Cobordism and the Pontryagin-Thom Construction

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Exercise

(i) Classify all compact $n$-manifolds up to diffeomorphism.

(ii) Give computable invariants that distinguish all diffeomorphism classes of compact $n$-manifolds.
Classifying Manifolds

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In fact, (ii) is impossible.
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In fact, (ii) is impossible.

Idea: Replace diffeomorphism with simpler equivalence relation.
Exercise

(i) Classify all compact n-manifolds up to cobordism.
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Surprisingly, this can be done.

**Goal:** Explain how (i) is equivalent to computing some homotopy groups (via the *Pontryagin-Thom construction*). Thom did this calculation using methods of homotopy theory unrelated to manifolds.
A **cobordism** between compact $n$-manifolds $M$ and $N$ is an $(n + 1)$-manifold $B$ with

$$\partial B \cong M \sqcup N.$$ 

$M$ and $N$ are **cobordant** if $\exists$ a cobordism between them.
What is Cobordism?

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$M$ and $N$ are **cobordant** if $\exists$ a cobordism between them. This is an equivalence relation:

- **Reflexive**: $M \times I$
- **Symmetric**: obvious
- **Transitive**: use analysis to smoothly glue cobordisms
The Cobordism Ring

Definition

\[ \mathcal{N}_n := \text{set of cobordism classes of compact } n\text{-manifolds} \]
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\partial(A \sqcup B) \cong (M \sqcup N) \sqcup (M' \sqcup N')
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Everything has order 2:

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\partial((M \times I)) \cong (M \sqcup M) \sqcup [\emptyset].
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$X$ a space.

A **vector bundle** on $X$ is:

\[ p : V \to X \quad \text{(continuous, surjective)} \]

$p^{-1}(x)$ is a vector space $\forall x \in X$.

Each $x \in X$ has a neighbourhood $U$ with $p$ the same as $U \times \mathbb{R}^k \to U$.

**Trivial bundle:** $\mathbb{R}^k : \mathbb{R}^k \times X \to X$.

A smooth manifold $M$ has a **tangent bundle** $TM$. 

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A smooth manifold $M$ has a tangent bundle $TM$. 

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Thom Spaces

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**Definition**

- $\mathbb{D}(V) := \text{subspace with norm} \leq 1$
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Point “at infinity” in $T(V)$ from $\mathbb{S}(V)$ is canonical base point.
Example

Zero bundle $\mathbb{R}^0 = X \times \{0\}$ on $X$: 

$$D(\mathbb{R}^0) \simeq X \quad S(\mathbb{R}^0) \simeq \emptyset$$

So $T(\mathbb{R}^0) = X / \emptyset = X \amalg \{\ast\}$, with added base point.

Lemma

$T(V \oplus \mathbb{R}) \simeq \Sigma T(V)$

$(X, x)$ a pointed space, reduced suspension is $\Sigma X := X \times I \cup X \times \{0\} \cup X \times \{1\} \cup \{x\} \times I$. 
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Grassmannians

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Grassmannians

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- \( \text{Gr}_{n+k}(n) \subseteq \text{Gr}_{n+k+1}(n) \) and \( \gamma_{n+k}^n \subseteq \gamma_{n+k+1}^n \) via
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Take a colimit — glue together spaces along these maps:

$$\text{Gr}_n(n) \hookrightarrow \text{Gr}_{n+1}(n) \hookrightarrow \text{Gr}_{n+2}(n) \hookrightarrow \cdots \hookrightarrow BO(n)$$

$$\gamma_{n}^n \hookrightarrow \gamma_{n+1}^n \hookrightarrow \gamma_{n+2}^n \hookrightarrow \cdots \hookrightarrow \gamma^n$$
Fact

$BO(n)$ is a classifying space for rank-$n$ vector bundles: For a nice space $X$, there's a natural bijection

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BO(n) is a classifying space for rank-n vector bundles: For a nice space X, there’s a natural bijection

\[ [X, BO(n)] \cong \{ \text{isomorphism classes of rank-n vector bundles on } X \} \]

