INTRODUCTION TO TOTAL POSITIVITY

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In his paper [C], Chevalley has shown that, given a semisimple algebraic group over $\mathbf{C}$, one can replace $\mathbf{C}$ by any field, and consider the group of points with values in that field. The question arises whether one can replace $\mathbf{C}$ by something less than a field, for instance, by the subsets $\mathbf{R}_{>0}$ or $\mathbf{R}_{\geq 0}$ of the real numbers. It turns out (see [L2]) that this can indeed be done in a meaningful way and leads to a monoid instead of a group. (Following the terminology of Bourbaki, a monoid does not necessarily have 1.)

This generalizes the classical theory of total positivity for SL_n. The study of $n \times n$ matrices with entries in $\mathbf{R}_{>0}$ (say, with determinant 1) goes back to Perron's 1907 paper [P] where it was shown that, for such a matrix, the eigenvalue that is largest in absolute value is simple, real and > 0. In their papers [GK],[KG] (in 1935,1937), Gantmacher and Krein observed that, if one assumes that not only the entries, but all minors of the matrix are > 0, then all eigenvalues of the matrix are simple, real and > 0. The same class of matrices (called totally positive) has in fact appeared earlier, in Schoenberg's 1930 paper [S] in connection with a problem of Pólya, and some of the key ideas of the theory are already present in a 1912 letter of M. Fekete to G. Pólya [FP]. Totally positive matrices are a very useful tool in various areas of applied mathematics (statistics, theory of oscillations of mechanical systems, etc.)

The totally positive matrices form obviously a submonoid of $SL_n(\mathbf{R})$ which could be denoted by $SL_n(\mathbf{R}_{>0})$. Its closure $SL_n(\mathbf{R}_{\geq 0})$ is again a submonoid. In her 1952 paper [W], A. Whitney (apparently a student of Schoenberg in Philadelphia) proved a reduction theorem for totally positive matrices which was later interpreted by Loewner [Lo] as providing a description of $SL_n(\mathbf{R}_{\geq 0})$ as the submonoid generated by certain explicit, rather simple matrices.

This last definition makes in fact sense for any split semisimple group over $\mathbf{R}$ and this is the definition that was adopted in [L2].

In this paper we will give an exposition of some of the results in [L2],[L3] and indicate some open problems. Many of these results have proofs that, at the present state of knowledge, rely on the positivity properties of canonical bases [L1] (even though the results themselves can be stated without reference to canonical bases) and thus are far from elementary. It would be interesting to try to reprove these results independently of the theory of canonical bases.

Supported in part by the National Science Foundation

1. Preliminaries

1.1. Let $\mathbf{R}$ be the field of real numbers. We shall often identify an algebraic variety X defined over $\mathbf{R}$ with the set of its $\mathbf{R}$-rational points. We generally consider X with the usual (metrizable) topology.

Let G be a split, semisimple algebraic group defined over $\mathbf{R}$. Let $\mathfrak{g}$ be the Lie algebra of G (over $\mathbf{R}$). Let $(e_i, f_i, h_i)_{i\in I}$ be a set of Chevalley generators of $\mathfrak{g}$. Then h_i span the Lie algebra $\mathfrak{t}$ of an $\mathbf{R}$-split maximal torus T of G. There is a unique homomorphism of algebraic groups $\chi_i : \mathbf{R}^* \to T$ whose tangent map takes $1 \in \mathbf{R}$ to h_i. We set

$x_i(a) = \exp(ae_i) \in G, \quad y_i(a) = \exp(af_i) \in G$

for any $i \in I, a \in \mathbf{R}$.

There is a unique isomorphism $\Psi : G \xrightarrow{\sim} G^{\text{opp}}$ (the opposite group structure) such that $\Psi(x_i(a)) = y_i(a)$, $\Psi(y_i(a)) = x_i(a)$ for all $i \in I, a \in \mathbf{R}$ and $\Psi(t) = t$ for all $t \in T$.

Let B^+ be the Borel subgroup of G that contains T and $x_i(a)$ for all i, a. Let B^- be the Borel subgroup of G that contains T and $y_i(a)$ for all i, a. Let U^+, U^- be the unipotent radicals of B^+, B^-.

If H is a subgroup of G and $g \in G$, we write ${}^gH$ instead of gHg^{-1}.

1.2. Let $\mathcal{N}$ be the normalizer of T (in G). For $i \in I$, we set

$\dot{s}_i = x_i(1)y_i(-1)x_i(1) \in \mathcal{N}$.

Let $W = \mathcal{N}/T$ and let s_i be the image of $\dot{s}_i$ in W. Then W together with $(s_i)_{i\in I}$ is a Coxeter group. Let $l : W \to \mathbf{N}$ be the standard length function. For $w \in W$, let I_w be the set of all sequences $(i_1, i_2, \dots, i_p)$ in I such that $p = l(w)$ and $s_{i_1}s_{i_2}\dots s_{i_p} = w$. Let $\dot{w} = \dot{s}_{i_1}\dot{s}_{i_2}\dots\dot{s}_{i_p} \in \mathcal{N}$ where $(i_1, i_2, \dots, i_p) \in I_w$. This is independent of the choice of $(i_1, i_2, \dots, i_p)$.

Let $w_0 \in W$ be the element such that $l(w_0)$ is maximum.

1.3. Let $\mathcal{B}$ be the (real) flag manifold of G, that is the real algebraic variety whose points are the Borel subgroups of G defined over $\mathbf{R}$.

