Lie theory over a semifield

George Lusztig (M.I.T.)
Let $C = (i : j)$ be a (positive definite) Cartan matrix of simply laced type ($i, j$ run through $I$). For any field $k$,

Chevalley (1950’s) associated to $C$ a (simply connected) group $G_k$. We often assume $k = \mathbb{C}$ and write $G = G_{\mathbb{C}}$.

The definition of $G$ includes a torus $T \subset G$,

the Borel subgroups $B^+, B^-$, their “unipotent radicals” $U^+, U^-$.
and injective (root) homomorphisms $x_i : \mathbb{C} \to U^+$, $y_i : \mathbb{C} \to U^-$

(with $i \in I$). Let $W$ be the Weyl group of $G$, $\{s_i; i \in I\}$ the simple reflections, $l : W \to \mathbb{N}$ the length function, $w_0$ the longest element of $W$. Let $\mathcal{B}$ be the variety of Borel subgroups of $G$. For $B, B'$ in $\mathcal{B}$ the relative position $\text{pos}(B, B') \in W$ is well defined.
A *semifield* is a set with two operations, +, \times, which is an abelian group with respect to \times, an abelian semigroup with respect to + and with \((a + b)c = ac + bc\) for all \(a, b, c\). Thus addition, multiplication, division (but no substraction) are defined.
Examples of semifields:

(i) $K = \mathbb{R}_{>0}$; sum and product are induced from $\mathbb{C}$;

(ii) $K = \mathbb{Z}$; new sum $(a, b) \mapsto \min(a, b)$,

new product $(a, b) \mapsto a + b$;

(iii) $K = \{1\}$ with $1 + 1 = 1, 1 \times 1 = 1$. 
The main theme of this talk is that $G_k$ and various related objects can also be defined when the field $k$ is replaced by a semifield $K$. For evidence of this, assume $G = SL_n$.

Then there is a classical submonoid of $G$, the “totally positive” (TP) part $G^{TP}$ of $G$ introduced by Schoenberg (1930), Gantmacher-Krein (1935). It consists of all matrices in $G$ all of whose $s \times s$ minors are in $\mathbb{R}_{\geq 0}$ for $s = 1, 2, \ldots, n - 1$. 
We can view $G^{TP}$ as being obtained from $G$ by replacing $\mathbb{C}$ by the semifield $\mathbb{R}_{>0}$. Return to the general case. Assume $K = \mathbb{R}_{>0}$. In [L1994] I defined the TP-part $G_K$ of $G$ as the submonoid of $G$ generated by

$$\{x_i(a), y_i(a); i \in I, a \in K\}$$

and by $\{\chi(a); \chi \in \text{Hom}(\mathbb{C}^*, \mathbb{T}), a \in K\}$.

(When $G = SL_n$ this is the same as $G^{TP}$ by results of Whitney, Loewner in the 1950’s.)
-the TP-part $U_K^+ \subset U^+$ as the submonoid generated by

$$\{x_i(a); i \in I, a \in K\}$$

- the TP-part $U_K^- \subset U^-$ as the submonoid generated by

$$\{y_i(a); i \in I, a \in K\}.$$  

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$G_K$ is closed in $G$. (The proof uses the theory of canonical bases [L1990].)
The theory in [L1994] was a starting point for

-the theory of cluster algebras: Fomin, Zelevinsky 2002;

-higher Teichmüller theory: Goncharov, Fock 2006;

-the use of the TP Grassmannian in physics: Postnikov 2007,

For any semifield $K$, we define $U^+_K$ (or $U^-(K)$) as the monoid (with 1) with generators $i^a$ with $i \in I$, $a \in K$ and relations (similar to those of a Coxeter group):

$$i^a i^b = i^{a+b} \text{ for } i \in I, \ a, b \text{ in } K;$$

$$i^a j^b i^c = j^{bc/(a+c)} i^{a+c} j^{ab/(a+c)} \text{ for } i, j \in I \text{ with } i : j = -1, \ a, b, c \text{ in } K;$$

$$i^a j^b = j^b i^a \text{ for } i, j \in I \text{ with } i : j = 0, \ a, b \text{ in } K.$$

When $K = \mathbb{R}_{>0}$ we recover $U^+_K$ defined earlier.
(This definition makes sense even if $C$ is not positive definite.)

