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Non-Euclidean Analysis

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1 The Non-Euclidean Plane

In case the work of Bolyai [Bo] and Lobatschevsky [Lo] left any doubts about the existence of non-Euclidean geometry these doubts were removed by the work [Be] of Beltrami. With a modification made possible by hindsight one can state the following result.

Theorem 1.1. *Given a simply connected region* $D \subset \mathbf{R}^2$ $(D \neq \mathbf{R}^2)$ *there exists a Riemannian metric* g *on* D *which is invariant under all conformal transformations of* D*. Also* g *is unique up to a constant factor.*

Because of the Riemann mapping theorem we can assume D to be the unit disk. Given $a \in D$ the mapping $\varphi : z \to \dfrac{a-z}{1-\bar{a}z}$ is conformal and $\varphi(a) = 0$. The invariance of g requires

$$g_a(u,u) = g_0(d\varphi(u),\, d\varphi(u)) \tag{1.1}$$

for each $u \in D_a$ (the tangent space to D at a). Since g_0 is invariant under rotations around 0,

$$g_0(z,z) = c|z|^2\,, \tag{1.2}$$

where c is a constant. Here D_0 is identified with $\mathbf{C}$. Let $t \to z(t)$ be a curve with $z(0) = a$, $z'(0) = u \in \mathbf{C}$. Then $d\varphi(u)$ is the tangent vector

$$\left\{\frac{d}{dt}\varphi(z(t))\right\}_{t=0} = \left(\frac{d\varphi}{dz}\right)_a \left(\frac{dz}{dt}\right)_0 = \left\{\frac{1-|a|^2}{(1-\bar{a}z)^2}\right\}_{z=a} u$$

so by (1.1), (1.2)

$$g_a(u,u) = c\frac{1}{(1-|a|^2)^2}|u|^2\,.$$

Thus g is the Riemannian structure

$$ds^2 = c\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2} \tag{1.3}$$

and the proof shows that it is indeed invariant.

We shall now take D as the unit disk $|z| < 1$ with $g = ds^2$ given by (1.3) with $c = 1$. In our analysis on D we are mainly interested in the *geodesics* in D (the arcs orthogonal to the boundary $B = \{z \in \mathbf{C} : |z| = 1\}$) and the *horocycles* in D which are the circles inside D tangential to B. Note that a horocycle tangential to B at b is orthogonal to all the geodesics in D which end at b.

2 The Non-Euclidean Fourier Transform

We first recall some of the principal results from Fourier analysis on $\mathbf{R}^n$. The Fourier transform $f \to \tilde{f}$ or $\mathbf{R}^n$ is defined by

$$\tilde{f}(u) = \int\limits_{\mathbf{R}^n} f(x) e^{-i(x,u)}\, dx \tag{2.1}$$

where (,) denotes the scalar product and dx the Lebesgue measure. In polar coordinates $u = \lambda w$ $\lambda \in \mathbf{R}$, $w \in SS^{n-1}$ we can write

$$\tilde{f}(\lambda w) = \int\limits_{\mathbf{R}^n} f(x) e^{-i\lambda(x,w)}\, dx\,. \tag{2.2}$$

It is then inverted by

$$f(x) = (2\pi)^{-n} \int\limits_{\mathbf{R}^+ \times SS^{n-1}} \tilde{f}(\lambda w) e^{i\lambda(x,w)} \lambda^{n-1}\, d\lambda\, dw \tag{2.3}$$

say for $f \in \mathcal{D}(\mathbf{R}^n) = \mathcal{C}_c^\infty(\mathbf{R}^n)$, dw denoting the surface element on the sphere SS^{n-1}. The Plancherel formula

$$\int\limits_{\mathbf{R}^n} |f(x)|^2\, dx = (2\pi)^{-n} \int\limits_{\mathbf{R}^+ \times SS^{n-1}} |\tilde{f}(\lambda, w)|^2 \lambda^{n-1}\, d\lambda\, dw \tag{2.4}$$

expresses that $f \to \tilde{f}$ is an isometry of $L^2(\mathbf{R}^2)$ onto $L^2(\mathbf{R}^+ \times SS^{n-1}, (2\pi)^{-n}\lambda^{n-1}\, d\lambda\, dw)$.

The range of the mapping $f(x) \to \tilde{f}(\lambda w)$ as f runs through $\mathcal{D}(\mathbf{R}^n)$ is expressed in the following theorem [He7]. A vector $a = (a_1, \ldots, a_n) \in \mathbf{C}^n$ is said to be *isotropic* if $(a, a) = a_1^2 + \cdots + a_n^2 = 0$.