For any $w \in W$, let $\mathcal{O}_w$ be the G-orbit on $\mathcal{B}\times\mathcal{B}$ (diagonal action) that contains $(B^+, \dot{w}B^+\dot{w}^{-1})$. For any w, w' in W, let

$$\mathcal{R}_{w,w'} = \{B \in \mathcal{B} | (B^-, B) \in \mathcal{O}_{w'}, (B, B^+) \in \mathcal{O}_{w^{-1}w_0}\}.$$

The sets $\mathcal{R}_{w,w'}$ form a partition of $\mathcal{B}$. It is known (see [KL]) that $\mathcal{R}_{w,w'}$ is non-empty if and only if $w \le w'$ for the standard partial order in W.

1.4. Assume now that $w \le w'$. Let

$$\tilde{\mathcal{R}}_{w,w'} = \{B \in \mathcal{B} | (B^-, B) \in \mathcal{O}_{w'}, (B, {}^{\dot{w}_0\dot{w}}B^-) \in \mathcal{O}_{w_0}\}.$$

This is an open set in $\{B | (B^-, B) \in \mathcal{O}_{w'}\} \cong \mathbf{R}^{l(w')}$. Clearly, the map $(B, u) \mapsto {}^uB$ is an isomorphism

$$\mathcal{R}_{w,w'} \times (U^- \cap {}^{\dot{w}_0\dot{w}}U^-) \xrightarrow{\sim} \tilde{\mathcal{R}}_{w,w'}.$$

It follows that $\mathcal{R}_{w,w'}$ is a smooth manifold of pure dimension $l(w') - l(w)$.

1.5. For $w \in W$, $u \mapsto {}^uB^-$ is an isomorphism $U^+ \cap B^-\dot{w}B^- \xrightarrow{\sim} \mathcal{R}_{1,w_0ww_0^{-1}}$ and $u \mapsto {}^uB^+$ is an isomorphism $U^- \cap B^+\dot{w}B^+ \xrightarrow{\sim} \mathcal{R}_{w_0w,w_0}$. Hence, by 1.4,

(a) $U^\pm \cap B^\mp\dot{w}B^\mp$ is a smooth manifold of pure dimension $l(w)$.

Let $U^\pm_w$ be the closure of $U^\pm \cap B^\mp\dot{w}B^\mp$ in $U^\pm$, that is,

$U^\pm_w = \cup_{w' \le w}(U^\pm \cap B^\mp\dot{w}'B^\mp)$.

2. Definitions

2.1. Let $\mathbf{R}_{>0} = \{a \in \mathbf{R} | a > 0\}$, $\quad \mathbf{R}_{\ge 0} = \{a \in \mathbf{R} | a \ge 0\}$.

If X is a real algebraic variety, it is sometimes possible to give a meaning to the subsets $X_{>0}$ and $X_{\ge 0}$ of X. For example, if $X = \mathbf{R}^p$, these subsets can be taken to be

$\mathbf{R}^p_{>0} = (\mathbf{R}_{>0})^p, \quad \mathbf{R}^p_{\ge 0} = (\mathbf{R}_{\ge 0})^p$.

We will define subsets $X_{>0}, X_{\ge 0}$ of X where X is one of the following real algebraic varieties:

(a) $U^\pm_w, (w \in W)$, $U^\pm$, T,G,$\mathcal{B}$.

2.2. For $w \in W$ with $l(w) = p$, we choose $(i_1, i_2, \dots, i_p) \in I_w$ and define $\phi^\pm : \mathbf{R}^p_{\ge 0} \to U^\pm$ by

$\phi^+(a_1, a_2, \dots, a_p) = x_{i_1}(a_1)x_{i_2}(a_2)\dots x_{i_p}(a_p)$,

$\phi^-(a_1, a_2, \dots, a_p) = y_{i_1}(a_1)y_{i_2}(a_2)\dots y_{i_p}(a_p)$.

Let

$U^\pm_{w,\ge 0} = \phi^\pm(\mathbf{R}^p_{\ge 0}) \subset U^\pm, \quad U^\pm_{w,>0} = \phi^\pm(\mathbf{R}^p_{>0}) \subset U^\pm$.

Then $U^\pm_{w,\ge 0}$ and $U^\pm_{w,>0}$ are independent of the choice of $(i_1, i_2, \dots, i_p)$. (See [L2, 2.7], [L2, 2.10]; in the last reference it is assumed that $w = w_0$ but the same proof applies to any w.)

2.3. Let

$$U^+_{\ge 0} = U^+_{w_0,\ge 0}, \quad U^+_{>0} = U^+_{w_0,>0}, \quad U^-_{\ge 0} = U^-_{w_0,\ge 0}, \quad U^-_{>0} = U^-_{w_0,>0}.$$

These are submonoids of G. More precisely, $U^+_{\ge 0}$ (resp. $U^-_{\ge 0}$) is the submonoid with 1 of U^+ (resp. U^-) generated by the elements $x_i(a)$ (resp. $y_i(a)$) for various $i \in I$ and $a \in \mathbf{R}_{\ge 0}$; $U^+_{>0}, U^-_{>0}$ are submonoids without 1. See [L2, §2].

2.4. Let $T_{>0} = T_{\ge 0}$ be the connected component that contains 1 in the real Lie group T. This is a subgroup of T.