In the case where $K = \mathbb{Z}$, relations of the type considered above involve piecewise-linear functions; they first appeared in [L1990] in connection with the parametrization of the canonical basis.
Example: $U^\pm_{\{1\}}$ is the monoid with generators $i^1$ with $i \in I$ and relations

$$i^1 i^1 = i^1 \text{ for } i \in I;$$

$$i^1 j^1 i^1 = j^1 i^1 j^1 \text{ for } i, j \in I \text{ with } i : j = -1;$$

$$i^1 j^1 = j^1 i^1 \text{ for } i, j \in I \text{ with } i : j = 0.$$

We can identify $U^\pm_{\{1\}} = W$ as a set (not as a monoid)

by $i_1^1 \ldots i_m^1 \mapsto s_{i_1} \ldots s_{i_m}$ whenever $l(s_{i_1} \ldots s_{i_m}) = m.$
We consider besides $I$, two other copies $-I = \{ -i; i \in I \}$, $I = \{ i; i \in I \}$ of $I$, in obvious bijection with $I$. For $\epsilon = \pm 1$, $i \in I$ we write $\epsilon i = i$ if $\epsilon = 1$, $\epsilon i = -i$ if $\epsilon = -1$.

For any semifield $K$, we define $G_K$ as

the monoid (with 1) with generators $i^a, (-i)^a, \bar{i}^a$

with $i \in I, a \in K$ and the relations below.
\[(\epsilon i)^a(\epsilon i)^b = (\epsilon i)^{a+b} \text{ for } i \in I, \epsilon = \pm 1, a, b \in K;\]

\[(\epsilon i)^a(\epsilon j)^b(\epsilon i)^c = (\epsilon j)^{bc/(a+c)}(\epsilon i)^{a+c}(\epsilon j)^{ab/(a+c)}\]

for \(i, j \in I\) with \(i : j = -1, \epsilon = \pm 1, a, b, c \in K;\)

\[(\epsilon i)^a(\epsilon j)^b = (\epsilon j)^b(\epsilon i)^a\]

for \(i, j \in I\) with \(i : j = 0, \epsilon = \pm 1, a, b \in K;\)

\[(\epsilon i)^a(-\epsilon i)^b = (-\epsilon i)^{b/(1+ab)}i^{(1+ab)^{\epsilon}}(\epsilon i)^{a/(1+ab)}\]

for \(i \in I, \epsilon = \pm 1, a, b \in K;\)
\[ i^a_i^b = i^{ab}, \ i^{(1)} = 1 \text{ for } i \in I, \ a, b \text{ in } K; \]

\[ i^a_j^b = j^b i^a \text{ for } i, j \in I, \ a, b \text{ in } K; \]

\[ j^a (\epsilon i)^b = (\epsilon i)^a \epsilon^{(i:j)} j^a \text{ for } i, j \in I, \ \epsilon = \pm 1, \ a, b \text{ in } K; \]

\[ (\epsilon i)^a (\epsilon j)^b = (\epsilon j)^b (\epsilon i)^a \text{ for } i \neq j \in I, \ \epsilon = \pm 1, \ a, b \text{ in } K. \]

When \( K = \mathbb{R}_{>0} \) we recover \( G_K \) defined earlier.

(This definition makes sense even if \( C \) is not positive definite.)
We have $G_{\{1\}} = W \times W$ (as sets, not as monoids).

Tits has said that $W$ ought to be regarded as the Chevalley group $G_k$ where $k$ is the (non-existent) field with one element.