Theorem 2.1. *The Fourier transform* $f(x) \to \tilde{f}(\lambda w)$ *maps* $\mathcal{D}(\mathbf{R}^n)$ *onto the set of functions* $\tilde{f}(\lambda w) = \varphi(\lambda, w) \in \mathcal{C}^\infty(\mathbf{R} \times SS^{n-1})$ *satisfying:*

(i) There exists a constant $A > 0$ *such that for each* $w \in SS^{n-1}$ *the function* $\lambda \to \varphi(\lambda, w)$ *extends to a holomorphic function on* $\mathbf{C}$ *with the property*

$$\sup_{\lambda \in \mathbf{C}\,,\, w \in SS^{n-1}} |\varphi(\lambda, w)|(1 + |\lambda|)^N e^{-A|\operatorname{Im} \lambda|} < \infty \tag{2.5}$$

for each $N \in \mathbf{Z}$. *(*$\operatorname{Im} \lambda =$ *imaginary part of* λ*).*

(ii) For each $k \in \mathbf{Z}^+$ *and each isotropic vector* $a \in \mathbf{C}^n$ *the function*

$$\lambda \to \lambda^{-k} \int\limits_{SS^{n-1}} \varphi(\lambda, w)(a, w)^k \, dw \tag{2.6}$$

is even and holomorphic in $\mathbf{C}$.

Condition (2.5) expresses that the function $\lambda \to \varphi(\lambda, w)$ is of *uniform exponential type*: The classical Paley–Wiener theorem states that $\mathcal{D}(\mathbf{R}^n)^\sim$ consists of entire functions of exponential type in n variables whereas in the description above only λ enters.

Formula (2.2) motivates a Fourier transform definition on D. The inner product (x, ω) equals the (signed) distance from 0 to the hyperplane through x with normal ω. A horocycle in D through b is perpendicular to the (parallel) family of geodesics ending at b so is an analog of a hyperplane in $\mathbf{R}^n$. Thus if $z \in D$, $b \in B$ we define $\langle z, b \rangle$ as the (signed) distance from 0 to the horocycle through z and b. The Fourier transform $f \to \tilde{f}$ on D is thus defined by

$$\tilde{f}(\lambda, b) = \int\limits_D f(z) e^{(-i\lambda + 1)\langle z, b \rangle} \, dz \tag{2.7}$$

for all $b \in B$ and $\lambda \in \mathbf{C}$ for which integral converges. Here dz is the invariant surface element on D

$$dz = (1 - x^2 - y^2)^{-2} \, dx \, dy \,. \tag{2.8}$$

The +1 term in 2.7 is included for later technical convenience.

The Fourier transform (2.7) is a special case of the Fourier transform on a symmetric space $X = G/K$ of the non-compact type, introduced in [He3]. Here G is a semisimple connected Lie group with finite center and K is a maximal compact subgroup. In discussing the properties of $f \to \tilde{f}$ below we stick to the case $X = D$ for notational simplicity but shall indicate (with references) the appropriate generalizations to arbitrary X. Some of the results require a rank restriction on X.

Theorem 2.2. *The transform $f \to \tilde{f}$ in (2.7) is inverted by*

$$f(z) = \frac{1}{4\pi} \int\limits_{\mathbf{R}} \int\limits_{B} \tilde{f}(\lambda, b) e^{(i\lambda+1)\langle z,b\rangle} \lambda \,\mathrm{th} \left(\frac{\pi\lambda}{2}\right) d\lambda\, db\,, \quad f \in \mathcal{D}(D)\,. \tag{2.9}$$

Also the map $f \to \tilde{f}$ extends to an isometry of $L^2(D)$ onto $L^2(\mathbf{R}^+ \times B, \mu)$ where μ is the measure

$$\mu = \frac{\lambda}{2\pi} \,\mathrm{th} \left(\frac{\pi\lambda}{2}\right) d\lambda\, db \tag{2.10}$$

and db is normalized by $\int db = 1$.

This result is valid for arbitrary $X = G/K$ ([He3] [He4]), suitably formulated in terms of the fine structure of G. While this result resembles (2.3)—(2.4) closely the range theorem for D takes a rather different form.

Theorem 2.3. *The range $\mathcal{D}(D)^\sim$ consists of the functions $\varphi(\lambda, b)$ which (in λ) are holomorphic of uniform exponential type and satisfy the functional equation*

$$\int\limits_{B} \varphi(\lambda, b) e^{(i\lambda+1\langle z,b\rangle)}\, db = \int\limits_{B} \varphi(-\lambda, b) e^{(-i\lambda+1)\langle z,b\rangle}\, db\,. \tag{2.11}$$

One can prove that condition (2.11) is equivalent to the following conditions (2.12) for the Fourier coefficients $\varphi_k(\lambda)$ of φ

$$\begin{aligned} \varphi_k(\lambda) &= \frac{1}{2\pi} \int\limits_0^{2\pi} \varphi(\lambda, e^{i\theta}) e^{-ik\theta}\, d\theta \\ \varphi_k(-\lambda) p_k(-i\lambda) &= \varphi_k(\lambda) p_k(i\lambda)\,, \quad k \in \mathbf{Z}\,, \end{aligned} \tag{2.12}$$

where $p_k(x)$ is the polynomial

$$p_k(x) = \frac{\Gamma(\frac{1}{2}(x+1) + |k|)}{\Gamma(\frac{1}{2}(x+1))}\,.$$

Again these results are valid for arbitrary $X = G/K$ ([He5] and [He7]).

