial. In addition, $\pi(\lambda)q^j(i\lambda)$ is a polynomial so the operators $\Lambda_{j,\sigma}$ are differential operators. Since

$$d(o, a \cdot o) \leq d(o, na \cdot o), \quad a \in A, n \in N, \quad (4.29)$$

we thus deduce from (4.27) that $u(x, H) = 0$ if $|H + A(x, b)| \geq R$ for $b \in B$. But if $x = gK, b = kM, k^{-1}g = nak_0$, (4.29) implies

$$|A(x, b)| = |A(k^{-1}g)| = |\log a| \leq d(o, na \cdot o) \leq d(o, x). \quad (4.30)$$

Thus if $|H| > R + d(o, x)$ we have for $b \in B$, $|H + A(x, b)| \geq |H| - |A(x, b)| \geq R$ so $u(x, H) = 0$ for $d(o, x) < |H| - R$. This, together with (4.15) implies the corollary.

5 Incoming Waves and Supports

In this section we shall prove the symmetric space analog of Cor. 1.3, relating incoming waves to support properties of the sources S_σ . Results of this type are given in §§ 4.1-4.2 in [STS] and §§4-5 in [PS] but we have not understood these papers well enough to make a detailed comparison. However, the proof of Theorem 5.1 below is mostly a variation, with some details added, of the proof of Theorem 4.10 in [PS].

As usual, we put

$${}^+\mathfrak{a} = \{H \in \mathfrak{a} : \langle H, H' \rangle > 0 \text{ for } H' \in \mathfrak{a}^+\}$$

and $\mathfrak{a}^- = -\mathfrak{a}^+$. Note that ${}^+\mathfrak{a} \cup \{0\}$ is closed ([H6], p. 438) and $\mathfrak{a}^+ \subset {}^+\mathfrak{a}$.

Definition Fix $\sigma \in W$. The wave $u(x, H)$ in (2.5) is said to be σ -incoming if

$$u(x, H) = 0 \text{ for } d(o, x) < |H|, \quad \sigma H \in \mathfrak{a}^-. \quad (5.1)$$

To distinguish from balls in X , an open ball in \mathfrak{a} of radius r and center H is denoted $V_r(H)$.

We shall now, in analogy with Cor. 1.3, characterize σ -incoming waves in terms of their σ -sources.

Theorem 5.1. Fix $\sigma \in W$ and $f_j \in \mathcal{D}(X)$, ($1 \leq j \leq w$). Then u is σ -incoming if and only if

$$\text{supp}(S_\sigma f)(b, \cdot) \subset \exp(\overline{{}^+\mathfrak{a}}) \text{ for all } b \in B. \quad (5.2)$$

Naturally, formula (4.27) is basic to the proof. First we prove (5.2) \Rightarrow (5.1). Suppose $H \in \mathfrak{a}$, $\sigma H \in \mathfrak{a}^-$ and $d(o, x) < |H|$. Then by (4.30), $A(x, b) \in V_{|H|}(0)$. Now let $H_0 \in {}^+\mathfrak{a} + (-\sigma H)$. Then $\langle H_0 + \sigma H, -\sigma H \rangle \geq 0$ so

$$-|H_0||H| \leq \langle H_0, \sigma H \rangle \leq -|H|^2$$

so $H_0 \notin V_{|H|}(0)$. Hence $A(x, b) \notin {}^+\mathfrak{a} + (-\sigma H)$ so by (5.2) and (4.27), $u(x, H) = 0$. This proves (5.2) \Rightarrow (5.1).

For the converse we assume (5.1). For $\varphi \in \mathcal{D}(\mathfrak{a})$ we consider the integral

$$v(x) = \int_{\mathfrak{a}} u(x, H) \varphi(H) dH. \quad (5.3)$$

Then by (4.21) we obtain

$$v(x) = \int_{\mathfrak{a}^* \times B} F_\lambda^\sigma(b) \psi^*(\lambda) e^{(i\lambda + \rho)(A(x, b))} d\lambda db, \quad (5.4)$$

$$\text{where } \psi(H) = \varphi(-\sigma^{-1}H) \text{ and } F_\lambda^\sigma(b) = \sum_k j_k^\sigma(\lambda) \tilde{f}_k(\lambda, b).$$

Note that

$$F_\lambda^\sigma(b) = h_\sigma(b, \cdot)^*(\lambda),$$

the Fourier transform (in A) of the function

$$h_\sigma(b, a) = \sum_k J_k^\sigma(e^\rho \widehat{f}_k)(b, a), \quad b \in B, a \in A. \quad (5.5)$$

Let $\sigma^* \in W$ be the element interchanging \mathfrak{a}^+ and \mathfrak{a}^- and put $\tau = \sigma^{-1}\sigma^*$. Then $\sigma H \in \mathfrak{a}^- \iff H \in \tau\mathfrak{a}^+$ so by (5.1)

$$u(x, H) = 0 \quad \text{for } d(o, x) < |H|, H \in \tau\mathfrak{a}^+. \quad (5.6)$$

Now fix $H_0 \in \tau\mathfrak{a}^+$ and consider the solution $(x, H) \longrightarrow u(x, H + H_0)$ to (2.7) whose initial data are

$$\left\{ \partial(p_i)_H (u(x, H + H_0)) \right\}_{H=0} = (\partial(p_i)u)(x, H_0).$$

We denote these by $F_i(x)$ ($1 \leq i \leq w$). Then (5.6) implies $F_i(x) = 0$ for $d(o, x) < |H_0|$, i.e., $\text{supp}(F_i) \cap B_{|H_0|}(o) = \emptyset$. Hence by Prop. 4.6,

$$u(x, H + H_0) = 0 \quad \text{for } x \in B_{|H_0|-|H|}(o). \quad (5.7)$$

Of course, $H + H_0 \in V_{|H_0|-\epsilon}(H_0) \iff |H| < |H_0| - \epsilon$ which in turn implies

$$B_{|H_0|-|H|}(o) \supset B_\epsilon(o).$$

Thus (5.7) implies the following result.

Lemma 5.2. *Let $\tau = \sigma^{-1}\sigma^*$, fix $H_0 \in \tau\mathfrak{a}^+$ and let $\epsilon < |H_0|$. Then*

$$u(x, H + H_0) = 0 \quad \text{for } d(o, x) < \epsilon \quad \text{if } H + H_0 \in V_{|H_0|-\epsilon}(H_0).$$

Now following [PS], Lemma 4.8 let

$$\mathfrak{a}_\epsilon(\tau) = \{H \in \mathfrak{a} : \langle H, H' \rangle > \epsilon |H'| \text{ for some } H' \in \tau\mathfrak{a}^+\}. \quad (5.8)$$

Then $\mathfrak{a}_\epsilon(\tau) = \tau \cdot \mathfrak{a}_\epsilon(e)$. Let H_0 in Lemma 5.2 run along a fixed ray $\{te_0 : t > 0, |e_0| = 1\}$ in $\tau\mathfrak{a}^+$. The balls $V_{|H_0|-\epsilon}(H_0)$ will then fill up the half space $\langle H, e_0 \rangle > \epsilon$. As H_0 varies in $\tau\mathfrak{a}^+$ (and $|H_0| > \epsilon$) these half spaces have union $\mathfrak{a}_\epsilon(\tau)$. Thus Lemma 5.2 implies the following result (Lemma 4.8 in [PS]).