One direction: pull back \( \gamma^n \)

\[
\begin{array}{ccc}
  f^* \gamma^n & \rightarrow & \gamma^n \\
  \downarrow \quad & & \downarrow \\
  X & \longrightarrow & BO(n)
\end{array}
\]
Some justification:
If \( X \) is compact, rank-\( n \) bundle \( V \to X \) embeds in \( \mathbb{R}^N \) for \( N \gg 0 \).
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If $X$ is compact, rank-$n$ bundle $V \to X$ embeds in $\mathbb{R}^N$ for $N \gg 0$. This gives map $X \to \text{Gr}_N(n) \hookrightarrow BO(n)$:

$$x \in X \mapsto V_x \text{ as subspace of } \mathbb{R}^N$$
The Universal Thom Space

\[ MO(k) := T(\gamma^k) \]
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- Get map $\gamma^k \oplus \mathbb{R} \to \gamma^{k+1}$
- Apply $T$: get $T(\gamma^k \oplus \mathbb{R}) \simeq \Sigma MO(k) \to MO(k + 1)$
Homotopy Groups

- $(X, x)$ and $(Y, y)$ pointed spaces
- $f, g : X \to Y$ pointed maps
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- \(f, g : X \to Y\) pointed maps
- A **pointed homotopy** from \(f\) to \(g\) is

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h : X \times I \to Y
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with

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h(-, 0) = f, \quad h(-, 1) = g, \quad h(x, t) = y \forall t
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- $\pi_n(X, x) :=$ pointed homotopy classes of pointed maps
  \[ (S^n, *) \to (X, x) \]
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- $f + g : S^n \to S^n \vee S^n \xrightarrow{f \vee g} X$
The Universal Thom Space

Have

\[ \sigma_{k,n+k} : \pi_{n+k} \text{MO}(k) \xrightarrow{\Sigma} \pi_{n+k+1} \Sigma \text{MO}(k) \to \pi_{n+k+1} \text{MO}(k+1) \]
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**Definition**

\[ \pi_n MO := \text{colim}_k \pi_{n+k} MO(k) \]

I.e. \( \bigoplus_k \pi_{n+k} MO(k)/(\phi - \sigma_{k,n+k}(\phi)) \)
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Or: Take adjoint \( \text{MO}(n) \to \Omega \text{MO}(n+1) \), get

\[ \text{MO}(0) \to \Omega \text{MO}(1) \to \Omega^2 \text{MO}(2) \to \cdots \]
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Could build a space \( \text{MO} \) with homotopy groups \( \pi_* \text{MO} \) by “fattening” these maps into inclusions, then glue.
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Could build a space \( \text{MO} \) with homotopy groups \( \pi_* \text{MO} \) by “fattening” these maps into inclusions, then glue. \( \text{MO} \) should really be a **spectrum**, but that’s another story...
Theorem (Thom, 1954)

$\mathcal{N}_* \cong \pi_* MO$
Normal Bundles

- $M$ a compact $n$-manifold
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$$i : M \hookrightarrow \mathbb{R}^{n+k}$$
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- By definition, this means \( TM \) injects into

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i^* \mathbb{R}^{n+k} \cong \mathbb{R}^{n+k}
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- $\nu := \textbf{normal bundle}$ of embedding
The embedding $i$ extends to an embedding $\nu \hookrightarrow \mathbb{R}^{n+k}$. The image $N$ of $\nu$ is called a tubular neighbourhood of $i(M)$. 

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- Then $\bar{N}/\partial \bar{N} \simeq T(\nu)$
The Pontryagin-Thom Collapse Map

Theorem (Tubular Neighbourhood Theorem)

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- Then $\tilde{N}/\partial \tilde{N} \cong T(\nu)$
- Pontryagin-Thom collapse map:
  \[
  C_i : S^{n+k} := (\mathbb{R}^{n+k})^+ \to \tilde{N}/\partial \tilde{N}
  \]
  Identity on $N$, everything outside $N$ goes to “point at $\infty$”
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- This induces $T(\nu) \to T(\gamma^k) = MO(k)$
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- But $\nu$ is a rank-$k$ vector bundle, so is pulled back from $\gamma^k$
- This induces $T(\nu) \to T(\gamma^k) = MO(k)$
- Composing we’ve got $\alpha(i): S^{n+k} \to MO(k)$
The Pontryagin-Thom Construction