The Paley–Wiener type theorems can be extended to the Schwartz spaces $\mathcal{S}^p(D)$ $(0 < p \leq 2)$. Roughly speaking, f belongs to $\mathcal{S}^p(D)$ if each invariant derivative Df belongs to $L^p(D)$, more precisely, it is rapidly decreasing in the distance from 0 even after multiplication by the p^{th} root of the volume element. Let S_p denote the strip $|\operatorname{Im}\lambda| < \frac{2}{p} - 1$ in $\mathbf{C}$ and $\mathcal{S}(S_p \times B)$ the space of smooth functions on $\mathcal{S}(S_p \times B)$ holomorphic (in λ) in S_p and rapidly decreasing (uniformly for $b \in B$) on each line $\lambda = \xi + i\eta$ $(|\eta| < \frac{2}{p} - 1)$. Then we have

Theorem 2.4. *The Fourier transform $f \to \tilde{f}$ on D is a bijection of $\mathcal{S}^p(D)$ onto the set of $\varphi \in \mathcal{S}(S_p \times B)$ satisfying (2.11).*

The theorem holds for all $X = G/K$ (Eguchi [Eg]). The proof is complicated. For the case of K-invariant functions (done for $p = 2$ by Harish–Chandra [H] and Trombi–Varadarajan [TV] for general p) a substantial simplification was done by Anker [A]. A further range theorem for the space of functions for which each invariant derivative has arbitrary exponential decay was proved by Oshima, Saburi and Wakayama [OSW]. See also Barker [Bar] (p. 27) for the operator Fourier transform of the intersection of all the Schwartz spaces on $\mathbf{SL}(2, \mathbf{R})$.

In classical Fourier analysis on $\mathbf{R}^n$ the Riemann–Lebesgue lemma states that for $f \in L^1(\mathbf{R})$, $\tilde{f}$ tends to 0 at ∞. For D the situation is a bit different.

Theorem 2.5. *Let $f \in L^1(D)$. Then there exists a null set N in B such that if $b \in B - N$, $\lambda \to \tilde{f}(\lambda, b)$ is holomorphic in the strip $|\operatorname{Im}\lambda| < 1$ and*

$$\lim_{\xi \to \infty} \tilde{f}(\xi + i\eta, b) = 0 \tag{2.13}$$

uniformly for $|\eta| \leq 1$.

The proof [HRSS] is valid even for symmetric spaces $X = G/K$ of arbitrary rank. Moreover

$$\|\tilde{f}(\lambda, \cdot)\|_1 \to 0 \text{ as } \lambda \to \infty\,, \tag{2.14}$$

uniformly in the strip $|\operatorname{Im}\lambda| \leq 1$, and this extends to $f \in L^p$ $(1 \leq p < 2)$ this time in the strip $|\operatorname{Im}\lambda| < \frac{2}{p} - 1$ ([SS]). In particular, if $f \in L^p(D)$ then there is a null set N in B such that $\tilde{f}(\lambda, b)$ exists for $b \notin N$ and all λ in the strip $|\operatorname{Im}\lambda| < \frac{2}{p} - 1$.

The classical inversion formula for the Fourier transform on $\mathbf{R}^n$ now extends to $f \in L^p(D)$ $(1 \leq p < 2)$ as follows.

Theorem 2.6. *Let* $f \in L^p(D)$ *and assume* $\tilde{f} \in L^1(\mathbf{R} \times B, \mu)$ *(with* μ *as in (2.10)). Then the inversion formula (2.9) holds for almost all* $z \in D$ *(the Lebesgue set for* f*).*

Again this holds for all $X = G/K$. A result of this type was proved by Stanton and Thomas [ST] without invoking $\tilde{f}$ explicitly (since the existence had not been established). The version in Theorem 2.6 is from [SS].

In Schwartz's theory of mean–periodic functions [Sc] it is proved that any closed translation–invariant subspace of $\mathcal{C}^\infty(\mathbf{R})$ contains an exponential $e^{\mu x}$. The analogous question here would be:

Does an arbitrary closed invariant subspace of $\mathcal{C}^\infty(D)$ *contain an exponential*

$$e_{\mu,b}(z) = e^{\mu\langle z,b\rangle} \tag{2.15}$$

for some $\mu \in \mathbf{C}$ *and some* $b \in B$*?*

Here the topology of $\mathcal{C}^\infty(D)$ is the usual Fréchet space topology and "invariant" refers to the action of the group $G = \mathbf{SU}(1,1)$ on D. The answer is yes.

Theorem 2.7. *Each closed invariant subspace* E *of* $\mathcal{C}^\infty(D)$ *contains an exponential* $e_{\mu,b}$.