Lemma 5.3. *Let $\tau = \sigma^{-1}\sigma^*$, $\epsilon > 0$ and $H \in \mathfrak{a}_\epsilon(\tau)$. Then*

$$u(x, H) = 0 \quad \text{for } d(o, x) < \epsilon.$$

Now let φ in (5.3) satisfy $\varphi \in \mathcal{D}(\mathfrak{a}_\epsilon(\tau))$. Then

$$v(x) = 0 \quad \text{for } d(o, x) < \epsilon, \quad (5.9)$$

which by (5.4) amounts to

$$v(x) = \int_{\mathfrak{a}^*} \psi^*(\lambda) \mathcal{P}_\lambda(F_\lambda^\sigma)(x) d\lambda = 0, \quad (5.10)$$

where \mathcal{P}_λ is the Poisson transform defined by

$$(\mathcal{P}_\lambda F)(x) = \int_B e^{(i\lambda+\rho)(A(x,b))} F(b) db. \quad (5.11)$$

As in the proof in [H5] of the injectivity criterion for the Poisson transform we exploit (5.10) by means of the equation

$$\{D_g(v(g \cdot o))\}_{g=e} = 0 \quad (5.12)$$

for all $D \in \mathbf{D}(G)$, the algebra of left-invariant differential operators on G . The kernel in (5.11) is given by

$$e^{(i\lambda+\rho)(A(gK, kM))} = \zeta_{-\lambda}(k^{-1}g),$$

where $\zeta_\lambda(g) = e^{(-i\lambda+\rho)(A(g))}$. Fix $\delta \in \widehat{K}_M$ (as in §3) and consider the operators $\epsilon_j(v_i)$ in §3. By [H7], p. 239 they satisfy

$$(\epsilon_j(v_i)\xi_{-\lambda})(k^{-1}) = \sum_{p=1}^{\ell(\delta)} \langle v_i, \delta(k)v_p \rangle Q^\delta(-\lambda)_{pj}. \quad (5.13)$$

Writing (5.4) and (5.10) in the form (with $dk_M = db$)

$$v(g \cdot o) = \int_{\mathfrak{a}^*} d\lambda \int_{K/M} \zeta_{-\lambda}(k^{-1}g) F(\lambda, kM) dk_M, \quad (5.14)$$

where $F(\lambda, b) = h_\sigma(b, \cdot)^*(\lambda)\psi^*(\lambda)$, we thus have

$$0 = \{\epsilon_j(v_i)_g(v(g \cdot o))\}_{g=e} = \int_{\mathfrak{a}^*} \sum_{p=1}^{\ell(\delta)} Q^\delta(-\lambda)_{pj} F_{pi}^\delta(\lambda) d\lambda.$$

Here

$$F_{pi}^\delta(\lambda) = \int_{K/M} \langle v_i, \delta(k)v_p \rangle F(\lambda, kM) dk_M.$$

Now $F(\lambda, b) = (h_\sigma(b, \cdot) \times \psi)^*(\lambda)$, \times denoting convolution on A (and \mathfrak{a}). Thus, defining

$$\zeta_{pi}(H) = \int_{K/M} h_\sigma(kM, \exp H) \langle v_i, \delta(k)v_p \rangle dk_M \quad (H \in \mathfrak{a})$$

we have for $1 \leq i, j \leq \ell(\delta)$,

$$\sum_{p=1}^{\ell(\delta)} \int_{\mathfrak{a}^*} Q^\delta(-\lambda)_{pj} (\zeta_{pi} \times \psi)^*(\lambda) d\lambda = 0. \quad (5.15)$$

Define $Q_{pj}^\delta \in S(\mathfrak{a})$ by $Q_{pj}^\delta(\lambda) = Q^\delta(\lambda)_{pj}$ and $P_{pj}^\delta \in S(\mathfrak{a})$ by $P_{pj}^\delta(\lambda) = Q_{pj}^\delta(i\lambda)$. Then, since $P^\delta(i\lambda) = Q^\delta(-\lambda)$, (5.15) implies by the Fourier inversion formula,

$$\sum_{p=1}^{\ell(\delta)} [(\partial(P_{pj}^\delta)\zeta_{pi}) \times \psi](0) = 0,$$

so, if $\check{\psi}(H) = \psi(-H)$,

$$\sum_{p=1}^{\ell(\delta)} \int_{\mathfrak{a}} (\partial(P_{pj}^\delta)\zeta_{pi})(H) \check{\psi}(H) dH = 0. \quad (5.16)$$

This holds whenever $\varphi \in \mathcal{D}(\mathfrak{a}_\epsilon(\tau))$, and $\epsilon > 0$.

Let $\mathfrak{a}_0(\tau) = \lim_{\epsilon \rightarrow 0} \mathfrak{a}_\epsilon(\tau)$. Then $\mathfrak{a}_0(\tau) = \tau \mathfrak{a}_0(e)$ so the complement of $\mathfrak{a}_0(\tau)$ is $-\tau(+\mathfrak{a}) \cup \{0\} = \sigma^{-1}(+\mathfrak{a}) \cup \{0\}$. Since $\check{\psi}(H) = \varphi(\sigma^{-1}H)$, (5.16) implies

$$\sum_{p=1}^{\ell(\delta)} \partial(P_{pj}^\delta)\zeta_{pi} = 0 \quad \text{on } \mathfrak{a} \setminus +\mathfrak{a}. \quad (5.17)$$

The matrix P^δ with entries $P_{pj}^\delta \in S(\mathfrak{a})$ satisfies

$$\det(P^\delta) = P_c P^\delta, \quad (5.18)$$

where P_c is a matrix whose entries $P_{pi}^c \in S(\mathfrak{a})$ are the cofactors of P^δ . Applying $\partial(P_{pj}^c)$ to (5.17) and summing on j we get by (5.18),

$$(\partial(\det P^\delta)\zeta_{pi})(H) = 0 \quad \text{for } H \in \mathfrak{a} \setminus +\mathfrak{a}.$$