2.5. Let

$$G_{\ge 0} = U^+_{\ge 0}T_{>0}U^-_{\ge 0} = U^-_{\ge 0}T_{>0}U^+_{\ge 0}, \quad G_{>0} = U^+_{>0}T_{>0}U^-_{>0} = U^-_{>0}T_{>0}U^+_{>0}.$$

These are submonoids of G. More precisely, $G_{\ge 0}$ is the submonoid with 1 of G generated by the elements $x_i(a), y_i(a)$ for various $i \in I$ and $a \in \mathbf{R}_{\ge 0}$ and by $T_{>0}$; $G_{>0}$ is a submonoid without 1. See [L2, §2].

We have $\Psi(G_{\ge 0}) = G_{\ge 0}, \Psi(G_{>0}) = G_{>0}$.

2.6. Let
$$\mathcal{B}_{>0}=\{{}^uB^+|u\in U^-_{>0}\}=\{{}^{u'}B^-|u'\in U^+_{>0}\}\subset\mathcal{B}.$$
Let $\mathcal{B}_{\ge 0}$ be the closure of $\mathcal{B}_{>0}$ in $\mathcal{B}$.

Proposition 2.7. *If X is as in 2.1(a), then $X_{>0}$ is open in X and $X_{\ge 0}$ is closed in X. More precisely, $X_{\ge 0}$ is the closure of $X_{>0}$ in X.*

The fact that $X_{\ge 0}$ is the closure of $X_{>0}$ is proved in [L2, 4.2, 4.3, 4.4] for $X=U^+,G$. The proof for $X=U^\pm_w$ is entirely similar to that for U^+, while for $X=T,\mathcal{B}$ there is nothing to prove.

The fact that $X_{>0}$ is open in X is proved in [L2, 4.5, 4.7] for $X=U^+,G$. The proof for $X=U^\pm_w$ is entirely similar to that for U^+ (using 1.5(a)). The proof for $X=\mathcal{B}$ follows from the fact that $U^+_{>0}$ is open in U^+; for $X=T$ there is nothing to prove.

3. Algebraic cell decompositions

3.1. The sets $X_{>0}$, for X in 2.1(a), are cells (that is, they are homeomorphic with $\mathbf{R}^p_{>0}$ for a suitable p). We illustrate this in each case.

3.2. Let $w\in W$. Choose $(i_1,i_2,\dots i_p)\in I_w$. Then $\phi^\pm:\mathbf{R}^p_{>0}\to U^\pm_{w,>0}$ (as in 2.2) is a homeomorphism.

This applies in particular for $w=w_0$, so that $U^+_{>0},U^-_{>0}$ are cells.

3.3. $T_{>0}$ is clearly a cell.

3.4. For any w,w' in W, the map $U^+_{w,>0}\times T_{>0}\times U^-_{w',>0}\to G$, given by multiplication in G, is a homeomorphism onto a subset $G_{w,w'}$ of G. Thus, $G_{w,w'}$ is a cell. In particular, $G_{>0}=G_{w_0,w_0}$ is a cell.

3.5. The map $u\to {}^uB^+$ is a homeomorphism of $U^-_{>0}$ onto $\mathcal{B}_{>0}$. In particular, $\mathcal{B}_{>0}$ is a cell.

3.6. The sets $X_{\ge 0}$, for $X=U^\pm_w,G$ are disjoint unions of finitely many cells.

3.7. More precisely, $U^\pm_{w,\ge 0}=\sqcup_{w';w'\le w}U^\pm_{w',>0}$. In particular, $U^\pm_{\ge 0}=\sqcup_{w'\in W}U^\pm_{w',>0}$.

3.8. We have $G_{\ge 0}=\sqcup_{w,w'\in W}G_{w,w'}$.

3.9. For $X=\mathcal{B}$ the situation is not completely understood. For $w\le w'$, let $\mathcal{R}_{w,w';>0}=\mathcal{B}_{\ge 0}\cap\mathcal{R}_{w,w'}$.

Problem 3.10. *Assume that $w\le w'$. Show that $\mathcal{R}_{w,w';>0}$ is open in $\mathcal{R}_{w,w'}$.*

Since $\mathcal{R}_{w,w';>0}$ is clearly closed in $\mathcal{R}_{w,w'}$, it would follow that $\mathcal{R}_{w,w';>0}$ is a union of connected components of $\mathcal{R}_{w,w'}$. It would also follow that the sets $\mathcal{R}_{w,w';>0}$ (with $w\le w'$) form a partition of $\mathcal{B}_{\ge 0}$ whose pieces are smooth manifolds of pure dimension $l(w')-l(w)$ (see 1.4). For example,

(a) $\mathcal{R}_{1,w_0;>0}=\mathcal{B}_{>0}$ is a connected component of $\mathcal{R}_{1,w_0}$.

See [L2, 8.14]. Moreover, for any $w\in W$, $\mathcal{R}_{w,w;>0}=\mathcal{R}_{w,w}$ is one point, namely a Borel subgroup containing T. See [L2, 8.13].

