Problem Set 2

Due: 28 February 2013 in class

1. (10 points) Show that
   \[ L = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & 0 & 1 \end{pmatrix} \] is the inverse of \[ S = \begin{pmatrix} 1 & 0 & 0 \\ -\ell_{21} & 1 & 0 \\ -\ell_{31} & 0 & 1 \end{pmatrix} \]

2. (10 points) By trial and error, find examples of 2 by 2 matrices such that
   (a) \( AB \neq BA \)
   (b) \( A^2 = -I \), with only real entries in \( A \)
   (c) \( B^2 = 0 \), with no zeros in \( B \)

3. (10 points) By hand, factor the matrix \( A = LU \), where \( L \) is lower triangular, \( U \) is upper triangular, and
   \[ A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \]

4. (10 points) Use back substitution twice by hand to solve \( LUx = f \), where
   \[ L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} , \quad U = \begin{pmatrix} 2 & 8 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & 7 \end{pmatrix} , \quad \text{and} \quad f = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix} \]

5. (20 points) Consider a line of \( n \) nodes, each connected to its neighbors by a resistor of resistance \( R \). At the first node, potential is set to 1. At the \( n \)th node, potential is set to 0.

   (a) Write down \( n \) equations relating \( v_1, v_2, \ldots, v_n \). For \( n = 5 \), write out by hand the equations in the form \( Ax = b \).

   (b) Write a Matlab program that, for arbitrary \( n \), forms \( A \) and \( b \) and solves for \( x \). Solve \( Ax = b \) in the case of \( n = 10,000 \). What is the computed value of \( v_{5000} \)? Provide 6 digits. How long does it take to solve \( Ax = b \) in this case? Ignore the time it takes to build the matrix \( A \). Print out your Matlab code.

6. (20 points) Consider the 2d lattice of points from (1, 1) to \((n, n)\). Each is connected to its neighbors by a resistor of resistance \( R \). At the first node \( v_1 = 1 \). At the last node, \( v_{n^2} = 0 \).
(a) In the \( n = 3 \) case, write out by hand the 9 linear equations in the form \( Ax = b \).

(b) Write a Matlab program that, for arbitrary \( n \), forms \( A \) and \( b \) and solves for \( x \). Solve \( Ax = b \) in the case of \( n = 100 \). What is the computed value of \( v_{50} \)? Provide 6 digits. How long does it take to solve \( Ax = b \) in this case? Ignore the time it takes to build the matrix \( A \). Print out your Matlab code.

(c) Based on the results of 5b and 6b: is a one-dimensional problem involving 10,000 nodes more, less, or equally expensive as a two-dimensional problem involving 10,000 nodes?