Now $\det P^\delta(\lambda) = \det Q^\delta(i\lambda)$ which by [H7], III, Theorem 4.2 and Cor. 11.3 is a product of factors $\langle \lambda, \alpha \rangle + c_{j,\alpha}$ where $c_{j,\alpha} > 0$. Here $\alpha \in \sum_0^+$ and j runs through a certain finite set. (Apart from variable shifts this is contained in [K], Theorems 5 and 7.) With $H_\alpha \in \mathfrak{a}$ given by $\langle H_\alpha, H \rangle = \alpha(H)$ we thus have for $F = \zeta_{pi}$, which belongs to $\mathcal{S}(\mathfrak{a})$,

$$\prod_{\alpha,j} (\partial(H_\alpha) + c_{j,\alpha})F = 0 \quad \text{on } \mathfrak{a} \setminus +\mathfrak{a}. \quad (5.19)$$

We pull out a single factor $\partial(H_\alpha) + c$ ($c > 0$) and consider the equation

$$(\partial(H_\alpha) + c)f = 0 \quad \text{in } \mathfrak{a} \setminus +\mathfrak{a}, \quad f \in \mathcal{S}(\mathfrak{a}).$$

Then the function

$$g(H) = e^{c_0 \alpha(H)} f(H) \quad c_0 = c / \langle \alpha, \alpha \rangle \quad (5.20)$$

satisfies

$$\partial(H_\alpha)g = 0 \quad \text{in } \mathfrak{a} \setminus +\mathfrak{a}. \quad (5.21)$$

Fix $H_0 \in \mathfrak{a} \setminus +\mathfrak{a}$. Since $-H_\alpha \in \mathfrak{a} \setminus +\mathfrak{a}$ and since $\mathfrak{a} \setminus +\mathfrak{a}$ is star-shaped with respect to 0 we see that $H_0 - tH_\alpha = t(\frac{H_0}{t} - H_\alpha)$ belongs to $\mathfrak{a} \setminus +\mathfrak{a}$ for $t > 0$ sufficiently large. Then (5.21) implies

that the function $t \rightarrow g(H_0 - tH_\alpha)$ is a constant C for large $t > 0$ so by (5.20)

$$f(H_0 - tH_\alpha) = Ce^{-C_0\alpha(H_0)}e^{C_0\alpha(H_\alpha)t}. \quad (5.22)$$

This contradicts $f \in \mathfrak{S}(\mathfrak{a})$ unless $f \equiv 0$ on $\mathfrak{a} \setminus^+ \mathfrak{a}$. Iterating this argument for the factors in (5.19) we deduce

$$\zeta_{pi} \equiv 0 \quad \text{on } \mathfrak{a} \setminus^+ \mathfrak{a}, \quad 1 \leq i, p \leq \ell(\delta). \quad (5.23)$$

Observe now that for each $k_0 \in K$ assumption (5.1) is valid for the function $u_{k_0} : (x, H) \rightarrow u(k_0 \cdot x, H)$. Since $A(k_0 \cdot x, b) = A(x, k_0^{-1} \cdot b)$ it is clear from (4.25) that replacing u by u_{k_0} amounts to replacing h_σ by the function $(b, a) \rightarrow h_\sigma(k_0 \cdot b, a)$. Thus (5.23) implies

$$\int_{K/M} h_\sigma(k_0 k M, \exp H) \langle v_i, \delta(k) v_p \rangle dk_M = 0, \quad H \in \mathfrak{a} \setminus^+ \mathfrak{a}, \quad (5.24)$$

and here we put $u = k_0 k, k = k_0^{-1} u$. Since the vectors $\delta(k_0) v_i (k_0 \in K, 1 \leq i \leq \ell(\delta))$ span V_δ , (5.24) shows that the function $uM \rightarrow h_\sigma(uM, \exp H)$ is orthogonal to all $\langle v, \delta(k) v_p \rangle (1 \leq p \leq \ell(\delta), v \in V_\delta)$. Since $\delta \in \widehat{K}_M$ is arbitrary this proves

$$h_\sigma(b, \exp H) = 0 \quad \text{for } b \in B, H \in \mathfrak{a} \setminus^+ \mathfrak{a}$$

so (5.2) is proved.

6 Energy and Spectral Representation

In [S] §2, Shahshahani transfers the Cauchy problem (2.7)(2.8) to a vector formulation and uses this to generalize the classical energy for the wave equation to the present context. We recall the definition.

Let $p \in S(\mathfrak{a})$. By Lemma 4.3 (i) we have a matrix $L_p = (L_p)_{ij}$ with entries in $I(\mathfrak{a})$ such that

$$pp_j = \sum_1^w (L_p)_{ij} p_i, \quad 1 \leq j \leq w.$$

Let $D_p = (D_p)_{ij}$ denote the corresponding matrix with entries in $\mathbf{D}(G/K)$, i.e., $\Gamma((D_p)_{ij}) = (L_p)_{ij}$ (cf. (2.5)). Let ${}^t D_p$ denote the transpose of D_p .

Lemma 6.1. *The Cauchy problem (2.7)-(2.8) is equivalent to the problem*

$${}^t D_p \mu = \partial(p)\mu, \quad p \in S(\mathfrak{a}), \quad (6.1)$$

where the column vector μ is a smooth map from $X \times \mathfrak{a}$ to \mathbf{C}^w and

$${}^t \mu(x, 0) = (f_1(x), \dots, f_w(x)). \quad (6.2)$$

More precisely, if u satisfies (2.7)-(2.8) then

$${}^t \mu(x, H) = ((\partial(p_1)u)(x, H), \dots, (\partial(p_w)u)(x, H)) \quad (6.3)$$

satisfies (6.1)(6.2) above and if μ satisfies (6.1)(6.2) then $u = \mu_1$ (first component) satisfies (2.7)(2.8).

Proof: Since $p_k = p_k p_1 = \sum_i (L_{p_k})_{i1} p_i$ we have $(L_{p_k})_{i1} = \delta_{ik}$. Thus, if μ satisfies (6.1)(6.2) then

$$\partial(p_k)\mu_1 = (\partial(p_k)\mu)_1 = ({}^t D_{p_k} \mu)_1 = \sum_i (D_{p_k})_{i1} \mu_i = \mu_k$$

so (2.8) holds. Also, if $D \in \mathbf{D}(G/K)$, $L_{\Gamma(D)}$ is a diagonal matrix, ${}^t D_{\Gamma(D)} = D I_w$ so (6.1) implies $D\mu_1 = \partial(\Gamma(D))\mu_1$. Thus $u = \mu_1$ satisfies (2.7).

On the other hand, assume u satisfies (2.7)(2.8). Define μ by (6.3). Then if $p \in S(\mathfrak{a})$,

$$\begin{aligned} (\partial(p)\mu)_j &= \partial(p)\mu_j = \partial(p)\partial(p_j)u = \partial(pp_j)u = \partial\left(\sum_i (L_p)_{ij} p_i\right)(u) = \sum_i \partial(p_i)\partial(\Gamma(D_p)_{ij})u \\ &= \sum_i \partial(p_i)(D_p)_{ij}u = \sum_i (D_p)_{ij}\partial(p_i)u = ({}^t D_p \mu)_j \end{aligned}$$

so (6.1) holds; also (6.2) is obvious.