But $G_{\{1\}}$ is defined for the semifield $\{1\}$. The bijection $W \times W \to G_{\{1\}}$ almost realizes the wish of Tits.
For any semifield $K$ the obvious map $K \to \{1\}$ is compatible with the semifield structure. It induces homomorphisms of monoids $U_K^\pm \to U_{\{1\}}^\pm = W$ (with fibre $U_K^\pm(w)$ over $w$),

$G_K \to G_{\{1\}} = W \times W$. Assume $K = \mathbb{R}_{>0}$. In each case $X = G, U^+, U^-$, the fibres of $X_K \to X_{\{1\}}$ are cells ($\cong K^m$ for some $m$); they give a canonical cell decomposition of $X_K$ and $X_{\{1\}}$ can be viewed as the set of cells.
This pattern extends to other basic objects of Lie theory.

Let $\mathcal{U}$ be the set of unipotent elements in $G$. Assume $K = \mathbb{R}_{>0}$.

The TP-part of $\mathcal{U}$ is by definition $\mathcal{U}_K = \mathcal{U} \cap G_K$. For $w \in W$, let $\text{supp}(w) = \{i \in I; s_i \text{ appears in a reduced expression of } w\}$. By [L1994],

$$\mathcal{U}_K = \sqcup_{(w, w') \in W \times W; \text{supp}(w) \cap \text{supp}(w') = \emptyset} \mathcal{U}_K(w, w') \subset G_K$$

where $\mathcal{U}_K(w, w') = U_K^+(w)U_K^-(w') = U_K^-(w')U_K^+(w) \subset G_K$ are cells.
The same formula can be used to define $U_K$ for any semifield $K$.

For example $U_\{1\} = \{(w, w') \in W \times W; \text{supp}(w) \cap \text{supp}(w') = \emptyset\}$.

From now on assume $K = \mathbb{R}_{>0}$. In [L1994] I defined the TP-part $B_K$ of $B$ as the closure in $B$ of the set

$$\{uB^+u^{-1}; u \in U^-_K(w_0)\} = \{u'B^-u'^{-1}; u' \in U^+_K(w_0)\}.$$

When $G = SL_2$, $B_K$ is a closed half circle.
Following [L1994] we give a second definition of $\mathcal{B}_K$.

Let $V$ be the irreducible $G$-module over $\mathbb{C}$ with highest weight $\rho$ (which takes value 1 at any simple coroot). Let $\mathcal{B}$ be the canonical basis [L1990] of $V$. Let $V_+ = \sum_{b \in B} \mathbb{R}_{\geq 0} b \subset V$.

Let $\mathcal{X}$ be the set of lines $L$ in $V$ such that $L$ contains some vector in the $G$-orbit of a highest weight vector of $V$. Let

$$\mathcal{X}_K = \{ L \in \mathcal{X}; L \cap (V_+ - \{0\}) \neq \emptyset \}. $$
We can identify $\mathcal{X} = \mathcal{B}, \mathcal{X}_K = \mathcal{B}_K$ by $L \mapsto$ stabilizer of $L$ in $G$.

This second definition of $\mathcal{B}_K$ makes sense even if $C$ is not positive definite. (The first one doesn’t.)

Example: $G = SL_3$. The canonical basis of $V$ can be denoted by $X_{-12}, X_{-1}, X_{-2}, t_1, t_2, X_1, X_2, X_{12}.$
The set $\mathcal{X}_K$ consists of all $a_{-12}X_{-12} + a_{-1}X_{-1} + a_{-2}X_{-2} + c_1 t_1 + c_2 t_2 + a_1 X_1 + a_2 X_2 + a_{12} X_{12} \in V$

with $a_{-12}, a_{-1}, a_{-2}, c_1, c_2, a_1, a_2, a_{12}$ in $\mathbb{R}_{\geq 0}$ (not all 0) such that

\[
    a_2 a_{-12} = c_2 a_{-1}, \quad a_1 a_{-12} = c_1 a_{-2}, \quad a_{-1} a_{12} = c_1 a_2,
\]

\[
    a_{-2} a_{12} = c_2 a_1, \quad a_{12} (c_1 + c_2) = a_1 a_2, \quad a_{-12} (c_1 + c_2) = a_{-1} a_{-2},
\]

\[
    c_1 c_2 = a_{12} a_{-12}, \quad c_1 (c_1 + c_2) = a_1 a_{-1}, \quad c_2 (c_1 + c_2) = a_2 a_{-2},
\]