This was proved in [HS] for all symmetric $X = G/K$ of rank one. Here is a sketch of the proof. By a result of Bagchi and Sitaram [BS] E contains a spherical function

$$\varphi_\lambda(z) = \int_B e^{(i\lambda+1)\langle z,b\rangle}\, db\,, \quad \varphi_\lambda = \varphi_{-\lambda}\,. \tag{2.16}$$

For either λ or $-\lambda$ it is true ([He9], Lemma 2.3, Ch. III) that the Poisson transform $P_\lambda : F \to f$ where

$$f(z) = \int_B e^{(i\lambda+1)\langle z,b\rangle} F(b)\, db\,, \tag{2.17}$$

maps $L^2(B)$ into the closed invariant subspace of E generated by φ_λ. On the other hand it is proved in [He9] (Ex. B1 in Ch. III) that $e_{i\lambda+1,b}$ is a series of terms $P_\lambda(F_n)$ where $F_n \in L^2(B)$ and the series converges in the topology of $\mathcal{C}^\infty(D)$. Thus $e_{i\lambda+1,b} \in E$ as desired.

The following result for the Fourier transform on $\mathbf{R}^n$ is closely related to the Wiener Tauberian theorem.

Let $f \in L^1(\mathbf{R}^n)$ be such that $\tilde{f}(u) \neq 0$ for all $u \in \mathbf{R}^n$. Then the translates of f span a dense subspace of $L^1(\mathbf{R}^n)$.

There has been considerable activity in establishing analogs of this theorem for semisimple Lie groups and symmetric spaces. See e.g. [EM], [Sa], [Si1], [Si2]. The neatest version for D seems to me to be the following result from [SS] [MRSS] which remains valid for $X = G/K$ of rank one.

Let $d(z, w)$ denote the distance in D and if $\epsilon > 0$, let $\text{Ł}_\epsilon(D)$ denote the space of measurable functions f on D such that $\int_D |f(z)| e^{\epsilon d(0,z)}\, dz < \infty$. Let T_ϵ denote the strip $|\operatorname{Im} \lambda| \leq 1 + \epsilon$.

Theorem 2.8. *Let $f \in \text{Ł}_\epsilon(X)$ and assume f is not almost everywhere equal to any real analytic function. Let*

$$Z = \{\lambda \in T_\epsilon : \tilde{f}(\lambda, \cdot) \equiv 0\}.$$

If $Z = \emptyset$ then the translates of f span a dense subspace of $L^1(D)$.

A theorem of Hardy's on Fourier transforms on $\mathbf{R}^n$ asserts in a precise fashion that f and its Fourier transform cannot both vanish too fast at infinity. More precisely ([Ha]):

Assume

$$|f(x)| \leq A e^{-\alpha |x|^2}, \quad |\tilde{f}(u)| \leq B e^{-\beta |u|^2},$$

where A, B, α and β are positive constants and $\alpha\beta > \frac{1}{4}$. Then $f \equiv 0$.

Variations of this theorem for L^p spaces have been proved by Morgan [M] and Cowling–Price [CP].

For the Fourier transform on D the following result holds.

Theorem 2.9. *Let f be a measurable function on D satisfying*

$$|f(x)| \leq C e^{-\alpha d(0,x)^2} \qquad |\tilde{f}(\lambda, b)| \leq C e^{-\beta |\lambda|^2}$$

where C, α, β are positive constants. If $\alpha\beta > 16$ then $f \equiv 0$.

This is contained in Sitaram and Sundari [SiSu] § 5 where an extension to certain symmetric spaces $X = G/K$ is also proved. The theorem for all such X was obtained by Sengupta [Se], together with refinements in terms of $L^p(X)$.

Many such completions of Hardy's theorem have been given, see [RS], [CSS], [NR], [Shi].

3 Eigenfunctions of the Laplacian

Consider first the plane $\mathbf{R}^2$ and the Laplacian

$$L^0 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \, .$$

Given a unit vector $\omega \in \mathbf{R}^2$ and $\lambda \in \mathbf{C}$ the function $x \to e^{i\lambda(x,\omega)}$ is an eigenfunction

$$L_x^0 e^{i\lambda(x,\omega)} = -\lambda^2 e^{i\lambda(x,\omega)} \, . \tag{3.1}$$

Because of (2.3) one might expect all eigenfunctions of L to be a "decomposition" into such eigenfunctions with fixed λ but variable ω.

Note that the function $\omega \to e^{i\lambda(x,\omega)}$ is the restriction to SS^1 of the holomorphic function

$$z \to \exp\left[\tfrac{1}{2}(i\lambda)x_1(z+z^{-1}) + \tfrac{1}{2}\lambda x_2(z-z^{-1})\right] \qquad z \in \mathbf{C} - (0) \, ,$$

which satisfies a condition

$$\sup_z \left(|f(z)| e^{-a|z| - b|z|^{-1}} \right) < \infty \, , \tag{3.2}$$

with $a, b \geq 0$. Let $E_{a,b}$ denote the Banach space of holomorphic functions satisfying (3.2), the norm being the expression in (3.2). We let E denote the union of the spaces $E_{a,b}$ and give it the induced topology. We identify the elements of E with their restrictions to SS^1 and call the members of the dual space E' *entire functionals.*

Theorem 3.1. *([He6]) The eigenfunctions of L^0 on $\mathbf{R}^2$ are precisely the harmonic functions and the functions*

$$f(x) = \int_{SS^1} e^{i\lambda(x,\omega)} \, dT(\omega) \tag{3.3}$$

where $\lambda \in \mathbf{C} - (0)$ and T is an entire functional on SS^1.