Remark If (6.1) holds for $p \in \mathfrak{a}$ it holds for all $p \in S(\mathfrak{a})$.

In fact, the map $p \rightarrow L_p$ is an isomorphism of $S(\mathfrak{a})$ onto an algebra of $w \times w$ matrices with entries in $I(\mathfrak{a})$. Thus $\partial(pq)\mu = {}^t D_{pq}\mu$, justifying the remark.

Consider now the matrix $A = (A_{ij})$ with entries in $I(\mathfrak{a})$ given by

$$A_{ij} = (\pi q^j, \theta(\pi q^i)), \quad (6.4)$$

where $(,)$ is defined in (4.4). Let $\mathcal{A} = (\mathcal{A}_{ij})$ be the matrix with entries in $\mathbf{D}(G/K)$ given by $\Gamma(\mathcal{A}_{ij}) = A_{ij}$.

Definition Given $u, v \in \mathcal{E}(X \times \mathfrak{a})$ the *energy form* is defined by

$$E(u, v; H) = \int_X ({}^t \mu \mathcal{A} \bar{\nu})(x, H) dx, \quad (6.5)$$

the integral assumed convergent. Here ${}^t \mu, \nu$ are given by (6.3) from u and v , respectively.

It is a routine matter to verify that the function ${}^t \mu \mathcal{A} \bar{\nu}$ is independent of the choice of basis of $H(\mathfrak{a})$. However, the homogeneity of each p_i plays a role in the proof of Theorem 6.2 and in Theorem 6.3.

In the rank-one example discussed before this reduces to

$$E(u, v; tH_0) = \frac{\langle \pi, \pi \rangle}{2} \int_X (u_t \bar{v}_t - u(L + |\rho|^2) \bar{v}) dx$$

which up to the factor $\langle \pi, \pi \rangle$ agrees with the usual formula ([H7], p. 501).

As used in the proof of (4.1) we have

$$\Gamma(D^*) = \theta \Gamma(D) \quad D \in \mathbf{D}(G/K).$$

Hence

$$\Gamma(\mathcal{A}_{ij}^*)(i\lambda) = \Gamma(\mathcal{A}_{ij})(-i\lambda) = A_{ij}(-i\lambda) = A_{ji}(i\lambda) = \Gamma(\mathcal{A}_{ji})(i\lambda)$$

so $\mathcal{A}_{ij}^* = \mathcal{A}_{ji}$. This implies the Hermitian symmetry,

$$E(u, v; H) = \overline{E(v, u; H)}.$$

While this relation is a simple formality, the positivity

$$E(u, u; H) \geq 0$$

which is a consequence of (6.11) below in connection with Theorem 6.2 is more subtle and seems to require the Fourier transform theory. Because of (6.11) we have a norm

$$\|F\| = \left\{ \int_X ({}^t F \mathcal{A} \overline{F})(x) dx \right\}^{1/2}, \quad {}^t F = (f_1, \dots, f_w)$$

on $\mathcal{D}(X) \times \dots \times \mathcal{D}(X)$ (w times) which by the injectivity in Theorem 6.4 is strictly positive for $F \neq 0$.

Theorem 6.2. *Let u and v be the solutions to (2.7)(2.8) with initial data $(f_i) \subset \mathcal{D}(X)$, $(g_j) \subset \mathcal{E}(X)$. (Theorem 4.2). Then the energy form $E(u, v; H)$ is constant in the time $H \in \mathfrak{a}$.*

Proof: (Shahshahani [S]). Consider the matrix $B = (B_{ij})$ given by

$$B_{ij} = (\pi q^i, \pi q^j) \in I(\mathfrak{a})$$

and J the diagonal matrix whose i^{th} entry is $(-1)^{d_i}$ where $d_i = \deg p_i = -\deg(q^i)$. Then $A_{ij} = (\pi q^j, \theta(\pi q^i)) = (-1)^{\deg(\pi)+d_i} B_{ij}$ so if $\mathcal{B} = (\mathcal{B}_{ij})$ with $\Gamma(\mathcal{B}_{ij}) = B_{ij}$,

$$(-1)^{\deg(\pi)} J B = A, \quad (-1)^{\deg(\pi)} J \mathcal{B} = \mathcal{A}.$$

Taking inner product with q^m and p_k , respectively, we derive the relations

$$\sum_i B_{ik} p_i = \pi^2 q^k, \quad p q^j = \sum_i (L_p)_{ji} q^i, \quad p \in S(\mathfrak{a}),$$

which imply quickly that the matrix $L_p B$ is symmetric. Thus

$$L_p B = B^t L_p, \quad D_p \mathcal{B} = \mathcal{B}^t D_p.$$

In particular, take $p = H \in \mathfrak{a}$ and observe that $\partial(p) {}^t \mu = {}^t(\partial(p)\mu)$. Then if u and v are solutions and μ, ν determined by (6.3) we have (by (6.1))

$$\begin{aligned} \partial(p)_H(E(u, v; H)) &= \int_X [{}^t({}^t D_p \mu) \mathcal{A} \overline{\nu} + {}^t \mu \mathcal{A} \partial(p) \overline{\nu}] (x, H) dx \\ &= (-1)^{\deg(\pi)} \int_X [{}^t({}^t D_p \mu) J \mathcal{B} \overline{\nu} + {}^t \mu J \mathcal{B}^t D_p \overline{\nu}] (x, H) dx. \end{aligned}$$

In the last term we replace $\mathcal{B} {}^t D_p$ by $D_p \mathcal{B}$ and put $\xi = \mathcal{B}\bar{v}$. The integrand is a sum of the terms

$$\begin{aligned} \sum_j ({}^t({}^t D_p \mu))_j (J\xi)_j &= \sum_{i,j} (D_p)_{i,j} \mu_i (-1)^{d_j} \xi_j \\ \sum_i ({}^t \mu)_i (J D_p \xi)_i &= \sum_{i,j} \mu_i (-1)^{d_i} (D_p)_{i,j} \xi_j \end{aligned}$$

so it would suffice to prove

$$\int_X [(D_p)_{i,j} \mu_i \xi_j + (-1)^{d_i - d_j} \mu_i (D_p)_{i,j} \xi_j] (x, H) dx = 0.$$

Since $pp_j = \sum_i (L_p)_{i,j} p_i$, $(L_p)_{i,j}$ is either 0 or homogeneous of degree $1 + d_j - d_i$. Also $\Gamma(D^*) = \theta\Gamma(D)$ as used before, so

$$\Gamma((D_p)_{i,j}^*) = \theta((L_p)_{i,j}) = (-1)^{1+d_j-d_i} \Gamma((D_p)_{i,j}),$$

whence

$$(D_p)_{i,j}^* = (-1)^{1+d_j-d_i} (D_p)_{i,j}.$$

Thus the last integral equals

$$\int_X [(D_p)_{i,j} \mu_i \xi_j - \mu_i (D_p)_{i,j}^* \xi_j] (x, H) = 0.$$

Hence $\partial(p)_H(E(u, v; H)) = 0$ as claimed.