4. Contractibility

4.1. The sets $X_{\geq 0}$ for $X = U^{\pm}, G, \mathcal{B}$ are contractible. We illustrate this in each case.

4.2. For $r \in \mathbf{R}_{\geq 0}$ we set $u_r = \exp(r \sum_i e_i) \in U^+$. Then $(r, u) \to u_r u$ is a continuous map

$f : \mathbf{R}_{\geq 0} \times U^+_{\geq 0} \to U^+_{\geq 0}$

such that $f(0, u) = u$ and $f(1, u) \in U^+_{>0}$ for all $u \in U^+_{\geq 0}$. See [L2, 5.9, 2.12]. Using the fact that $U^+_{>0}$ is a cell, it follows that $U^+_{\geq 0}$ is contractible. A similar proof shows that $U^-_{\geq 0}$ is contractible.

4.3. For $r \in \mathbf{R}_{\geq 0}$, we set $g_r = \exp(r \sum_i (e_i + f_i)) \in G$. Then $(r, g) \to g_r g$ is a continuous map

$f' : \mathbf{R}_{\geq 0} \times G_{\geq 0} \to G_{\geq 0}$

such that $f'(0, g) = g$ and $f'(1, g) \in G_{>0}$ for all $g \in G_{\geq 0}$. See [L2, 5.9, 2.12]. Using the fact that $G_{>0}$ is a cell, it follows that $G_{\geq 0}$ is contractible.

4.4. The map

$f'' : \mathbf{R}_{\geq 0} \times \mathcal{B}_{\geq 0} \to \mathcal{B}_{\geq 0}$

given by $(r, B) \mapsto {}^{g_r}B$ is continuous and satisfies $f''(0, B) = B$ and $f''(1, B) \in \mathcal{B}_{>0}$ for all $B \in \mathcal{B}_{\geq 0}$. (We use the fact that $g \in G_{>0}, B \in \mathcal{B}_{\geq 0}$ implies ${}^gB \in \mathcal{B}_{>0}$ which follows easily from [L2, 8.17].) Using the fact that $\mathcal{B}_{>0}$ is a cell, it follows that $\mathcal{B}_{\geq 0}$ is contractible.

4.5. If $g \in G_{\geq 0}$ and $B \in \mathcal{B}_{\geq 0}$ then ${}^gB \in \mathcal{B}_{\geq 0}$. See [L2, 8.12]. Thus, the monoid $G_{\geq 0}$ acts on $\mathcal{B}_{\geq 0}$. For any $g \in G_{\geq 0}$, the fixed point set $\{B \in \mathcal{B}_{\geq 0} | g \in B\}$ is a non-empty set. See [L2, 8.11]. It is obviously compact.

Problem 4.6. *Show that $\{B \in \mathcal{B}_{\geq 0} | g \in B\}$ is a contractible space.*

This is certainly true if $g = 1$ by 4.4. On the other hand, if $g \in G_{>0}$, then the fixed point set of g on $\mathcal{B}_{\geq 0}$ is exactly one point, hence is again contractible. See [L2, 8.9(a)].

5. Semialgebraicity

5.1. For any X as in 2.1(a), the subsets $X_{>0}, X_{\geq 0}$ of X are semialgebraic, that is, they can be defined by a finite set of algebraic equalities or inequalities. We illustrate this in the case of $G_{>0}, \mathcal{B}_{>0}$.

In this section we assume that the Coxeter graph of G is simply laced. (The general case can be reduced to this.)

5.2. A homomorphism of algebraic groups $\lambda : T \to \mathbf{R}^*$ will be called a character. If λ is a character then, for any $i \in I$, there is a unique $\lambda_i \in \mathbf{Z}$ such that $\lambda \chi_i(x) = x^{\lambda_i}$ for all $x \in \mathbf{R}^*$. We say that λ is *dominant* (resp. *strictly dominant*) if $\lambda_i \geq 0$ for all i (resp. $\lambda_i > 0$ for all i).

For λ dominant, let Λ_λ be a simple algebraic G-module of finite dimension over $\mathbf{R}$ with a non-zero vector η such that $x_i(a)\eta = \eta$ for all $i \in I, a \in \mathbf{R}$ and $t\eta = \lambda(t)\eta$

for all $t \in T$. Let ${}_\lambda\mathbf{B}$ be the canonical basis of Λ_λ that contains η. See [L1]. For $g \in G$, let $M_\lambda(g)$ be the matrix of $g : \Lambda_\lambda \to \Lambda_\lambda$ with respect to the canonical basis ${}_\lambda\mathbf{B}$. For $B \in \mathcal{B}$, let L_B be the unique B-stable line in Λ_λ.

5.3. If $g \in G_{>0}$ then, for any dominant λ, all entries of $M_\lambda(g)$ and $M_\lambda(\Psi(g))$ are > 0. See [L2, 3.2].

5.4. For a fixed strictly dominant λ, the following converse of 5.3 holds.

If $g \in G$ and all entries of $M_\lambda(g)$ and $M_\lambda(\Psi(g))$ are > 0, then we have $g \in G_{>0}$. See [L3, 5.5].

The hypotheses here can be considerably weakened. See [BZ].

5.5. If $B \in \mathcal{B}_{>0}$, then for any dominant λ, the coordinates of a non-zero vector of L_B with respect to ${}_\lambda\mathbf{B}$ are either all > 0 or all < 0.

5.6. For a fixed strictly dominant λ, the following converse of 5.5 holds.

If $B \in \mathcal{B}$ is such that the coordinates of a non-zero vector of L_B with respect to ${}_\lambda\mathbf{B}$ are either all > 0 or all < 0, then $B \in \mathcal{B}_{>0}$. See [L2, 8.17].