For the non-Euclidean metric (1.3) (with $c = 1$) the Laplacian is given by

$$L = (1 - x^2 - y^2)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \tag{3.4}$$

and the exponential function $e_{\mu,b}(z) = e^{\mu\langle z,b\rangle}$ is an eigenfunction:

$$L_z(e^{(i\lambda+1)\langle z,b\rangle}) = -(\lambda^2+1) e^{(i\lambda+1)\langle z,b\rangle} \, . \tag{3.5}$$

In particular, the function $z \to e^{2\langle z,b\rangle}$ is a harmonic function and in fact coincides with the classical Poisson kernel from potential theory:

$$e^{2\langle z,b\rangle} = \frac{1-|z|^2}{|z-b|^2}\,. \tag{3.6}$$

Again the eigenfunctions of L are obtained from the functions $e_{\mu,b}$ by superposition. To describe this precisely consider the space $\mathcal{A}(B)$ of analytic functions on B. Each $F \in \mathcal{A}(B)$ extends to a holomorphic function on a belt $B_\epsilon : 1-\epsilon < |z| < 1+\epsilon$ around B. The space $\mathcal{H}(B_\epsilon)$ of holomorphic functions on B_ϵ is topologized by uniform convergence on compact subsets. We can view $\mathcal{A}(B)$ as the union $\cup_n^\infty \mathcal{H}(B_{1/n})$ and give it the inductive limit topology. The dual space $\mathcal{A}'(B)$ then consists of the *analytic functionals* on B (or hyperfunctions in B).

Theorem 3.2. *([He4], IV, §1). The eigenfunctions of L are precisely the functions*

$$u(z) = \int\limits_B e^{\mu\langle z,b\rangle}\, dT(b)\,, \tag{3.7}$$

where $\mu \in \mathbf{C}$ and $T \in \mathcal{A}'(B)$.

Lewis in [L] has proved (under minor restriction on μ) that T in (3.7) is a distribution if and only if u has at most an exponential growth (in $d(0,z)$). On the other hand, Ban and Schlichtkrull proved in [BaS] that $T \in \mathcal{C}^\infty(B)$ if and only if all the invariant derivatives of u have the same exponential growth.

We consider now the natural group representations on the eigenspaces. The group $\mathbf{M}(2)$ of isometries of $\mathbf{R}^2$ acts transitively on $\mathbf{R}^2$ and leaves the Laplacian L^0 invariant: $L^0(f\circ\tau) = (L^0 f)\circ\tau$ for each $\tau \in \mathbf{M}(2)$. If $\lambda \in \mathbf{C}$ the eigenspace

$$\mathcal{E}_\lambda = \{f \in \mathcal{C}^\infty(\mathbf{R}^2) : L^0 f = -\lambda^2 f\}$$

is invariant under the action $f \to f\circ\tau^{-1}$ so we have a representation T_λ of $\mathbf{M}(2)$ on $\mathcal{E}_\lambda$ given by $T_\lambda(\tau)(f) = f\circ\tau^{-1}$, the *eigenspace representation.*

Theorem 3.3. *([He6]) The representation T_λ is irreducible if and only if $\lambda \neq 0$.*

Similarly the group $G = \mathbf{SU}(1,1)$ of conformal transformations

$$z \to \frac{az+b}{\bar{b}z+a} \quad (|a|^2 - |b|^2 = 1)$$

leaves (1.3) and the operator L in (3.4) invariant. Thus we get again an eigenspace representation τ_λ of G on each eigenspace

$$\mathcal{E}_\lambda = \{f \in \mathcal{C}^\infty(D) : Lf = -(\lambda^2 + 1)f\}\,.$$

Theorem 3.4. *([He4]) The representation τ_λ is irreducible if and only if $i\lambda + 1 \notin 2\mathbf{Z}$.*

Again all these results extend to Euclidean spaces of higher dimensions and suitably formulated, to all symmetric spaces G/K of the noncompact type.

4 The Radon Transform

A. The Euclidean Case.

Let d be a fixed integer, $0 < d < n$ and let $\mathbf{G}(d,n)$ denote the space of d-dimensional planes in $\mathbf{R}^n$. To a function f on $\mathbf{R}^n$ we associate a function $\hat{f}$ on $\mathbf{G}(d,n)$ by

$$\hat{f}(\xi) = \int\limits_\xi f(x)\,dm(x)\,, \quad \xi \in \mathbf{G}(d,n)\,, \tag{4.1}$$

dm being the Euclidean measure on ξ. The transform $f \to \hat{f}$ is called the *d-plane transform.* For $d = 1$, $n = 2$ it is the classical Radon transform. The parity of d turns out to be important.

The inversion of the transform $f \to \hat{f}$ is well known (case $d = n - 1$ in [R], [J], [GS], general d in [F], [He1], [He2]). We shall give another group-theoretic method here, resulting in alternative inversion formulas.