Remark In [STS], Prop. 1 a similar transfer of (2.7)(2.8) to a vector formulation is described. The corresponding energy is given by (4) §1.2 in [STS]. It differs from formula (6.5) above in that the Cartan involution in (6.4) is absent. Thus the positivity of the energy stated in Theorem 1 of [STS] seems doubtful. The unproved statement in Prop. 3 in [STS] that this energy is time-independent seems to us not to be compatible with Theorem 6.2.

As in [STS] and [S] we consider now for each $\sigma \in W$, $\lambda \in \mathfrak{a}_C^*$, $b \in B$ the particular solution $\mu^\sigma(x, H; \lambda, b)$ of (6.1) given by (6.3) for

$$u^\sigma(x, H; \lambda, b) = e^{i\lambda(H) + (i\sigma\lambda + \rho)(A(x,b))}. \quad (6.6)$$

This is indeed a solution for by (3.2),

$$(Du^\sigma) = \Gamma(i\sigma\lambda)u^\sigma = \Gamma(i\lambda)u^\sigma, \quad D \in \mathbf{D}(G/K)$$

and

$$\partial(\Gamma(D))u^\sigma = \Gamma(i\lambda)u^\sigma.$$

We consider then the linear map

$$\mathcal{E}^\sigma : F(x) \rightarrow \int_X ({}^t F \mathcal{A} \bar{\mu}^\sigma)(x, 0; \lambda, b) dx, \quad (6.7)$$

where ${}^t F(x) = (f_1(x), \dots, f_w(x))$, $f_i \in \mathcal{D}(X)$. Thus \mathcal{E}^σ maps $\mathcal{D}(X) \times \dots \times \mathcal{D}(X)$ into a function space on $\mathfrak{a}^* \times B$. The following result shows that \mathcal{E}^σ is intimately related to the Fourier transform on X .

Theorem 6.3. For each $\sigma \in W$

$$\mathcal{E}^\sigma(F)(\lambda, b) = \pi(\lambda)^2 \sum_k q^k(i\lambda) \tilde{f}_k(\sigma\lambda, b).$$

Proof: For $\lambda \in \mathfrak{a}^*$ (real) we have

$$\mu_j^\sigma(x, 0; \lambda, b)^- = p_j(-i\lambda) e^{(-i\sigma\lambda + \rho)(A(x, b))}$$

so

$$\mathcal{E}^\sigma(F) = \int_X \sum_{i,j} f_i(x) A_{ij}(-i\sigma\lambda) p_j(-i\lambda) e^{(-i\sigma\lambda + \rho)(A(x, b))} dx, \quad (6.8)$$

which by the W -invariance of A_{ij} equals

$$\sum_k \tilde{f}_k(\sigma\lambda, b) \sum_j A_{kj}(-i\lambda) p_j(-i\lambda). \quad (6.9)$$

Here the sum \sum_j equals

$$\sum_j \left[\sum_{\tau \in W} (\pi q^j)(-\tau i\lambda) (\pi q^k)(\tau i\lambda) \right] p_j(-i\lambda) = \sum_{\tau \in W} (\pi q^k)(\tau i\lambda) \sum_j (\pi q^j)(-\tau i\lambda) p_j(-i\lambda).$$

Now if $d_j = \text{degree } p_j$ then $\text{degree } q^j = -d_j$ so the last \sum_j equals

$$(-1)^{\deg \pi} \sum_j (\pi q^j)(\tau i\lambda) p_j(i\lambda).$$

Thus

$$\begin{aligned} \sum_j A_{kj}(-i\lambda) p_j(-i\lambda) &= (-1)^{\deg(\pi)} \sum_j p_j(i\lambda) \sum_\sigma (\pi q^k)(\sigma i\lambda) (\pi q^j)(\sigma i\lambda) \\ &= (-1)^{\deg(\pi)} \sum_j p_j(i\lambda) (\pi q^k, \pi q^j)(i\lambda). \end{aligned}$$

However, $\sum_j p_j(\pi q^k, \pi q^j) = \pi^2 q^k$ as we see by taking inner product with q^m and noting that $(\pi^2 q^k, q^m) = (\pi q^k, \pi q^m)$. Putting these formulas together we see that expression (6.9) equals

$$\sum_k \tilde{f}_k(\sigma\lambda, b) (-1)^{\deg \pi} \pi(i\lambda)^2 q^k(i\lambda)$$

and this proves the theorem.

Given $H_0 \in \mathfrak{a}$ let U_{H_0} denote the map of $\mathcal{D}(X) \times \dots \times \mathcal{D}(X)$ into itself given by

$$U_{H_0} : \mu(x, 0) \rightarrow \mu(x, H_0), \quad (6.10)$$

μ satisfying (6.1). Since $\partial(p)$ is invariant under translations of \mathfrak{a} it follows that for each $H \in \mathfrak{a}$, U_{H_0} maps the function $x \rightarrow \mu(x, H)$ into the function $x \rightarrow \mu(x, H + H_0)$.

In view of Theorem 6.3 the following result can be viewed as an enlargement of the Plancherel theorem for the Fourier transform on X . In fact it will be proved by the tools described in §3.

Theorem 6.4. For each $\sigma \in W$ the map \mathcal{E}^σ is an injective norm-preserving map of $\mathcal{D}(X) \times \dots \times \mathcal{D}(X)$ onto a dense subspace of $L^2(\mathfrak{a}^* \times B, d\lambda db / |\pi(\lambda)\mathbf{c}(\lambda)|^2)$. Thus

$$\int_{\mathfrak{a}^* \times B} \mathcal{E}^\sigma(F) \overline{\mathcal{E}^\sigma(F)}(\lambda, b) \frac{d\lambda db}{|\pi(\lambda)\mathbf{c}(\lambda)|^2} = \int_X ({}^t F \mathcal{A} \overline{F})(x) dx. \quad (6.11)$$

Moreover, \mathcal{E}^σ intertwines U_{H_0} and the endomorphism

$$e(H_0) : \varphi(\lambda, b) \rightarrow e^{i\lambda(H_0)} \varphi(\lambda, b) \quad \text{of } L^2(\mathfrak{a}^* \times B, \frac{d\lambda db}{|\pi(\lambda)\mathbf{c}(\lambda)|^2}),$$

i.e.,

$$\mathcal{E}^\sigma U_{H_0} = e(H_0) \mathcal{E}^\sigma.$$

Proof: We separate the various statements.