6. Miscellaneous results

6.1. If $g \in G_{>0}$, then g is regular, semisimple. More precisely, the unique maximal torus of G containing g is split over $\mathbf{R}$ and g lies in the identity component of the group of $\mathbf{R}$-points of that torus. See [L2, 5.6]. This generalizes the result of Gantmacher and Krein (for SL_n) mentioned earlier.

6.2. Let $\Omega^+ = B^+\dot{w}_0B^+, \Omega^- = B^-\dot{w}_0B^-$.

(a) $U^\pm_{>0}$ is a connected component of the open subset $U^\pm \cap \Omega^\mp$ of $U^\pm$. See [L2, 4.6].

(b) $G_{>0}$ is a connected component of the open subset $\Omega^+ \cap \Omega^-$ of G. See [L2, 4.8].

6.3. The following generalization of 6.2(a) is proved in the same way as 6.2(a).

For any $w \in W$, $U^\pm_{w,>0}$ is a connected component of $U^\pm \cap B^\mp\dot{w}B^\mp$.

6.4. Let $\mathfrak{g}_{\geq 0}$ be the set of elements in $\mathfrak{g}$ of the form $\sum_i a_ie_i + \sum_i b_if_i + h$ where $a_i, b_i \in \mathbf{R}_{\geq 0}$ for all $i \in I$ and $h \in \mathfrak{t}$.

The following two conditions for $x \in \mathfrak{g}$ are equivalent:

(a) $\exp(tx) \in G_{\geq 0}$ for all $t \in \mathbf{R}_{\geq 0}$.

(b) $x \in \mathfrak{g}_{\geq 0}$.

(The fact that (b) implies (a) is proved in [L2, 5.8]; the fact that (a) implies (b) is proved by K. Rietsch in [R].) In the case of SL_n, the equivalence of (a) and (b) is due to Loewner [Lo].

6.5. Let $\mathfrak{g}_{>0}$ be the set of elements in $\mathfrak{g}$ of the form $\sum_i a_ie_i + \sum_i b_if_i + h$ where $a_i, b_i \in \mathbf{R}_{>0}$ for all $i \in I$ and $h \in \mathfrak{t}$.

The following two conditions for $x \in \mathfrak{g}$ are equivalent:

(a) $\exp(tx) \in G_{>0}$ for all $t \in \mathbf{R}_{>0}$.

(b) $x \in \mathfrak{g}_{>0}$.
We shall prove this assuming that the Coxeter graph of W is simply laced. (The general case can be reduced to this, in a standard way.)

For the fact that (b) implies (a), see [L2, p.549,550].

Assume now that (a) holds. Using 6.4, we see that $x \in \mathfrak{g}_{\geq 0}$. Assume that $b_i = 0$ for some i. Then $\exp(x)$ belongs to a parabolic subgroup P of G such that $B^+ \subset P \neq G$. We can find a dominant character $\lambda : T \to \mathbf{R}^*$ such that $\Lambda_\lambda \neq 0$ and $p\eta = \eta$ for all $p \in P$. (Notation of 5.2.) Since $\eta \in {}_\lambda\mathbf{B}$, it follows that for any $p \in P$ the matrix $M_\lambda(p)$ has some zero entries. Hence, by 5.3, we must have $P \cap G_{>0} = \emptyset$. This contradicts the fact that $\exp(x) \in P \cap G_{>0}$. Thus, we have $b_i \neq 0$ for all i. Since $b_i \in \mathbf{R}_{\geq 0}$, it follows that $b_i > 0$ for all i. One proves in the same way that $a_i > 0$ for all i. Thus, $x \in \mathfrak{g}_{>0}$ and (b) holds.

7. Partial flag manifolds

7.1. In this section we fix a subset J of I. Let P^J be the parabolic subgroup of G generated by B^+ and by $\{\dot{s}_i | i \in J\}$. Let $\mathcal{P}^J$ be the real algebraic variety whose points are the G-conjugates of P^J (a partial flag manifold). Let $\pi^J : \mathcal{B} \to \mathcal{P}^J$ be the canonical map, that is, $B \mapsto P$ where $P \in \mathcal{P}^J$ contains B. We set

$$\mathcal{P}^J_{>0} = \pi^J(\mathcal{B}_{>0}), \quad \mathcal{P}^J_{\geq 0} = \pi^J(\mathcal{B}_{\geq 0}).$$

Then $\mathcal{P}^J_{>0}$ is open in $\mathcal{P}^J$ and $\mathcal{P}^J_{\geq 0}$ is closed in $\mathcal{P}^J$. It is clear that $\mathcal{P}^J_{\geq 0}$ is the closure of $\mathcal{P}^J_{>0}$ in $\mathcal{P}^J$.

A dominant character $\lambda : T \to \mathbf{R}^*$ is said to be *of type* J if $\lambda_i = 0$ for all $j \in J$ and J-*regular* if $\{i \in I | \lambda_i = 0\} = J$. If λ is of type J then, for any $P \in \mathcal{P}^J$, there is a unique line L_P in Λ_λ that is P-stable. (We have $L_P = L_B$ for $B \in \mathcal{B}, P = \pi^J(B)$). If, in addition, λ is J-regular, then the map $P \mapsto L_P$ is an imbedding of $\mathcal{P}^J$ into the projective space of Λ_λ.

7.2. Now assume that W has a simply laced Coxeter graph.

(a) If $P \in \mathcal{P}^J_{>0}$ then, for any dominant λ of type J, the coordinates of a non-zero vector of L_P with respect to ${}_\lambda\mathbf{B}$ are are either all > 0 or all < 0.