The group $G = \mathbf{M}(n)$ acts transitively both on $\mathbf{R}^n$ and on $\mathbf{G}(d,n)$. In particular, $\mathbf{R}^n = G/K$ where $K = \mathbf{O}(n)$. Let $p > 0$. Consider a pair $x \in \mathbf{R}^n$, $\xi \in \mathbf{G}(d,n)$ at distance $p = d(x,\xi)$. Let $g \in G$ be such that $g \cdot 0 = x$. Then the family $kg^{-1} \cdot \xi$ constitutes the set of elements in $\mathbf{G}(d,n)$ at distance p from 0. Along with the transform $f \to \hat{f}$ we consider the *dual transform* $\varphi \to \overset{\vee}{\varphi}$ given by

$$\overset{\vee}{\varphi}(x) = \int\limits_{\xi \ni x} \varphi(\xi)\,d\mu(\xi)\,, \tag{4.2}$$

the average of φ over the set of d-planes passing through x. More generally we put

$$\overset{\vee}{\varphi}_p(x) = \int\limits_{d(\xi,x)=p} \varphi(\xi)\,d\mu(\xi) \tag{4.3}$$

the average of φ over the set of d-planes at distance p from x. Since K acts transitively on the set of d-planes through 0 we see by the above that

$$\overset{\vee}{\varphi}_p(g\cdot 0)=\int\limits_K \varphi(gkg^{-1}\cdot\xi)\,dk\,, \tag{4.4}$$

dk being the normalized Haar measure on K. Let $(M^r f)(x)$ denote the mean-value of f over the sphere $S_r(x)$ of radius r with center x. If $z\in\mathbf{R}^n$ has distance r from 0 we then have

$$(M^r f)(g\cdot 0)=\int\limits_K f(gk\cdot z)\,dk\,. \tag{4.5}$$

We thus see that since $d(0,g^{-1}\cdot y)=d(x,y)$,

$$\begin{aligned}(\hat{f})^{\vee}_p(x) &= \int\limits_K \hat{f}(gkg^{-1}\cdot\xi)\,dk=\int\limits_\xi dk\int\limits_\xi f(gkg^{-1}\cdot y)\,dm(y)\\ &= \int\limits_\xi dm(y)\int\limits_K f(gkg^{-1}\cdot y)\,dk=\int\limits_\xi (M^{d(x,y)}f)(x)\,dm(y)\,.\end{aligned}$$

Let x_0 be the point in ξ at minimum distance p from x. The integrand $(M^{d(x,y)}f(x))$ is constant in y on each sphere in ξ with center x_0. It follows that

$$(\hat{f})^{\vee}_p(x)=\Omega_d\int\limits_0^\infty (M^q f)(x)r^{d-1}\,dr\,, \tag{4.6}$$

where $r=d(x_0,y)$, $q=d(x,y)$, Ω_d denoting the area of the unit sphere in $\mathbf{R}^d$. We have $q^2=p^2+r^2$ so putting $F(q)=(M^q f)(x)$, $\hat{F}(p)=(\hat{f})^{\vee}_p(x)$ we have

$$\hat{F}(p)=\Omega_d\int\limits_p^\infty F(q)(q^2-p^2)^{d/2-1}q\,dq\,. \tag{4.7}$$

This Abel-type integral equation has an inversion

$$F(r)=c(d)\left(\frac{d}{d(r^2)}\right)^d\int\limits_r^\infty p(p^2-r^2)^{d/2-1}\hat{F}(p)\,dp\,, \tag{4.8}$$

where $c(d)$ is a constant, depending only on d. Putting $r=0$ we obtain the inversion formula

$$f(x)=c(d)\left[\left(\frac{d}{d(r^2)}\right)^d\int\limits_r^\infty p(p^2-r^2)^{d/2-1}(\hat{f})^{\vee}_p(x)\,dp\right]_{r=0}\,. \tag{4.9}$$

Note that in (4.8)

$$p(p^2-r^2)^{d/2-1} = \frac{d}{dp}(p^2-r^2)^{(d/2)} \cdot \frac{1}{d}$$

so in (4.8) we can use integration by parts and the integral becomes

$$C\int_r^\infty (p^2-r^2)^{d/2}\hat{F}'(p)\,dp\,.$$

Applying $\dfrac{d}{d(r^2)} = \dfrac{1}{2r}\dfrac{d}{dr}$ to this integral reduces the exponent $d/2$ by 1. For d odd we continue the differentiation $\frac{d+1}{2}$ times until the exponent is $-\frac{1}{2}$. For d even we continue until the exponent is 0 and then replace $\int_r^\infty \hat{F}'(p)\,dp$ by $-\hat{F}(r)$. This $\hat{F}(r)$ is an even function so taking $(d/d(r^2))^{d/2}$ at $r=0$ amounts to taking a constant multiple of $(d/dr)^d$ at $r=0$. We thus get the following refinement of (4.9) where we recall that $(\hat{f})_r^\vee(x)$ is the average of the integrals of f over the d-planes tangent to $S_r(x)$.

