Lie Group Analysis at the Mittag Leffler Institute.

I am deeply grateful for the invitation to participate in this festive occasion. It has been most gratifying to refresh my memories from the years 1970–1971 and Fall 1995.

According to Stubhaug's splendid book on Sophus Lie, he travelled to Uppsala in 1881 and met Mittag Leffler in Stockholm. Here Lie suggested the start of a Nordic Mathematical Periodical with Mittag-Leffler as editor. ML was in favor of the idea but set his aims higher, namely an that of an International Journal which shortly after became Acta Mathematica. Lie considered ML 1000 times more of a diplomat than himself as regards obtaining moral and financial support: "It has more clout when you write".

I like to think that if ML and Lie were to suggest a topic for a year's concentration at the Institute they might have considered "Analysis on Lie group" a suitable topic. This was in fact the topic for 1970–1971 and again 25 years later 1995–1996.

While several topics in analysis appeared during 1970–1971, for example Feffermann's approach to Carleson's L^2 convergence theorem and Gundy's one semester lectures on probabilistic analysis, Analysis on Lie Groups was the main topic. I gave lectures on it during the whole year and several experts, Korányi, Eymard, Weiss, Sherman, Guzman and others came for short visits. Personally this was a very happy and eventful year for me. I received many rewarding invitations for lectures various places in Europe but managed not to have these trips interfere with my usual activity at the Institute. In those days the members were younger and for the most part stayed here for the whole year. Four of these, M. Flensted-Jensen, L. Lindahl, A. Hole and A. Melin were just starting research in this area and I gave them some research problems on topics related to my lectures. They were all very dedicated, were brimming with energy combined with ravenous appetite for mathematics. They assimilated the basic Lie group theory very quickly and at the end of the year each of them produced a respectable research paper published either in Arkiv för Matematik or Mathematica Scandinavica. Let me indicate the main topics where I suggested to them research problems.

1. Spherical Function Theory. Here the basic results were obtained by Harish Chandra during 1958–1966. These can be described very concretely in the case when the symmetric space $X = G/K$ has rank one or equivalently, X is a two-point homogeneous space. The radial eigenfunctions φ_λ of the Laplacian L can be parametrized by λ in $\mathbb{C}$ and one considers the **spherical transform**

$$\widetilde{f}(\lambda) = \int_0^\infty f(r)\varphi_{-\lambda}(r)A(r)$$

where $A(r) =$ the surface area of a sphere of radius r. Actually

$$A(r) = \sinh^p r \sinh^q 2r$$

the constants p and q being certain integers which characterize X.

The radial part of the Laplacian has the form

$$L_r = \frac{d^2}{dr^2} + \frac{A'(r)}{A(r)}\frac{d}{dr}.$$

The main result is the Plancherel formula

$$\int_0^\infty |f(r)|^2 A(r)\,dr = \int_0^\infty |\tilde{f}(\lambda)|^2 \delta(\lambda)\,d\lambda$$

where the density $\delta(\lambda)$ has a very interesting interpretation. He also proved the Schwartz type theorem $\mathcal{S}(\mathbb{R}^+)^{\sim} = \mathcal{S}(\mathbb{R}^+)$.

Later I proved the analog of the Paley Wiener type theorem $\mathcal{D}(\mathbb{R}^+)^{\sim} = \mathcal{H}(\mathbb{C})$ describing the image of the space $\mathcal{D}$ under the spherical transform as the space of even holomorphic functions of exponential type.

Harish Chandra had indicated to me that the Plancherel formula (which he proved group theoretically for G/K of all ranks) could perhaps in this rank one case be derived from Weyl's thesis work on singular second order operators. I was not sure but suggested to Flensted-Jensen to try to carry out such a proof. He managed to do so and in the process proved a more general result in that in the formula for $A(r)$ the two constants p and q can be any positive real numbers. In this generality he proved both the Schwartz theorem and the Paley-Wiener theorem. This required new methods since the group theory was no longer present. This also led to an extensive collaboration with Koornwinder (who was here for the whole year) on the Jacobi transform which also is a generalization of the rank one spherical transform.

This was further generalized substantially by Chebli who allowed much more general $A(r)$ but also added a potential $q(r)$ to L. He proved the Plancherel formula as well as the Paley Wiener theorem in this greater generality. Thus one can say that the spherical function theory in the simplest case of rank one has led to a new chapter in the theory of Sturm-Liouville operators.

For X of higher rank a related and quite explosive development took place primarily in Holland through the work of Beerends, Heckman and Opdam and others. This has become a long term project of theirs. Again while the work is motivated by the group theory, at a certain stage the group disappears and the subject becomes a topic in several complex variables.

II. Harmonic Functions. Consider again the symmetric space $X = G/K$ and the differential operators D on X which are invariant under the action of G and have no zero order term. A function u on X is said to be *harmonic* if $Du = 0$ for all D.