modulo the homothety action of $K = \mathbb{R}_{>0}$. 
In [L1994] I described a decomposition of $\mathcal{B}_K$ into pieces

$$\mathcal{B}_{K; a \leq b} = \{ B \in \mathcal{B}_K; pos(B^+, B) = b, pos(B^-, B) = w_0a \}$$

indexed by pairs $(a, b) \in W \times W$ such that $a \leq b$ ($\leq$ is the standard partial order on $W$) and conjectured that

$$\mathcal{B}_{K; a \leq b} \cong K^{l(b)-l(a)}.$$ (In the example of $SL_3$ there are 19 pieces.) The conjecture was proved by Rietsch [1998 MIT Ph.D. thesis]. Hence $\mathcal{B}_{\{1\}} = \{(a, b) \in W \times W; a \leq b \}$ is defined.
The natural action of $G$ on $B$ induces an action of the monoid $G_K$ on $B_K$. This induces an action of the monoid $G_{\{1\}} = W \times W$ on $B_{\{1\}}$. It can be described as follows (here $i \in I$):

$$(s_i, 1): (a, b) \mapsto (a, s_i b) \text{ if } s_ib \geq b$$

$$(s_i, 1): (a, b) \mapsto (a, b) \text{ if } s_ib \leq b$$

$$(1, s_i): (a, b) \mapsto (s_i a, b) \text{ if } s_i a \leq a$$

$$(1, s_i): (a, b) \mapsto (a, b) \text{ if } s_i a \geq a.$$
Let $\bar{G}$ be the De Concini-Procesi compactification of $G$. We can define the TP-part $\bar{G}_K$ of $\bar{G}$ as the closure of $G_K$ in $\bar{G}$.

In the early 2000’s I conjectured an explicit cell decomposition for $\bar{G}$ extending the cell decomposition of $B_K \times B_K \subset \bar{G}_K$; this was established by Xuhua He [2005 MIT Ph.D.Thesis].

Hence $\bar{G}_{\{1\}}$ is defined (in terms of $W$) as the indexing set of the set of cells.
Let $u \in G$ be a unipotent element. The Springer fibre

$\mathcal{B}_u = \{ B \in \mathcal{B}; u \in B \}$ is a much studied variety. (See for example Spaltenstein’s 1982 book, which is an extension of his Warwick 1977 Ph.D. thesis). It plays a key role in many questions of representation theory, such as character formulas of complex representations of finite reductive groups.
In 1985/86 (while I was on sabbatical in Rome) I was involved in a joint work with De Concini and Procesi where we showed that $B_u$ has something very close to a cell decomposition and that its homology is generated by algebraic cycles.
Now assume that $u \in G_K$ is unipotent. We define the TP-part of the Springer fibre $\mathcal{B}_u$ to be

$$\mathcal{B}_{u,K} = \{ B \in \mathcal{B}_K; u \in B \} = \mathcal{B}_u \cap \mathcal{B}_K.$$ 

One can show that $\mathcal{B}_{u,K} \neq \emptyset$. Surprisingly, $\mathcal{B}_{u,K}$ has a canonical cell decomposition. Now $u$ is contained in a unique cell

$$\mathcal{U}_K(z, z') = \mathcal{U}_K^+(z)\mathcal{U}_K^-(z') = \mathcal{U}_K^-(z')\mathcal{U}_K^+(z)$$

of $\mathcal{U}_K$ where $(z, z') \in W \times W$ and $J = \text{supp}(z)$, $J' = \text{supp}(z')$ are disjoint.
Let $Z_{J,J'} = \{(v, w) \in W \times W; \\
v \leq w; s_iw \leq w, v \not\leq s_iw \hspace{1em} \forall i \in J; v \leq s_jv, s_jv \not\leq w \hspace{1em} \forall j \in J'\}.$

Theorem: $B_{u,K} = \bigcup_{(v, w) \in Z_{J,J'}} B_{K;v,w}.$

Thus $B_{u,K}$ has a canonical cell decomposition with each cell being a part of the canonical cell decomposition of $B_K$. Hence