a. *The norm identity* (6.11). We have by Prop. 6.3,

$$\mathcal{E}^\sigma(F) \overline{\mathcal{E}^\sigma(F)} = \pi(\lambda)^4 \sum_{k, \ell} q^k(i\lambda) q^\ell(-i\lambda) \widetilde{f}_k(\sigma\lambda, b) \overline{\widetilde{f}_\ell(\sigma\lambda, b)} \quad (6.12)$$

and by [H7], III, (12) §1,

$$\int \widetilde{f}_k(\sigma\lambda, b) \overline{\widetilde{f}_\ell(\sigma\lambda, b)} db$$

is independent of σ . Thus the left hand side of (6.11) equals

$$\frac{1}{w} \int_{\mathfrak{a}^* \times B} \sum_{k, \ell} \pi(\lambda)^2 \widetilde{f}_k(\lambda, b) \overline{\widetilde{f}_\ell(\lambda, b)} \sum_{\sigma \in W} q^k(\sigma i\lambda) q^\ell(-\sigma i\lambda) \frac{d\lambda db}{|\mathbf{c}(\lambda)|^2}.$$

Writing $\pi(\lambda)^2 = \pi(\sigma i\lambda) \pi(-\sigma i\lambda)$ our expression becomes

$$\frac{1}{w} \int_{\mathfrak{a}^* \times B} \sum_{k, \ell} \widetilde{f}_k(\lambda, b) \overline{\widetilde{f}_\ell(\lambda, b)} A_{\ell k}(i\lambda) \frac{d\lambda db}{|\mathbf{c}(\lambda)|^2}. \quad (6.13)$$

On the other hand if $g \in \mathcal{D}(X)$ we have the inversion formula

$$g(x) = \frac{1}{w} \int_{\mathfrak{a}^* \times B} \widetilde{g}(\lambda, b) e^{(i\lambda + \rho)(A(x, b))} |\mathbf{c}(\lambda)|^{-2} d\lambda db.$$

Also $A_{\ell k}$ is real so $\mathcal{A}_{\ell k}$ has real coefficients and

$$(\mathcal{A}_{\ell k} g)^- = \mathcal{A}_{\ell k} \overline{g}, \quad \Gamma(\mathcal{A}_{\ell k})(i\lambda) = A_{\ell k}(i\lambda). \quad (6.14)$$

By the Plancherel formula and (6.14)

$$\int_X f(x) (\mathcal{A}_{k\ell} \overline{g})(x) dx = \frac{1}{w} \int_{\mathfrak{a}^* \times B} \widetilde{f}(\lambda, b) \left(A_{k\ell}(i\lambda) \widetilde{g}(\lambda, b) \right)^- |\mathbf{c}(\lambda)|^2 d\lambda db.$$

However, the definition (6.4) shows easily that for λ real, $\overline{A_{k\ell}(i\lambda)} = A_{\ell k}(i\lambda)$ and we introduce this in the last integral. Taking now $f = f_k$, $g = f_\ell$ and summing on k, ℓ we see that the right hand side of (6.11) equals (6.13). This proves (6.11).

b. *The injectivity.* For this suppose $\mathcal{E}^\sigma(F) = 0$. Then by Theorem 6.3

$$\sum_k q^k(i\lambda) \tilde{f}_k(\sigma\lambda, b) = 0. \quad (6.15)$$

Here we multiply by $e^{(i\sigma\lambda+\rho)(A(x,b))}$ and integrate over B . Since this integral

$$\int_B \tilde{f}_k(\lambda, b) e^{(i\lambda+\rho)(A(x,b))} db \quad (6.16)$$

is W -invariant in λ we derive from (6.15) for each $s \in W$,

$$\sum_k q^k(si\lambda) \int_B \tilde{f}_k(\lambda, b) e^{(i\lambda+\rho)(A(x,b))} db = 0.$$

We multiply by $p_\ell(si\lambda)$ and sum on s . This implies that (6.16) vanishes identically in x . By the injectivity (see [H5]) of the Poisson transform for $\lambda \in \mathfrak{a}^*$ we have $\tilde{f}_k(\lambda, b) \equiv 0$ whence by the injectivity of the Fourier transform, $f_k \equiv 0$.

The proof has the following consequence.

Corollary 6.5 (Schlichtkrull).

$$f_k(x) = \int_{\mathfrak{a}^* \times B} p_k(i\lambda) \mathcal{E}^\sigma(F)(\lambda, b) e^{(i\sigma\lambda+\rho)(A(x,b))} \frac{d\lambda db}{\pi(\lambda)^2 |\mathbf{c}(\lambda)|^2}.$$

In fact, substitute for $\mathcal{E}^\sigma(F)$ from Theorem 6.3 and use the W -invariance of (6.16). Again, replacing λ by $s\lambda$ and summing on s the right hand side reduces to $f_k(x)$ because of (3.4).

c. *The surjectivity.* Because of Theorem 6.3 this amounts to showing that if $g(\lambda, b)$ satisfies

$$\int_{\mathfrak{a}^* \times B} \tilde{f}(\sigma\lambda, b) \pi(\lambda)^2 q^j(i\lambda) g(\lambda, b) \frac{d\lambda db}{|\pi(\lambda) \mathbf{c}(\lambda)|^2} = 0$$

for all $f \in \mathcal{D}(X)$, and all $j(1 \leq j \leq w)$ then $g \equiv 0$. Putting $g^j(\lambda, b) = q^j(\sigma^{-1}i\lambda) g(\sigma^{-1}\lambda, b)$ we then have

$$\int_{\mathfrak{a}^* \times B} \tilde{f}(\lambda, b) g^j(\lambda, b) |\mathbf{c}(\lambda)|^{-2} d\lambda db = 0, \quad f \in \mathcal{D}(X), \quad 1 \leq j \leq w. \quad (6.17)$$

If we had this relation on $\mathfrak{a}_+^* \times B$ the problem would be solved since by the Plancherel theorem (§3) the functions $\tilde{f}(\lambda, b)$ are dense in $L^2(\mathfrak{a}_+^* \times B)$, $|\mathbf{c}(\lambda)|^{-2} d\lambda db$. The fact that (6.17) holds for all j is decisive in passing from $\mathfrak{a}_+^* \times B$ to $\mathfrak{a}^* \times B$.