Godement proved in 1952 that these functions are characterized by a certain mean value property. Furstenberg and Karpelevic then proved a Poisson integral formula

$$u(x) = \int_B P(x,b)F(b)\,db \qquad F \in L^\infty(B)$$

for bounded harmonic functions, B being a certain orbit of K and $P(x,b)$ a certain explicit Poisson kernel. They also proved that a bounded solution of $Lu = 0$ also satifies $Du = 0$ for all D.

The next problem was to establish the analog of the Fatou theorem. This was done by Korányi and myself in 1967 and can be stated:

If u is a bounded solution of $Lu = 0$ then for almost all geodesics $a(t)$ from o the limit

$$\lim_{t\to\infty} u(a(t)) \text{ exists.}$$

This raised many questions of analogs and generalizations: In the Poisson formula

$$u(x) = \int_B P(x,b)F(b)\,db$$

the bounded u correspond to $F \in L^\infty(B)$. For $F \in L^p(B)$, u is still harmonic. Is Fatou's theorem still valid? I raised this question with Lindahl and he made substantial progress on it. He showed that for a certain p_o this holds for all $F \in L^p$ for $p > p_o$. But there are also other "boundaries" associated with X besides the Furstenberg boundary B and Poisson integral representations for these and he proved similar results for these too. This was the substance of his Uppsala thesis.

III. Eigenspace Representations. Consider the Euclidean space $\mathbb{R}^n$ and the corresponding group $\mathbb{M}(n)$ of isometries. Let L denote the Laplacian and for each λ in $\mathbb{C}$ consider the eigenspace

$$E_\lambda = \{f \in \mathcal{C}^\infty(\mathbb{R}^n) | Lf = \lambda^2 f\}.$$

Since L is invariant under $\mathbb{M}(n)$ one has a natural representation T_λ of $\mathbb{M}(n)$ on E_λ .When is it irreducible? It turns out that it is irreducible if and only if $\lambda \neq 0$.

For the sphere S each eigenspace of the Laplacian on S is irreducible under the rotation group of the sphere. This comes from theory for spherical harmonics. For the non-Euclidean disk with the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

the Laplacian is given by $L = (1 - x^2 - y^2)(1 - x^2 - y^2)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

For each $\lambda \in \mathbb{C}$ consider the eigenspace

$$E_\lambda = \{f \in \mathcal{C}^\infty(D) | Lf = -(\lambda^2 + 1)f\}$$

and the corresponding representation T_λ of the isometry group $\mathbf{SU}(1,1)$ of D on the eigenspace E_λ. Here it turns out that

$$T \text{ is irreducible if and only if } i\lambda + 1 \notin 2\mathbb{Z}\,.$$

For a symmetric space X above of rank one there is a certain Gamma function Γ_X associated to X , namely

$$\Gamma_X(\lambda) = \Gamma\left(\tfrac{1}{2}\left(\tfrac{1}{2}p + 1 + i\lambda\right)\right)\Gamma\left(\tfrac{1}{2}\left(\tfrac{1}{2}p + q + i\lambda\right)\right)$$

and I had recently proved that

$$T_\lambda \text{ is irreducible if and only if } \frac{1}{\Gamma_X(\lambda)} \neq 0\,.$$

The eigenspace representation problem arises for every homogeneous space G/H of Lie groups. Let $\mathbb{D}(G/H)$ denote the algebra of differential operators on G/H which are invariant under the action of G. The most interesting case is when $\mathbb{D}(G/H)$ is commutative and this happens for example if G/H is symmetric. Given a homomorphism χ of $\mathbb{D}(G/H)$ into $\mathbb{C}$ consider the joint eigenspace

$$E_\chi\{f \in \mathcal{C}^\infty(X) | Df = \chi(D)f \text{ for all } D \in \mathbb{D}(G/H)$$

and the representation T_χ of G on E_χ. Again the question of irreducibility arises. Also: what are the representations that arise in this way?

Since Arne Hole had some experince with Kirillov's theory of representations of nilpotent Lie groups I suggested to him the eigenspace problem for G/H when G and H are nilpotent. He obtained some fairly definitive results (published in Math. Scand.) on this problem which then form a natural complement to Kirillov's theory. Further work in this direction was later done by Stetkaer and Jacobsen in Aarhus, even with extensions to some solvable groups.

IV. Invariant differential and pseudo differential operators on G and G/K.

Invariant differential operators are central to analysis on G and the associated symmetric space G/K. For G we consider invariance under both left and right translations on G and for G/K we consider invariance under the action $xK \to gxK$. Many questions arise as analogs to the theory of constant coefficient differential operators on $\mathbb{R}^n$. One such result is that for each invariant D on G/K the surjectivity (that is global solvability)

$$D\,\mathcal{C}^\infty(G/K) = \mathcal{C}^\infty(G/K)$$

holds. I worked on proving this during the M–L stay but did not find the complete proof until 1972. The analog remains valid for several spaces instead of $\mathcal{C}^\infty G/K)$ but it has not yet been proved for the space of all distributions. For each invariant D on G one has local solvability and for the Laplace-Beltrami operator even global solvability.

