Quiz 3 Practice

1. Consider $\frac{dy}{dt} = -y^2$. Let $y_n$ be the computed value of $y(n\Delta t)$. Write out an explicit formula for $y_{n+1}$ in terms of $y_n$ using
   (a) Forward Euler
   (b) Backward Euler
   (c) Trapezoidal rule

2. Find values of $c_0, c_1, c_2$ such that $c_0u(0) + c_1u(\Delta x) + c_2u(2\Delta x)$ is a first order approximation of $u''(0)$.

3. Consider the $2\pi$ periodic boundary value problem:
   $\frac{d}{dx} \left( \rho(x) \frac{du}{dx} \right) = f(x)$
   $u(0) = u(2\pi)$
   $u'(0) = u'(2\pi)$
   Assume $\rho(0) = \rho(2\pi)$. Show that any solution to this equation satisfies the following weak form:
   \[ \int_0^{2\pi} \rho(x) \frac{du}{dx} \frac{d\phi}{dx} dx = \int_0^{2\pi} f(x)\phi(x) dx \] for all $2\pi$ periodic $\phi(x)$

4. Consider the boundary value problem with Dirichlet boundary conditions:
   $\frac{d}{dx} \left( \rho(x) \frac{du}{dx} \right) = 1$
   $u(0) = 0$
   $u(1) = 0$
   where $\rho(x) = \begin{cases} 1 & \text{if } x < 1/2 \\ 2 & \text{if } x > 1/2 \end{cases}$
   (a) Write out the weak form of this boundary value problem
   (b) Let $N = 3, \Delta x = \frac{1}{4}, x_i = i\Delta x$. For $i = 1, 2, 3$, let $\phi_i(x)$ be the piecewise linear functions that satisfy $\phi_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. Write out the linear system that needs to be solved under a finite element method with basis functions $\phi_1, \phi_2, \phi_3$
5. What is the complex Fourier series expansion of \( f(x) = 1 + \sin(4x) \)?

6. What is the solution to

\[
\begin{align*}
- \frac{d^2 u}{dx^2} &= e^{4ix} + e^{-3ix} \\
u(0) &= u(2\pi) \\
u'(0) &= u'(2\pi)
\end{align*}
\]

7. Solve

\[
\begin{align*}
- \frac{d^2 u}{dx^2} &= \delta \left( x - \frac{L}{3} \right) \\
u(0) &= 0 \\
u(L) &= 0
\end{align*}
\]

(a) directly

(b) by a Fourier Sine series

8. The functions \( \phi_n(x) = \sin \frac{n\pi x}{L} \) for \( n = 1, 2, 3, \cdots \) form an orthogonal basis of functions from \( x = 0 \) to \( x = L \). This means \( f(x) = 1 \) can be written as a sum of sines. Find the expansion of \( f(x) = 1 \) in this basis.