(b) If $P \in \mathcal{P}^J_{\geq 0}$ then, for any dominant λ of type J, the coordinates of a non-zero vector of L_P with respect to ${}_\lambda\mathbf{B}$ are are either all ≥ 0 or all ≤ 0.

(a) follows immediately from 5.5 and (b) follows from (a) by a limit argument.

The following problem asks for a converse to (a),(b) above.

Problem 7.3. . *We fix a J-regular dominant λ. Show that*

(a) if $P \in \mathcal{P}^J$ is such that the coordinates of a non-zero vector of L_P with respect to ${}_\lambda\mathbf{B}$ are either all > 0 or all < 0, then $P \in \mathcal{P}^J_{>0}$;

(b) if $P \in \mathcal{P}^J$ is such that the coordinates of a non-zero vector of L_P with respect to ${}_\lambda\mathbf{B}$ are either all ≥ 0 or all ≤ 0, then $P \in \mathcal{P}^J_{\geq 0}$.

Note that one really only has to prove (a), since (b) can be deduced from (a) in the same way as in the case where $J = \emptyset$ (see the proof of [L2, 8.17]).

Note that (a) is a generalization of 5.6 (which corresponds to $J = \emptyset$).

8. i-CIRCLES IN $\mathcal{B}$

8.1. For any X as in 2.1(a), $X_{>0}$ can be viewed as a connected component of an open real algebraic subvariety of X. See 6.2, 6.3, 3.10(a).

Problem 8.2. *Show that, in the setup of 7.1, $\mathcal{P}^J_{>0}$ is a connected component of an open real algebraic subvariety of $\mathcal{P}^J$.*

8.3. In this section we will show that the problem above has an affirmative answer in the case where $J = \{i\}$ with $i \in I$.

We write $\mathcal{P}^i, \pi^i, \mathcal{P}^i_{>0}$ instead of $\mathcal{P}^J, \pi^J, \mathcal{P}^J_{>0}$. The fibres of $\pi^i : \mathcal{B} \to \mathcal{P}^i$ are called *i-circles.* They are homeomorphic to circles. Note that $\mathcal{P}^i$ may be naturally identified with the set of i-circles.

A subset of an i-circle is said to be an *open half circle* if it is open (in the i-circle), connected and is not the whole i-circle. Then it is homeomorphic with an open interval $(0,1)$. The *ends* of an open half circle are the two points in its boundary.

8.4. Let $\mathcal{L}$ be an i-circle. There is a unique $w \in W$ such that $ws_i < w$ and $(B^-, B) \in \mathcal{O}_w$ for all but one $B \in \mathcal{L}$ (say B'), for which $(B^-, B') \in \mathcal{O}_{ws_i}$.

Similarly, there is a unique $w' \in W$ such that $w's_i < w'$ and $(B^+, B) \in \mathcal{O}_{w'}$ for all but one $B \in \mathcal{L}$ (say B''), for which $(B^+, B'') \in \mathcal{O}_{w's_i}$.

We say that $\mathcal{L}$ is of *general type* if $w = w' = w_0$ and $B' \neq B''$. Let $\mathcal{P}^i_*$ be the set of all i-circles of general type. This is an open real algebraic subvariety of $\mathcal{P}^i$.

Lemma 8.5. *(a) The intersection of an i-circle $\mathcal{L}$ with $\mathcal{B}_{>0}$ is either empty or an open half circle.*

(b) If $\mathcal{L} \cap \mathcal{B}_{>0} \neq \emptyset$, then $\mathcal{L}$ is of general type.

Assume that the intersection is non-empty. Then it contains ${}^uB^+$ for some $u \in U^-_{>0}$. We have

$\mathcal{L} = \{{}^{uy_i(a)}B^+ | a \in \mathbf{R}\} \cup \{{}^{u\dot{s}_i}B^+\}$.

We can find $(i_1, i_2, \dots, i_n) \in I_{w_0}$ such that $i_n = i$. Then

$u = y_{i_1}(a_1)y_{i_2}(a_2)\dots y_{i_n}(a_n)$

where all $a_k \in \mathbf{R}_{>0}$. We have

${}^{uy_i(a)}B^+ = {}^{y_{i_1}(a_1)y_{i_2}(a_2)\dots y_{i_n}(a_n+a)}B^+$.

If $a > -a_n$, this belongs to $\mathcal{B}_{>0}$. If $a < -a_n$, this does not belong to $\mathcal{B}_{>0}$.

(Assume that it did. Then we would have

$y_{i_1}(a_1)y_{i_2}(a_2)\dots y_{i_n}(a_n + a) = y_{i_1}(b_1)y_{i_2}(b_2)\dots y_{i_n}(b_n)$

for some $b_1, b_2, \dots, b_n$ in $\mathbf{R}_{>0}$ and an argument in the proof of [L2, 2.7] would show that $a_k = b_k$ for $k < n$ and $a_n + a = b_n$. This would contradict $a_n + a < 0$.)

Thus, if we identify $\mathcal{L}$ with $\{a \in \mathbf{R}\} \sqcup \{\infty\}$, our intersection is the union of $\{a \in \mathbf{R} | a > -a_n\}$ with a subset of $\{-a_n\} \cup \{\infty\}$. But the intersection is open in the circle, hence it is exactly $\{a \in \mathbf{R} | a > -a_n\}$. Thus, it is an open half circle. This proves (a).