Theorem 4.1. *The d-plane transform is inverted as follows:*

(i) If d is even then

$$f(x) = C_1\left[\left(\frac{d}{dr}\right)^d (\hat{f})_r^\vee(x)\right]_{r=0}\,. \tag{4.10}$$

(ii) If d is odd then

$$f(x) = C_2\left[\left(\frac{d}{d(r^2)}\right)^{(d-1)/2}\int_r^\infty (p^2-r^2)^{-1/2}\frac{d}{dp}(\hat{f})_p^\vee(x)\,dp\right]_{r=0}\,. \tag{4.11}$$

(iii) If $d=1$ then

$$f(x) = -\frac{1}{\pi}\int_0^\infty \frac{1}{p}\,\frac{d}{dp}(\hat{f})_p^\vee(x)\,dp\,. \tag{4.12}$$

For $n=2$ formula (4.12) is proved in Radon's original paper [R]. Note that the constant $-1/\pi$ is the same for all n. In the case $d=n-1$ the formula in (i) coincides with formula (21) in Rouvière [Ro].

Another inversion formula ([He1], [He2]) valid for all d and n is

$$f \quad = \quad c(-L)^{d/2}((\hat{f})^{\vee}) \tag{4.13}$$

where

$$c \quad = \quad \frac{\Gamma\left(\frac{n-d}{2}\right)}{(4\pi)^{d/2}\Gamma\left(\frac{n}{2}\right)}\,.$$

Here the fractional power of L is defined in the usual way by the Fourier transform. The parity of d shows up in the same way as in Theorem 4.1.

For range questions for the transform $f \to \hat{f}$ see an account in [He10] and references there.

B. The Hyperbolic Case.

The hyperbolic space $\mathbf{H}^n$ is the higher-dimensional version of (1.3) and its Riemannian structure is given by

$$ds^2 = 4\frac{dx_1^2 + \cdots + dx_n^2}{(1 - x_1^2 - \cdots - x_n^2)^2} \tag{4.14}$$

in the unit ball $|x| < 1$. The constant 4 is chosen such that the curvature is now -1. The d-dimensional totally geodesic submanifolds are spherical caps perpendicular to the boundary $B : |x| = 1$. They are natural analogs of the d-planes in $\mathbf{R}^n$. We have accordingly a Radon transform $f \to \hat{f}$, where

$$\hat{f}(\xi) = \int\limits_{\xi} f(x)\, dm(x) \qquad \xi \in \Xi\,, \tag{4.15}$$

where Ξ is the space of d-dimensional totally geodesic submanifolds of $\mathbf{H}^n$.

The group G of isometries of $\mathbf{H}^n$ acts transitively on Ξ as well. As in (4.2)—(4.3) we consider the dual transform

$$\overset{\vee}{\varphi}(x) = \int\limits_{\xi \ni x} \varphi(\xi)\, d\mu(\xi) \tag{4.16}$$

and more generally for $p \geq 0$,

$$\overset{\vee}{\varphi}_p(x) = \int\limits_{d(\xi,x)=p} \varphi(\xi)\, d\mu(\xi)\,, \tag{4.17}$$

the mean value of φ over the set of $\xi \in \Xi$ at distance p from x. The formula

$$(\hat{f})^\vee_p(x) = \int\limits_\xi (M^{d(x,y)} f)(x)\, dm(y) \tag{4.18}$$

is proved just as before. Let x_0 be the point in ξ at minimum distance p from x and put $r = d(x_0, y)$, $q = d(x,y)$. Since the geodesic triangle (xx_0y) is right angled at x_0 we have by the cosine rule

$$\cosh q = \cosh p \, \cosh r\,. \tag{4.19}$$

Also note that since ξ is totally geodesic, distances between two points in ξ are the same as in $\mathbf{H}^n$. In particular $(M^{d(x,y)} f)(x)$ is constant as y varies on a sphere in ξ with center x_0. Therefore (4.18) implies

$$(\hat{f})^\vee_p(x) = \Omega_d \int\limits_0^\infty (M^q f)(x) \sinh^{d-1} r\, dr\,. \tag{4.20}$$

For x fixed we put

$$F(\cosh q) = (M^q f)(x)\,, \quad \hat{F}(\cosh p) = (\hat{f})^\vee_p(x)\,,$$

substitute in (4.20) and use (4.19). Writing $t = \cosh p$, $s = \cosh r$ we obtain the integral equation

$$\hat{F}(t) = \Omega_d \int\limits_1^\infty F(ts)(s^2-1)^{d/2\,-1}\, ds\,. \tag{4.21}$$

Putting here $u = ts$, $\quad ds = t^{-1}\, du$ we get the Abel–type integral equation

$$t^{d-1}\hat{F}(t) = \Omega_d \int\limits_t^\infty u^{-1}F(u)(u^2-t^2)^{d/2\,-1} u\, du\,,$$

which by (4.8) is inverted by

$$r^{-1}F(r) = c(d)\left(\frac{d}{d(r^2)}\right)^d \int\limits_r^\infty t(t^2-r^2)^{d/2\,-1} t^{d-1}\hat{F}(t)\, dt\,. \tag{4.22}$$

Here we put $r = 1$ and $s(p) = \cosh^{-1} p$. We then obtain the following variation of Theorem 3.12, Ch. I in [He9]:

Theorem 4.2. *The transform $f \to \tilde{f}$ is inverted by*

$$f(x) = C\left[\left(\frac{d}{d(r^2)}\right)^d \int\limits_r^\infty (t^2 - r^2)^{d/2\,-1} t^d (\hat{f})^\vee_{s(t)}(x)\,dt\right]_{r=1}. \tag{4.23}$$

As in the proof of Theorem 4.1 we can obtain the following improvement.