$B_{u,\{1\}} = Z_{J,J'} \subset B_{\{1\}}.$
Let \( \tilde{\mathcal{B}} = \{(u, B) \in \mathcal{U} \times \mathcal{B}; u \in B\} \). Let \( \tilde{\mathcal{B}}_{\{1\}} \) be the set of all 

\((z, z', v, w) \in W^4 \) such that \( J = \text{supp}(z), J' = \text{supp}(z') \) are disjoint and \((v, w) \in Z_{J,J'}\). We define the TP-part of \( \tilde{\mathcal{B}} \) to be

\[ \tilde{\mathcal{B}}_K = \{(u, B) \in \mathcal{U}_K \times \mathcal{B}_K; u \in B\}. \]

We have a canonical cell decomposition \( \tilde{\mathcal{B}}_K = \sqcup_{z,z',v,w} \tilde{\mathcal{B}}_{K,z,z',v,w} \)

where \( \tilde{\mathcal{B}}_{K,z,z',v,w} = \{(u, B) \in \mathcal{U}_K(z, z') \times \mathcal{B}_K; v,w\} \)

is a cell of dimension \( l(z) + l(z') + l(w) - l(v) \).
Another example of a semifield is $K' = \mathbb{R}(t)_{>0}$, the set of $f \in \mathbb{R}(t)$ of form $f = t^e f_0/f_1$ for some $f_0, f_1$ in $\mathbb{R}[t]$ with constant term in $\mathbb{R}_{>0}, e \in \mathbb{Z}$ ($t$ is an indeterminate); sum and product are induced from $\mathbb{R}(t)$.

Remark: The map $\alpha : K' \to \mathbb{Z}, t^e f_0/f_1 \to e$ is a semifield homomorphism.
Let $\mathcal{B} = \{ B \in \mathcal{B}; pos(B^+, B) = pos(B^-, B) = w_0 \}$ an open subset of $\mathcal{B}$. Define its TP-part as

$$\mathcal{B}_K = \{ uB^+u^{-1}; u \in U^-_K(w_0) \} = \{ u'B^-u'^{-1}; u' \in U^+_K(w_0) \}.$$  

Now $\mathcal{B}$ makes sense over any field, in particular over $\mathbb{C}(t)$ and then it contains

$$\mathcal{B}_{K'} := \{ uB^+u^{-1}; u \in U^-_{K'}(w_0) \} = \{ u'B^-u'^{-1}; u' \in U^+_{K'}(w_0) \}$$  

as a subset.
We have bijections $U_{K'}^{-}(w_0) \to \mathcal{B}_{K'}$, $u \mapsto uB^+u^{-1}$ and $U_{K'}^+(w_0) \to \mathcal{B}_{K'}$, $u' \mapsto u'B^{-}u'^{-1}$. The composition of the first bijection with the inverse of the second bijection is a bijection $U_{K'}^{-}(w_0) \to U_{K'}^+(w_0)$. 
One can show that there is a unique bijection $U^-_{Z}(w_0) \to U^-_{Z}(w_0)$ such that we have a commutative diagram

\[
\begin{array}{ccc}
U^-_{K'}(w_0) & \longrightarrow & U^+_{K'}(w_0) \\
\downarrow & & \downarrow \\
U^-_{Z}(w_0) & \longrightarrow & U^+_{Z}(w_0)
\end{array}
\]

with vertical maps induced by $\alpha : K' \to Z$. 
We define $\mathcal{B}_\mathbb{Z}$ to be the set of pairs $(\xi^+, \xi^-) \in U^-_\mathbb{Z}(w_0) \times U^+_\mathbb{Z}(w_0)$ such that $\xi^+, \xi^-$ correspond to each other under the bijection $U^-_\mathbb{Z}(w_0) \to U^+_\mathbb{Z}(w_0)$ above. Thus

(a) $\mathcal{B}_K, \mathcal{B}_{K'}, \mathcal{B}_\mathbb{Z}$

are defined. Note that $\mathcal{B}_\mathbb{Z}$ is some kind of flag manifold over the semifield $\mathbb{Z}$. One can show that $G_K, G_{K'}, G_\mathbb{Z}$ acts naturally on (a).