From the previous proof we see that the ends of our half circle are

$B' = {}^{u\dot{s}_i}B^+$ and $B'' = {}^{y_{i_1}(a_1)y_{i_2}(a_2)\dots y_{i_{n-1}}(a_{n-1})}B^+$.

Then $(B^-, B') = \mathcal{O}_{w_0s_i}$ and $(B^+, B'') \in \mathcal{O}_{w_0s_i}$ since

$y_{i_1}(a_1)y_{i_2}(a_2)\dots y_{i_{n-1}}(a_{n-1}) \in B^+\dot{w}_0\dot{s}_iB^+$.
The lemma is proved.

Lemma 8.6. *Let $\mathcal{L}$ be an i-circle of general type. Assume that $\mathcal{L}$ contains some point B of $\mathcal{B}_{\geq 0}$. Then $\mathcal{L}$ contains some point of $\mathcal{B}_{>0}$.*

If B is opposed to both B^+ and B^-, then $B \in \mathcal{B}_{>0}$ (by the equality

(a) $\mathcal{B}_{\geq 0} \cap \{\tilde{B} \in \mathcal{B} | \tilde{B}$ opposed to B^+ and $B^-\} = \mathcal{B}_{>0}$

in the proof of [L2, 8.14]).

Thus, we may assume that B is either B' or B'' (as in 8.4).

Assume for example that $B = B'$, so that $(B^-, B') \in \mathcal{O}_{w_0 s_i}$. Since $B' \neq B''$, we have $(B^+, B') \in \mathcal{O}_{w_0}$. Since $B' \in \mathcal{B}_{\geq 0}$ and B' is opposed to B^+, we must have $B' = {}^uB^+$ for a unique $u \in U^-_{\geq 0}$ (as in the proof of [L2, 8.14]). We can find $(i_1, i_2, \dots, i_n) \in I_{w_0}$ such that $i_n = i$. Then

$u = y_{i_1}(a_1)y_{i_2}(a_2)\dots y_{i_n}(a_n)$

with $a_k \in \mathbf{R}_{\geq 0}$ (not uniquely defined). For any $a \in \mathbf{R}_{>0}$ we have

${}^{uy_i(a)}B^+ = {}^{y_{i_1}(a_1)y_{i_2}(a_2)\dots y_{i_n}(a_n+a)}B^+ \in \mathcal{B}_{\geq 0}$

since $(a_1, \dots, a_{n-1}, a_n + a) \in \mathbf{R}^n_{\geq 0}$. Moreover, ${}^{uy_i(a)}B^+ \in \mathcal{L}$. Note also that $(B', {}^{uy_i(a)}B^+) = ({}^uB^+, {}^{uy_i(a)}B^+) \in \mathcal{O}_{s_i}$ since $a \neq 0$. From

$(B^-, B') \in \mathcal{O}_{w_0 s_i}, (B', {}^{uy_i(a)}B^+) \in \mathcal{O}_{s_i}$,

we deduce $(B^-, {}^{uy_i(a)}B^+) \in \mathcal{O}_{w_0}$. Since $(B^+, B') \in \mathcal{O}_{w_0}$, we must have

$(B^+, {}^{uy_i(a)}B^+) \in \mathcal{O}_{w_0}$

for all sufficiently small $a > 0$. For such a, ${}^{uy_i(a)}B^+ \in \mathcal{B}_{\geq 0}$ is opposed to both B^+ and B^-. Using again (a), we deduce that ${}^{uy_i(a)}B^+ \in \mathcal{B}_{>0}$. Thus, in this case, the conclusion of the lemma holds.

The case where $B = B''$ is entirely similar (in fact, symmetric) to that where $B = B'$. The lemma is proved.

Proposition 8.7. *$\mathcal{P}^i_{>0}$ is a connected component of $\mathcal{P}^i_*$.*

$\mathcal{P}^i_{>0}$ is an open connected non-empty subset of $\mathcal{P}^i$ (since $\mathcal{B}_{>0}$ is an open connected non-empty subset of $\mathcal{B}$). By 8.5, $\mathcal{P}^i_{>0}$ is an (open) subset of $\mathcal{P}^i_*$. It remains to show that $\mathcal{P}^i_{>0}$ is closed in $\mathcal{P}^i_*$. Since $\mathcal{P}^i_{\geq 0}$ is closed in $\mathcal{P}^i$, it suffices to show that $\mathcal{P}^i_{\geq 0} \cap \mathcal{P}^i_* = \mathcal{P}^i_{>0}$. But this follows from 8.6. The proposition is proved.

8.8. Now let J be any subset of I. Let W^J be the subgroup of W generated by $\{s_i | i \in J\}$ and let w_0^J be the longest element of W^J.

We want to define an open algebraic submanifold $\mathcal{P}^J_*$ of $\mathcal{P}^J$ that reduces to the one defined in 8.4 in the special case of 8.3.

Let $P \in \mathcal{P}^J$. As in 8.4, there is a unique Borel subgroup $B' \subset P$ such that $(B^-, B') \in \mathcal{O}_y$ for some $y \in W$ with $l(yw_0^J) = l(y) + l(w_0^J)$. Similarly, there is a unique Borel subgroup $B'' \subset P$ such that $(B^+, B'') \in \mathcal{O}_z$ for some $z \in W$ with $l(zw_0^J) = l(z) + l(w_0^J)$. We say that P is of general type if $yw_0^J = zw_0^J = w_0$ and if $(B', B'') \in \mathcal{O}_{w_0^J}$. Let $\mathcal{P}^J_*$ be the set of all $P \in \mathcal{P}^J$ that are of general type. This is an open real algebraic submanifold of $\mathcal{P}^J$.