- We’ve made $S^{n+k} \to T(\nu)$
- But $\nu$ is a rank-$k$ vector bundle, so is pulled back from $\gamma^k$
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Claim
$[\alpha(i)]$ in $\pi_n(MO)$ depends only on the cobordism class of $M$. 
Independent of Embedding

- \( i: M \hookrightarrow \mathbb{R}^{n+k} \) an embedding
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- \( i: M \hookrightarrow \mathbb{R}^{n+k} \) an embedding
- Then \( i': M \hookrightarrow \mathbb{R}^{n+k} \hookrightarrow \mathbb{R}^{n+k+1} \) is another embedding
Independent of Embedding

- $i: M \hookrightarrow \mathbb{R}^{n+k}$ an embedding
- Then $i': M \hookrightarrow \mathbb{R}^{n+k} \hookrightarrow \mathbb{R}^{n+k+1}$ is another embedding
- $\nu_{i'}$ is $\nu_i \oplus \mathbb{R}$ and $C_{i'}: S^{n+k+1} \to T(\nu_{i'}) \cong \Sigma T(\nu_i)$ is $\Sigma C_i$
Independent of Embedding

- $i : M \hookrightarrow \mathbb{R}^{n+k}$ an embedding
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- Then $\alpha(i')$ is $\sigma_{k,n+k}(\alpha(i))$ so same in $\pi_n MO$
Independent of Embedding

- $i: M \hookrightarrow \mathbb{R}^{n+k}$ an embedding
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Fact

- $i: M \hookrightarrow \mathbb{R}^{n+k}$ and $j: M \hookrightarrow \mathbb{R}^{n+l}$ embeddings
- For $m \gg 0$ get “same” tubular neighbourhoods from

$$M \overset{i}{\hookrightarrow} \mathbb{R}^{n+k} \overset{}{\hookrightarrow} \mathbb{R}^{n+m}$$

$$M \overset{j}{\hookrightarrow} \mathbb{R}^{n+l} \overset{}{\hookrightarrow} \mathbb{R}^{n+m}$$
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Fact

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M \xrightarrow{i} \mathbb{R}^{n+k} \hookrightarrow \mathbb{R}^{n+m} \\
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\]

- So $\alpha(i)$ and $\alpha(j)$ are same in $\pi_n MO$!
Homomorphism

- $M$ and $N$ two compact $n$-manifolds
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- Embedding $M \amalg N \hookrightarrow \mathbb{R}^{n+k}$ gives embeddings
  
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- $T(\nu_{M \amalg N}) \simeq T(\nu_M) \vee T(\nu_N)$
Homomorphism

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  $M \hookrightarrow \mathbb{R}^{n+k}$
  
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- $T(\nu_{M \amalg N}) \simeq T(\nu_M) \vee T(\nu_N)$
- $C_{M \amalg N}$ factors as

  $S^{n+k} \to S^{n+k} \vee S^{n+k} \xrightarrow{C_M \vee C_N} T(\nu_M) \vee T(\nu_N)$

  and $\alpha(M \amalg N)$ is

  $S^{n+k} \to S^{n+k} \vee S^{n+k} \xrightarrow{(\alpha(M), \alpha(N))} MO(k)$
Homomorphism

- $M$ and $N$ two compact $n$-manifolds
- Embedding $M \amalg N \hookrightarrow \mathbb{R}^{n+k}$ gives embeddings
  \[ M \hookrightarrow \mathbb{R}^{n+k} \]
  \[ N \hookrightarrow \mathbb{R}^{n+k} \]
- $T(\nu_{M\amalg N}) \simeq T(\nu_M) \vee T(\nu_N)$
- $C_{M\amalg N}$ factors as
  \[ S^{n+k} \to S^{n+k} \vee S^{n+k} \xrightarrow{C_M \vee C_N} T(\nu_M) \vee T(\nu_N) \]
  and $\alpha(M \amalg N)$ is
  \[ S^{n+k} \to S^{n+k} \vee S^{n+k} \xrightarrow{(\alpha(M), \alpha(N))} MO(k) \]
- But this defined addition in $\pi_{n+k}$ so
  \[ \alpha(M \amalg N) \simeq \alpha(M) + \alpha(N) \]
We’re left with showing that if $M$ is a boundary then $\alpha(M)$ is zero.
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**Theorem**

If $M = \partial W$ then any embedding of $M$ in $S^{n+k}$ extends to a nice embedding of $W$ in the disc $D^{n+k+1}$. 