Once again we invoke the representations $\delta \in \widehat{K}_M$ and the results for the δ -spherical transform described in §3. We specialize f in (6.17) to $f \in \mathcal{D}_\delta^\vee(X)$ and put

$$g_\delta^j(\lambda) = \int_K g^j(\lambda, kM)\delta(k) dk, \quad g_\delta(\lambda) = \int_K g(\lambda, kM)\delta(k) dk. \quad (6.18)$$

If $A, B(k) \in \text{Hom}(V_\delta, V_\delta)$ then

$$\int_K \text{Tr}(AB(k)) dk = \text{Tr}\left(A \int_K B(k) dk\right).$$

Using in addition $\mathfrak{a}^* = \cup_{s \in W} s\mathfrak{a}_+^*$ as well as formula (3.9) we derive from (6.17) the relation

$$\sum_{s \in W} \int_{\mathfrak{a}_+^*} \text{Tr}(\tilde{f}(s\lambda)g_\delta^j(s\lambda)) |\mathbf{c}(\lambda)|^{-2} d\lambda = 0. \quad (6.19)$$

Now according to (3.11) and (3.8)

$$\tilde{f}(s\lambda) = Q_\delta^\vee(s\lambda)F_\delta(\lambda) = Q_\delta^\vee(s\lambda)C_\delta^\vee C_\delta^{-1}F_\delta(\lambda), \quad (6.20)$$

where F_δ is a W -invariant holomorphic function on \mathfrak{a}_C^* , rapidly decreasing on \mathfrak{a}^* and with values in $\text{Hom}(V_\delta, E_\delta^\vee)$. We insert the expression (6.20) into (6.19). The map $Q_\delta^\vee(s\lambda)C_\delta^\vee$ is a linear transformation of V_δ^M into itself, but we extend it to V_δ defining it 0 on the orthocomplement of V_δ^M in V_δ . Then $g_\delta^j(s\lambda)$ can be shifted to the left in (6.19) and we obtain

$$\int_{\mathfrak{a}_+^*} \text{Tr}(G_j(\lambda)F_\delta^o(\lambda)) |\mathbf{c}(\lambda)|^{-2} d\lambda = 0, \quad (6.21)$$

where

$$G_j(\lambda) = \sum_{s \in W} g_\delta^j(s\lambda)Q_\delta^\vee(s\lambda)C_\delta^\vee, \quad \lambda \in \mathfrak{a}^*. \quad (6.22)$$

$$F_\delta^o(\lambda) = C_\delta^{-1}F_\delta(\lambda), \quad \lambda \in \mathfrak{a}^*. \quad (6.23)$$

Here $F_\delta^o(\lambda)$ maps V_δ into V_δ^M . By the Paley-Wiener theorem for the δ -spherical transform (see remarks following (3.11)) we can for each i and k ($1 \leq i \leq \ell(\delta)$), ($1 \leq k \leq d(\delta)$) take $F_\delta^o(\lambda) = \varphi(\lambda)P^{ik}$ where $\varphi(\lambda)$ is an arbitrary W -invariant holomorphic scalar-valued function of exponential type on \mathfrak{a}_C^* , rapidly decreasing on \mathfrak{a}^* , and $P^{ik}(v_m) = \delta_{km}v_i$ ($1 \leq m \leq d(\delta)$). Then

$$\text{Tr}(G_j(\lambda)F_\delta^o(\lambda)) = \varphi(\lambda)G_j(\lambda)_{ki}.$$

Thus by (6.21)

$$G_j(\lambda)_{ki} = 0 \quad 1 \leq i \leq \ell(\delta), \quad 1 \leq k \leq d(\delta), \quad (6.24)$$

for almost all $\lambda \in \mathfrak{a}_+^*$, hence by the W -invariance for almost all $\lambda \in \mathfrak{a}^*$. This means that $G_j(\lambda)$ maps V_δ^M into 0 and by definition it is 0 on $(V_\delta^M)^\perp$. Thus

$$G_j(\lambda) = 0 \text{ for almost all } \lambda \in \mathfrak{a}^*, \quad 1 \leq j \leq w. \quad (6.25)$$

We now have by (6.18)

$$g_\delta^j(s\lambda) = q^j(\sigma^{-1}si\lambda)g_\delta(\sigma^{-1}s\lambda), \quad 1 \leq j \leq w. \quad (6.26)$$

If $a_{mn}(\lambda)$ is an arbitrary matrix entry in $g_\delta(\sigma^{-1}\lambda)Q_\delta^\vee(\lambda)C_\delta^\vee$ and s_1, \dots, s_w the elements in W then by (6.25)(6.26),

$$\sum_{r=1}^w q^j(\sigma^{-1}s_r i\lambda)a_{mn}(s_r\lambda) = 0, \quad 1 \leq j \leq w. \quad (6.27)$$

The matrix $\{q^j(s_r\nu)\}_{1 \leq j, r \leq w}$ being non-singular for regular ν ([GV], Cor. 5.5.2), (6.27) implies

$$g_\delta(\sigma^{-1}\lambda)Q_\delta^\vee(\lambda)C_\delta^\vee = 0 \quad \text{for almost all } \lambda.$$

On V_δ^M , $\det(Q_\delta^\vee(\lambda)C_\delta^\vee) \neq 0$ so

$$g_\delta(\sigma^{-1}\lambda)v = 0 \quad \text{for } v \in V_\delta^M. \quad (6.28)$$

However, if $u \in V_\delta$ we have for $m \in M$

$$\begin{aligned} g_\delta(\lambda)u &= \int_K g(\lambda, kM)\delta(k)u \, dk \\ &= \int_K g(\lambda, kM)\delta(km)u \, dk = \int_M g(\lambda, kM)\delta(k) \, dk \int_M \delta(m)u \, dm \end{aligned}$$

which vanishes by (6.28). Thus $g_\delta(\lambda) = 0$ for all $\delta \in \widehat{K}_M$ whence $g(\lambda, b) = 0$ for almost all $\lambda \in \mathfrak{a}^*$ as desired.

- d. *Intertwining.* Fix $H_0 \in \mathfrak{a}$. As remarked after (6.10) U_{H_0} maps the function $x \rightarrow \mu(x, H)$ into the function $x \rightarrow \mu(x, H + H_0)$ and the solution u equals μ_1 . Passing from $u(x, H)$ in (4.21) to $u(x, H + H_0)$ amounts to replacing $\tilde{f}_k(\lambda, b)$ by $\tilde{f}_k(\lambda, b)e^{i\lambda(\sigma H_0)}$ (and $\tilde{f}_k(\sigma\lambda, b)$ by $\tilde{f}_k(\sigma\lambda, b)e^{i\lambda(H_0)}$). Now Theorem 6.3 shows that

$$\mathcal{E}^\sigma U_{H_0}(F) = e^{i\lambda(H_0)} \mathcal{E}^\sigma(F).$$

Remark 1 While Theorem 6.3 is new, a proof of Theorem 6.4 for the case $\sigma = e$ appears in [S] pp. 214-224. The proof is entirely different. Theorem 1 in [STS] is formally similar to Theorem 6.4 but differs from it in substance because the maps corresponding to \mathcal{E}^σ are given in terms of the different definition of energy indicated in the remark following Theorem 6.2. We observe that in contrast to our proof of Theorem 6.4 the sketched proof of Theorem 1 in [STS] relies on the time-invariance of the energy which as remarked before seems at odds with Theorem 6.2.