We could make Problem 8.2 more precise, by asking whether $\mathcal{P}^J_{>0}$ is a connected component of $\mathcal{P}^J_*$.

9. Compactifications

9.1. In this section we assume that G is adjoint. Let $\bar{G}$ be the Oshima-De Concini-Procesi compactification [DP] of G. Let us recall the definition of $\bar{G}$. We choose a strictly dominant character $\lambda : T \to \mathbf{R}^*$. Each element $g \in G$ defines an endomorphism $g : \Lambda_\lambda \to \Lambda_\lambda$; let L_g be the line in $\mathrm{End}(\Lambda_\lambda)$ spanned by $g : \Lambda_\lambda \to \Lambda_\lambda$. Then $g \mapsto L_g$ defines an imbedding of G into the projective space $P(\mathrm{End}(\Lambda_\lambda))$ and $\bar{G}$ is the closure of the image of G in $P(\mathrm{End}(\Lambda_\lambda))$. It is a smooth real algebraic variety, independent of λ, and containing G as an open set.

Let $\bar{G}_{\geq 0}$ be the closure of $G_{\geq 0}$ (or equivalently, of $G_{>0}$) in $\bar{G}$.

9.2. Assume now that the Coxeter graph of W is simply laced. We fix λ as in 9.1. Let ${}_\lambda\underline{\mathbf{B}}$ be the basis of $\mathrm{End}(\Lambda_\lambda)$ consisting of the endomorphisms $\tau_{b,b'}$ (with $b, b' \in {}_\lambda\mathbf{B}$) such that $\tau_{b,b'}$ maps b' to b and maps any other element of ${}_\lambda\mathbf{B}$ to 0.

Consider the following conditions for $L \in \bar{G}$:

(a) the coordinates of a non-zero vector of L with respect to the basis ${}_\lambda\underline{\mathbf{B}}$ of $\mathrm{End}(\Lambda_\lambda)$ are either all > 0 or all < 0;

(b) the coordinates of a non-zero vector of L with respect to the basis ${}_\lambda\underline{\mathbf{B}}$ of $\mathrm{End}(\Lambda_\lambda)$ are either all ≥ 0 or all ≤ 0.

By 5.3, if $g \in G_{>0}$, then $L = L_g$ satisfies (a), hence it satisfies (b). Since the set of all $L \in P(\mathrm{End}(\Lambda_\lambda))$ that satisfy (b) is closed, it follows that any $L \in \bar{G}_{\geq 0}$ satisfies (b).

Problem 9.3. *Let $L \in \bar{G}$. Show that*

(a) if L satisfies 9.2(a), then $L = L_g$ for some $g \in G_{>0}$;

(b) if L satisfies 9.2(b), then $L \in \bar{G}_{\geq 0}$.

Note that (a) would provide a strengthening of 5.4.

We will now show, by an argument similar to that in [L2, 8.17] that, if (a) holds, then (b) holds as well.

Assume that $L \in \bar{G}$ satisfies 8.2(b). Thus, we can find a vector $l \in L$ such that $l = \sum_{b,b' \in {}_\lambda\mathbf{B}} l_{b,b'} \tau_{b,b'}$
where $l_{b,b'} \in \mathbf{R}_{\geq 0}$ for all b, b' and $l_{b,b'} \in \mathbf{R}_{>0}$ for some b, b'.

As in 4.3, we set $g_r = \exp(r \sum_i (e_i + f_i)) \in G$ for any $r \in \mathbf{R}_{\geq 0}$.

By [L2, 5.9], for any $r > 0$ we have $g_r \in G_{>0}$ hence, by 5.3, we have $g_r = \sum_{b,b' \in {}_\lambda\mathbf{B}} c_{r;b,b'} \tau_{b,b'}$
where $c_{r;b,b'} \in \mathbf{R}_{>0}$ for all b, b'. It follows immediately that, for $r > 0$, $g_r l g_r$ (composition of endomorphisms of Λ_λ) is a linear combination of the basis elements $\tau_{b,b'}$ with > 0 coefficients. Since $\bar{G}$ is stable under left and right multiplication by elements in G, the line $g_r L g_r$ must be in $\bar{G}$. Since we assume that (a) holds, we can apply (a) to this line and we deduce that, for any $r > 0$, we have $g_r L g_r = L_g$ for some $g \in G_{>0}$. We now let $r \to 0$. Then $g_r L g_r \to L$. Thus L is in the closure of $G_{>0}$ in $\bar{G}$, hence in $\bar{G}_{\geq 0}$, as desired.

Problem 9.4. *Describe explicitly the intersection of $\bar{G}_{\geq 0}$ with the various strata of $\bar{G}$ (indexed, as in* [DP]*, by pairs J, J' of subsets of I).*

It is likely that the intersection of $\bar{G}_{\geq 0}$ with the lowest stratum $\mathcal{B} \times \mathcal{B}$ of $\bar{G}$ is exactly $\mathcal{B}_{\geq 0} \times \mathcal{B}_{\geq 0}$. (This is indeed so for $G = PGL_2(\mathbf{R})$.)

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