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Cobordism and the Pontryagin-Thom Construction
We’re left with showing that if $M$ is a boundary then $\alpha(M)$ is zero.

**Theorem**

If $M = \partial W$ then any embedding of $M$ in $S^{n+k}$ extends to a nice embedding of $W$ in the disc $D^{n+k+1}$.

- Tubular neighbourhood of $M$ in $S^{n+k}$ is intersection of sphere with a tubular neighbourhood of $W$ in the disc.
We’re left with showing that if $M$ is a boundary then $\alpha(M)$ is zero.

**Theorem**

If $M = \partial W$ then any embedding of $M$ in $S^{n+k}$ extends to a nice embedding of $W$ in the disc $D^{n+k+1}$.

- Tubular neighbourhood of $M$ in $S^{n+k}$ is intersection of sphere with a tubular neighbourhood of $W$ in the disc.
- Normal bundle of $M$ is pulled back from that of $W$. 
So we have

\[ S^{n+k} \rightarrow T(\nu_M) \rightarrow MO(k) \]

Factors through

\[ D^{n+k+1} \rightarrow T(\nu_W) \]
So we have

\[ S^{n+k} \to T(\nu_M) \]

\[ D^{n+k+1} \to T(\nu_W) \]

\[ \alpha(M) \]

\[ MO(k) \]

\[ \alpha(M) \text{ factors through } D^{n+k+1} \cong * \text{ so is zero.} \]
Have well-defined homomorphism $\alpha : \mathcal{N}_n \to \pi_n MO$. Want inverse $\beta : \pi_n MO \to \mathcal{N}_n$. 

WLOG $\phi : S^{n+k} \to MO(k)$ is compact so factors through $T(\gamma_{k+m+k}) \to Gr(m+k)(k)$ for $m \gg 0$ $T(\gamma_{k+m+k})$ is manifold away from $\infty$.

Zero section gives $\zeta : Gr(k+m+k) \to T(\gamma_{k+m+k})\{\infty\}$ $\zeta$ is embedding of codimension-$k$ submanifold.

WLOG pre-image of $\zeta(Gr(k+m+k))$ is submanifold $\beta(\phi)$ of $S^{n+k}$ of codimension $k$. 

Rune Haugseng
Cobordism and the Pontryagin-Thom Construction
Have well-defined homomorphism $\alpha : \mathcal{N}_n \to \pi_n MO$. Want inverse $\beta : \pi_n MO \to \mathcal{N}_n$.

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$S^{n+k}$ compact so factors through $T(\gamma_{m+k} \to Gr_{m+k}(k))$ for $m \gg 0$
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Take \( \phi : S^{n+k} \to MO(k) \)

\( S^{n+k} \) compact so factors through \( T(\gamma_{m+k}^k \to Gr_{m+k}(k)) \) for \( m \gg 0 \)

\( T(\gamma_{m+k}^k) \) is manifold away from \( \infty \)

Zero section gives \( \zeta : Gr_{m+k}^k \hookrightarrow T(\gamma_{m+k}^k) \setminus \{\infty\} \)