Because of Melin's background in pseudo-differential operators I suggested to him the problem of describing the invariant pseudo-differential operators on G. This had actually been solved by Stetkaer in his MIT thesis. It implies that such an operator is the sum of an invariant differential operator and an operator with a smooth kernel. The same conclusion was derived from the requirement of pseudo-local property. Melin clarified the situation by showing that the two requirements are indeed equivalent by proving the following result.

Definition. A Lie group G has the property $(*)$ if each conjugacy-invariant $\mathcal{C}^\infty$ function on $G-(e)$ is the restriction of a $\mathcal{C}^\infty$ function on G.

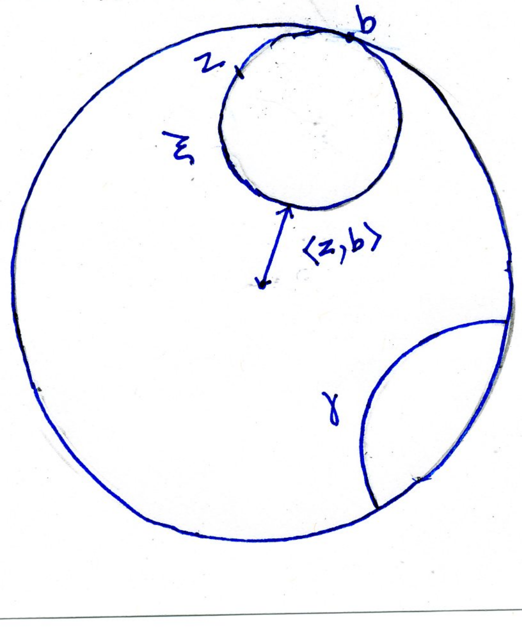
Theorem. G has property $(*)$ if and only if the adjoint group $\mathrm{Ad}\,(G)$ is not compact.

In the process Melin proved the following very concrete result on a finite-dimensional vector space V.

Theorem. Let $N \neq 0$ be a nilpotent linear transformation on V. Then every $\mathcal{C}^\infty$ function on $V-(0)$ which is conjugacy invariant under the group $\exp(tN)$ can be extended to $\mathcal{C}^\infty(V)$.

V. Fourier transforms on symmetric spaces.

On the hyperbolic space I have defined a natural Fourier transform as follows:



γ: Geodesics

ξ: Horocyle

$\langle z, b\rangle$: distance from o (with sign) to the horocycle through z and b.

I. then defines the Fourier transform as follows:

$$\widetilde{f}(\lambda, b) = \int_D f(z) e^{(-i\lambda+1\langle z,b\rangle)}\, dz\,.$$

It turns out that there is an inversion formula

$$f(z) = \int_{\mathbb{R}} \int_B \widetilde{f}(\lambda, b) e^{(i\lambda+1)\langle z,b\rangle} \delta(\lambda)\, d\lambda\, db$$

as well as a Plancherel formula.

$$\int_D |f(z)|^2\, dz = \int_{\mathbb{R}} \int_B |\widetilde{f}(\lambda, b)|^2 \delta(\lambda)\, d\lambda\, db\,.$$

There is also a Paley Wiener Theorem describing explicitly the image of the space $\mathcal{C}_c^\infty(X)$ under this Fourier transform. A similar Schwartz type theorem holds describing the image of the L^2 Schwartz space under the Fourier transform.

There is also an Integral Representation of all the eigenfunctions u of the Laplacian L

$$u(z) = \int_B e^{\mu\langle z,b\rangle}\, dT(b)$$

where $\mu \in \mathbb{C}$ is arbitrary and T is an arbitrary analytic functional (hyperfunction) on B. (Generalization of a distribution).

By the end of the seventies all these definitions and results had been extended to Riemannian Symmetric Spaces.

The next step would be to find analogs for non-Riemannian symmetric spaces G/H. Some examples had been developed in considerable detail but one of the first general results was the construction of the so-called discrete series for G/H, started by Flensted-Jensen followed by an important supplement by Oshima and Vogan. Gradually a definition of a Fourier transform emerged through work of Gestur Olafsson, E. van den Ban, Oshima and Schlichtkrull as well as a Plancherel formula. The main project of the 1995–1996 year organized by Flensted Jensen, P. Sjögren (Fall) and Gestur Olafsson and B. Örsted (Spring) was a development of this Fourier transform theory, particularly of the proof of the Paley Wiener theorem for this new Fourier transform. This project was led by van den Ban and Schlichtkrull who gave alternate lectures on the subject. The other principal contributors, Oshima and Delorme were also here for extended visits. Now the principal results, the Plancherel formula and the Paley Wiener theorem are finally in print with full proofs.

Thinking back one sees that the stimulating conditions at the Mittag Leffler Institute have played a prominent role in this development of Analysis on Lie Groups and Symmetric Spaces.