Theorem 4.3. *(i) If d is even the inversion can be written*

$$f(x) = C\left[\left(\frac{d}{d(r^2)}\right)^{d/2} (r^{d-1}(\hat{f})^\vee_{s(r)}(x))\right]_{r=1}. \tag{4.24}$$

(ii) If $d = 1$ then

$$f(x) = -\tfrac{1}{\pi}\int\limits_0^\infty \frac{1}{\sinh p}\frac{d}{dp}\Big((\hat{f})^\vee_p(x)\Big)\,dp\,. \tag{4.25}$$

Proof: Part (i) is proved as (4.10) except that we no longer can equate $(d/d(r^2))^{d/2}$ with $(d/dr)^d$ at $r = 1$.

For (ii) we deduce from (4.22) since $t(t^2 - r^2)^{-1/2} = \frac{d}{dt}(t^2 - r^2)^{1/2}$ that

$$\begin{aligned} F(1) &= -\frac{c(1)}{2}\left[\frac{d}{dr}\int\limits_r^\infty (t^2 - r^2)^{1/2}\frac{d}{dt}\hat{F}(t)\,dt\right]_{r=1} \\ &= \frac{c(1)}{2}\int\limits_1^\infty (t^2 - 1)^{-1/2}\frac{d}{dt}\hat{F}(t)\,dt\,. \end{aligned}$$

Putting again $t = \cosh p$, $dt = \sinh p\,dp$ our expression becomes

$$\frac{c(1)}{2}\int\limits_0^\infty \frac{1}{\sinh p}\,\frac{d}{dp}((\hat{f})^\vee_p)(x)\,dp\,.$$

Remark For $n = 2$, $d = 1$ formula (4.25) is stated in Radon [R], Part C. The proof (which is only indicated) is very elegant but would not work for $n > 2$.

For d even (4.24) can be written in a simpler form ([He1]) namely

$$f = c\, Q_d(L)((\hat{f})^\vee)\,, \tag{4.26}$$

where $c = \frac{\Gamma\left(\frac{n-d}{2}\right)}{(-4\pi)^{d/2}\Gamma\left(\frac{n}{2}\right)}$ and Q_d is the polynomial

$$Q_d(x) = (x + (d-1)(n-d))(x + (d-3)(n-d+2)) \cdots (x + 1\cdot(n-2))\,.$$

The case $d = 1$, $n = 2$ is that of the X-ray transform on the non-Euclidean disk ((4.15) for $n = 2$). Here are two further alternatives to the inversion formula (4.25). Let S denote the integral operator

$$(Sf)(x) = \int_D (\coth\, d(x,y) - 1) f(y)\, dy\,. \tag{4.27}$$

Then

$$LS(\hat{f})^\vee = -4\pi^2 f\,, \quad f \in \mathcal{D}(X)\,. \tag{4.28}$$

This is proved by Berenstein–Casadio [BC]; see [He10] for a minor simplification. By invariance it suffices to prove (4.28) for f radial and then it is verified by taking the spherical transform on both sides. Less explicit versions of (4.28) are obtained in [BC] for any dimension n and d.

One more inversion formula was obtained by Lissianoi and Ponomarev [LP] using (4.23) for $d = 1$, $n = 2$ as a starting point. By parameterizing the geodesics γ by the two points of intersection of γ with B they prove a hyperbolic analog of the Euclidean formula:

$$f(x) = \int_{SS^1} \left\{ \mathcal{H}_p \frac{d}{dp} \hat{f}(\omega, p) \right\}_{p=(\omega,x)} d\omega\,, \tag{4.29}$$

which is an alternative to (4.12). Here $\mathcal{H}_p$ is a normalized Hilbert transform in the variable p and $\hat{f}(\omega,p)$ is $\hat{f}(\xi)$ for the line $(x,\omega) = p$, where $|\omega| = 1$.

In the theorems in this section we have not discussed smoothness and decay at infinity of the functions. Here we refer to [Je], [Ru1], [Ru2], [BeR1] and [BeR2]as examples.

Additional inversion formulas for the transform $f \to \hat{f}$ can be found in [Sm], [Ru3] and [K]. The range problem for the transform $f \to \hat{f}$ is treated in [BCK] and [I].

Added in Proof:

I have since this was written proved an inversion formula for the X-ray transform on a noncompact symmetric space of rank $l > 1$. It is similar to (4.12) except that in (4.17) one restricts the averaging to the set of geodesics each of which lies in a flat l-dimensional totally geodesic submanifold through x and at distance p from x. On the other hand, Rouvière had proved earlier that the inversion formula (4.25) holds almost unchanged for the X-ray transform on a noncompact symmetric space of rank 1.

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