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WLOG \( \phi \) is smooth away from base point
• Have well-defined homomorphism $\alpha: \mathcal{N}_n \rightarrow \pi_n MO$. Want inverse $\beta: \pi_n MO \rightarrow \mathcal{N}_n$.
• Take $\phi: S^{n+k} \rightarrow MO(k)$
• $S^{n+k}$ compact so factors through $T(\gamma_{m+k}^k \rightarrow Gr_{m+k}(k))$ for $m \gg 0$
• $T(\gamma_{m+k}^k)$ is manifold away from $\infty$
• Zero section gives $\zeta: Gr_{m+k}^k \hookrightarrow T(\gamma_{m+k}^k) \setminus \{\infty\}$
• $\zeta$ is embedding of codimension-$k$ submanifold
• WLOG $\phi$ is smooth away from base point
• WLOG pre-image of $\zeta(Gr_{m+k}^k)$ is submanifold $\beta(\phi)$ of $S^{n+k}$ of codimension $k = \text{dimension } n$
If two such smooth $\phi$'s are homotopic can choose a nice, smooth homotopy $S^{n+k} \times I \to T(\gamma^k_{m+k})$. 

Then WLOG pre-image of $\zeta(Gr_k^{m+k})$ in $S^{n+k} \times I$ is a cobordism. Let's skip rest of proof that this is a well-defined homomorphism $\pi_n^{MO} \to \pi_n^N$. 

Rune Haugseng
• If two such smooth $\phi$’s are homotopic can choose a nice, smooth homotopy $S^{n+k} \times I \to T(\gamma_{m+k}^k)$

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Let’s show $\phi = \alpha(\beta(\phi))$:
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- $T(\gamma_{m+k}^k) \setminus \{\infty\}$ is tubular neighbourhood of $\zeta(\text{Gr}_{m+k}(k))$
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- Then $S^{n+k} \to T(\gamma^k_{m+k})$ factors through collapse map $S^{n+k} \to T(\nu_M)$
- Have pullback diagram

$$
\begin{array}{ccc}
\nu_M & \to & \gamma^k_{m+k} = T(\gamma^k_{m+k}) \setminus \{\infty\} \\
\downarrow & & \downarrow \\
M & \to & \text{Gr}_{m+k}(k)
\end{array}
$$
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- So $M \to \text{Gr}_{m+k}(k) \hookrightarrow BO(k)$ classifies $\nu_M$ and map $T(\nu_M) \to T(\gamma^k_{m+k}) \to MO(k)$ comes from this
Let’s show \( \phi = \alpha(\beta(\phi)) \):

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And now $\beta(\alpha(M)) = M$: 
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- $M \hookrightarrow \mathbb{R}^{n+k}$, get $\nu_M$ and $C_M: S^{n+k} \to T(\nu_M)$
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$$
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\end{array}
\end{align*}
$$

- So $T(\nu_M) \to MO(k)$ factors through $T(\gamma^k_{m+k})$
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And now $\beta(\alpha(M)) = M$:

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- I.e. $\beta(\alpha(M)) = M$. 

The Result

Theorem (Thom)

$\mathcal{N}_* \cong \mathbb{F}_2[x_2, x_4, x_5, x_6, x_8, \ldots]$ is a polynomial algebra with one generator in each degree not of the form $2^k - 1$. 

Theorem (Milnor)

For $k$ even, can take $x_k = \mathbb{RP}_k$

For $k$ odd, can take $x_k$ the hypersurface of degree $(1, 1)$ in $\mathbb{RP}_{2p+1} \times \mathbb{RP}_{2p}$ where $k = 2p(2q+1) - 1$. 

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Cobordism and the Pontryagin-Thom Construction
The Result

Theorem (Thom)

\[ \mathcal{N}_* \cong F_2[x_2, x_4, x_5, x_6, x_8, \ldots] \text{ is a polynomial algebra with one generator in each degree not of the form } 2^k - 1. \]

Theorem (Milnor)

- For \( k \) even, can take \( x_k = \mathbb{RP}^k \)
- For \( k \) odd, can take \( x_k \) the hypersurface of degree \((1, 1)\) in \( \mathbb{RP}^{2p+1}q \times \mathbb{RP}^{2p} \) where \( k = 2^p(2q + 1) - 1 \)
**Oriented Cobordism:** compact oriented manifolds $M$ and $N$
oriented cobordant: $\exists$ oriented $B$ with

$$\partial B \cong M \amalg \bar{N}$$
Oriented Cobordism: compact oriented manifolds $M$ and $N$ oriented cobordant: $\exists$ oriented $B$ with