Remark 2 In [HS] the range of \mathcal{E}^σ is determined more explicitly.

We conclude with the analog of Friedlander's Theorem 5.1. More general limit results appear in [STS] and except for the formulation, the result with the proof below coincides with Theorem 3.11 in [PS].

In analogy with (4.23) let Q_k , J and \bar{J} denote the operators on $\mathcal{S}(A)$ which under the Fourier transform on A correspond to multiplication by $q^k(i\lambda)\pi(i\lambda)$, $c(\lambda)^{-1}$ and $c(-\lambda)^{-1}$, respectively.

Theorem 6.6. *Let $\sigma \in W$, $H, H_0 \in \mathfrak{a}^+$ and put $a_t = \exp(tH)$. Then if $k \in K$,*

$$\lim_{t \rightarrow t^\infty} e^{t\rho(H)} (\partial(\pi)u)(ka_t \cdot o, -t\sigma H + \sigma H_0) = (\bar{J} \sum_k Q_k^{\sigma^{-1}}(e^\rho \hat{f}_k))(kM, H_0)$$

Proof: Because of (4.21),

$$(\partial(\pi)u)(x, H) = \int_{\mathfrak{a}^* \times B} F_\sigma(\lambda, b) e^{(i\lambda + \rho)(A(x, b))} e^{i\sigma\lambda(H)} \frac{d\lambda db}{|c(\lambda)|^2},$$

where

$$F_\sigma(\lambda, b) = \sum_k \tilde{f}_k(\lambda, b) q^k(i\sigma\lambda) \pi(i\sigma\lambda).$$

Thus if $\ell \in K$

$$\begin{aligned} & e^{t\rho(H)} (\partial(\pi)u)(\ell a_t \cdot o, -t\sigma H + \sigma H_0) \\ &= e^{t\rho(H)} \int_{\mathfrak{a}^* \times B} F_\sigma(\lambda, b) e^{(i\lambda + \rho)(A(a_t \cdot o, \ell^{-1}b))} e^{-it\lambda(H) + i\lambda(H_0)} \frac{d\lambda db}{|c(\lambda)|^2}. \end{aligned} \tag{6.29}$$

By the K -invariance of db we can replace

$$F_\sigma(\lambda, b) \text{ by } F_\sigma(\lambda, \ell \cdot b), \quad A(a_t \cdot o, \ell^{-1} \cdot b) \text{ by } A(a_t \cdot o, b). \tag{6.30}$$

In the Iwasawa decomposition $G = KAN$ we write

$$g = k(g) \exp H(g) n(g) \quad H(g) \in \mathfrak{a}. \tag{6.31}$$

Then $A(gK, kM) = -H(g^{-1}k)$. If $b = k'M$ and we define $k \in K$ by $k' = k(a_t k)$ then $kM \rightarrow k'M$ is a diffeomorphism and

$$dk' = e^{-2\rho(H(a_t k))} dk. \tag{6.32}$$

Since $a_t k = k' \exp H(a_t k) n$ and A normalizes N we have

$$H(a_{-t} k') = -H(a_t k). \quad (6.33)$$

After the substitution (6.30) and $A(a_t \cdot 0, k' M) = -H(a_{-t} k')$ relations (6.32)-(6.33) reduce the right hand side of (6.29) to

$$e^{t\rho(H)} \int_{\mathfrak{a}^*} e^{-i\lambda(tH-H_0)} \int_B F_\sigma(\lambda, \ell k(a_t k) M) e^{(i\lambda-\rho)(H(a_t k))} \frac{d\lambda dk_M}{|c(\lambda)|^2}. \quad (6.34)$$

Now we use the map $\bar{n} \rightarrow k(\bar{n})M$ of \bar{N} into K/M under which $dk_M = e^{-2\rho(H(\bar{n}))} d\bar{n}$ ([HC], p. 287) to transfer (6.34) to \bar{N} . Since A normalizes N we have

$$a\bar{n} = ak(\bar{n}) \exp H(\bar{n}) n_1 = k(ak(\bar{n})) \exp H(ak(\bar{n})) \exp H(\bar{n}) n_2$$

so we have Harish-Chandra's relations

$$k(a\bar{n}) = k(ak(\bar{n})) = k(\bar{n}^a), \quad H(a\bar{n}) = H(ak(\bar{n})) + H(\bar{n}) \quad (6.35)$$

$$H(a\bar{n}) - \log a = H(\bar{n}^a), \quad (6.36)$$

where $\bar{n}^a = a\bar{n}a^{-1}$. Thus (6.34) becomes

$$\int_{\mathfrak{a}^*} e^{-i\lambda(tH-H_0)} \int_{\bar{N}} F_\sigma(\lambda, \ell k(\bar{n}^{a_t}) M) L(a_t, \bar{n}) \frac{d\lambda d\bar{n}}{c(\lambda)c(-\lambda)}, \quad (6.37)$$

where the kernel $L(a, \bar{n})$ equals

$$L(a, \bar{n}) = e^{(i\lambda-\rho)(H(\bar{n}^a))} e^{-(i\lambda+\rho)(H(\bar{n}))}. \quad (6.38)$$

Let $\lambda = \xi + i\eta$ ($\xi, \eta \in \mathfrak{a}^*$) and $-\eta \in \mathfrak{a}_+^*$ so small that $(c(\lambda)c(-\lambda))^{-1}$ is holomorphic in $\mathfrak{a}^* + i\eta$ ($|\eta| \leq 1$). Let $\epsilon > 0$ be such that

$$0 < \epsilon < 1, \quad \rho + \epsilon\eta \in \mathfrak{a}_+^*.$$

Then by [H2] p. 447, (41)

$$|L(a_t, \bar{n})| \leq e^{(\epsilon\eta-\rho)(H(\bar{n}))},$$

which is integrable over \bar{N} . Since F_σ is holomorphic in λ and for a fixed η is rapidly decreasing in ξ the integral (6.37) can be shifted to $\mathfrak{a}^* + i\eta$. Letting then $t \rightarrow +\infty$ we get the expression

$$\begin{aligned} & \int_{\mathfrak{a}^* + i\eta} e^{i\lambda(H_0)} (c(\lambda)c(-\lambda))^{-1} F_\sigma(\lambda, \ell M) \int_{\bar{N}} e^{-(i\lambda+\rho)(H(\bar{n}))} d\bar{n} \\ &= \int_{\mathfrak{a}^* + i\eta} e^{i\lambda(H_0)} F_\sigma(\lambda, \ell M) (c(-\lambda))^{-1} d\lambda. \end{aligned}$$

Here we can shift the integration back and take $\eta = 0$. Using the definition of \bar{J} , Q_k and relation (4.24) the theorem follows.

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