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Theorem (Thom)

$$\Omega_{SO}^* \cong \pi_* MSO$$

$$\Omega_{SO}^* \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{CP}^{2i} | i \geq 1]$$
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Has been computed integrally, but complicated.
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(Too easy!)
Framed cobordism: $M$ with trivialization of $TM \oplus R^m$ for $m \gg 0$, cobordisms with trivialization
\[ \Omega_*^1 \cong \pi_* M \cong \pi_* S^0 = ??? \] — stable homotopy of spheres
**Framed cobordism:** $M$ with trivialization of $TM \oplus R^m$ for $m \gg 0$, cobordisms with trivialization

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(Too hard!)
Other Cobordisms

**Complex cobordism:** $M$ with almost complex structure on $TM \oplus R^m$ for $m \gg 0$

**Theorem (Milnor)**

$\Omega_*^U \cong \pi_* MU \cong \mathbb{Z}[x_1, x_2, \ldots]$ with $x_i$ in degree $2i$. 
**Complex cobordism:** $M$ with almost complex structure on $TM \oplus R^m$ for $m \gg 0$

**Theorem (Milnor)**

$$\Omega_*^U \cong \pi_* MU \cong \mathbb{Z}[x_1, x_2, \ldots] \text{ with } x_i \text{ in degree } 2i.$$  

(*Just right!*)
• $V \to M$ a rank-$n$ vector bundle, get $f_V : M \to BO(n)$

For $x \in H^n(BO(n); \mathbb{R})$, get $x^V := f_V^* V(x)$ — a characteristic class of $V$.

For $V = TM$, get $x^M := x^{TM}([M]) \in \mathbb{R}$, these are characteristic numbers of $M$. 
- $V \to M$ a rank-$n$ vector bundle, get $f_V : M \to BO(n)$
- $f_V^* : H^*(BO(n); R) \to H^*(M; R)$
• $V \to M$ a rank-$n$ vector bundle, get $f_V : M \to BO(n)$
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V → M a rank-\( n \) vector bundle, get \( f_V : M \to BO(n) \)

\( f_V^* : H^\ast(BO(n); R) \to H^\ast(M; R) \)

For \( x \in H^\ast(BO(n); R) \), get \( x_V := f_V^*(x) \) — a characteristic class of \( V \)

\( M \) a connected compact \( n \)-manifold, \( x \in H^n(BO(n); R) \), can evaluate on fundamental class — get \( x_V([M]) \in R \)
- \( V \rightarrow M \) a rank-\( n \) vector bundle, get \( f_V : M \rightarrow BO(n) \)
- \( f_V^* : H^*(BO(n); R) \rightarrow H^*(M; R) \)
- For \( x \in H^*(BO(n); R) \), get \( x_V := f_V^*(x) \) — a characteristic class of \( V \)
- \( M \) a connected compact \( n \)-manifold, \( x \in H^n(BO(n); R) \), can evaluate on fundamental class — get \( x_V([M]) \in R \)
- For \( V = TM \), get

\[
x_M := x_{TM}([M]) \in R,
\]

these are characteristic numbers of \( M \)
Stiefel-Whitney Classes

Theorem

\[ H^*(BO(n); \mathbb{F}_2) \cong \mathbb{F}_2[w_1, \ldots, w_n] \text{ with } w_i \text{ in degree } i \]
Stiefel-Whitney Classes

**Theorem**

\[ H^*(BO(n); \mathbb{F}_2) \cong \mathbb{F}_2[w_1, \ldots, w_n] \text{ with } w_i \text{ in degree } i \]

- So \( H^k(BO(n); \mathbb{F}_2) \) has generators

\[ w(i_1, \ldots, i_n) := w_1^{i_1} \cdots w_n^{i_n} \]

where \( i_1 + 2i_2 + \cdots + ni_n = k \).
Stiefel-Whitney Classes

Theorem

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Stiefel-Whitney Classes

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**Theorem**

*Two compact connected n-manifolds are cobordant if and only if they have the same Stiefel-Whitney numbers.*