Singualrity, character formulas, and a q-analog of weight multiplicities

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1. The purpose of this paper is to discuss examples in which the intersection cohomology theory of Deligne-Goretsky-MacPherson [4] enters in an essential way in the character formula for some irreducible representation of a semisimple group or Lie algebra. Thus, sections 3-5 are an exposition of the connection between singularities of Schubert varieties and multiplicities in Verma modules. In sections 6-11 we give an interpretation in terms of intersection cohomology for the multiplicities of weights in a finite dimensional representation of a simple Lie algebra. I wish to thank J. Bernstein for allowing me to use his unpublished results on the center of a Hecke algebra. (I learned about his results from D. Kazhdan.) These are used in the proof of Theorem 6.1; the original proof of that theorem was based on [10] and on Macdonald's formulas for spherical functions.

2. Notations. For an irreducible complex algebraic variety $X$, we denote by $H^i(X)$ the $i$-th cohomology sheaf of the intersection cohomology complex of $X$. Let $\mathfrak{g}$ be a simple complex Lie algebra, $\mathfrak{b} \subset \mathfrak{g}$ a Borel subalgebra, $\mathfrak{h} \subset \mathfrak{b}$ a Cartan subalgebra, $\mathfrak{h}^*$ its dual space. Let $W \subset \text{Aut}(\mathfrak{h}^*)$ be the Weyl group, and let $S \subset W$ be the set of simple reflections (with respect to $\mathfrak{b}$). $Q \subset \mathfrak{h}^*$ is the subgroup generated by the roots.

$P \subset \mathfrak{h}^*$ is the subgroup consisting of those elements of $\mathfrak{h}^*$ which take integral values on any coroot. Then $Q$ has finite index in $P$.

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$\tilde{W}_a \subset (\text{affi})$ (acting by trans. transform of $\lambda$)

$W_a$ is the group. It is a C with the reflect $Y_0 \in \mathfrak{h}$ is the h is a semi-direct

For $\lambda \in \mathfrak{g}$, the group law in $\lambda, \lambda' \in \mathfrak{p}$, $\tilde{W}_a$ by $x(y) = \lambda'$.

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Let $P^{++}$ is a possible cosets $\tilde{W}_a'$ in $\tilde{W}$, $\max_{\gamma} (q)$ is an indeterminate half the sum of $\gamma$ coroots.

The fundamental by the fixed hypo simplex in $P \otimes \mathbb{C}$ (left) action of $\lambda = (A_0 )\gamma \otimes \lambda$ For each the standard parts $s_1 s_2 \ldots s_n$ with $1 \leq i \leq n$. We extend $w \leq w'$, $(\gamma, \gamma') \in \Omega$.
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\[ \mathcal{W}_a \subset \text{affine transformations of } h^* \] is the semidirect of \( W \) and of \( P \) (acting by translations). We shall regard \( \mathcal{W}_a \) as acting on the right on \( h^* \). The transform of \( \lambda \in h^* \) under \( w = \mathcal{W}_a \) will be denoted \((\lambda)w\).

\( \mathcal{W}_a \) is the subgroup of \( \mathcal{W}_a \) generated by \( W \) and \( Q \). This is the affine Weyl group. It is a Coxeter group whose set \( S_a \) of simple reflections is \( S \) together with the reflection in \( a_a \) whose fixed point set is \( \{ x \in h^* \mid \langle x, a_a \rangle = 1 \} \); here \( a_\gamma \in h^* \) is the highest coroot. Let \( \Omega \) be the normalizer of \( S_a \) in \( \mathcal{W}_a \). Then \( \mathcal{W}_a \) is a semi-direct product \( \Omega \cdot \mathcal{W}_a \).

For \( \lambda \in P \), we denote by \( p_\lambda \) the same element, regarded in \( \mathcal{W}_a \). Since the group law in \( \mathcal{W}_a \) is written multiplicatively, we have \( p_{\lambda \lambda'} = p_{\lambda} p_{\lambda'} \) for \( \lambda, \lambda' \in P \). \( \ell \) is the length function on the Coxeter group \( \mathcal{W}_a \). We extend it to \( \mathcal{W}_a \) by \( \ell(\gamma w) = \ell(w) - \ell(\gamma) \), \( w \in \mathcal{W}_a \), \( \gamma \in \Omega \). For \( s \in S \), let \( a_s \in Q \) be the corresponding simple root and let \( a_s^\vee \in h \) be the corresponding simple coroot.

Let \( P^+ = \{ p \in P \mid \langle p, a_s^\vee \rangle > 0, \forall s \in S \} \). Then \( P^+ \) parametrizes the double cosets \( \mathcal{W}_a \backslash W / \mathcal{W}_a \). For \( \lambda \in P^+ \), \( \mathcal{W}_\lambda \) denotes the stabilizer of \( \lambda \) in \( W \), \( m_\lambda \) is the element of minimal length of \( \mathcal{W}_\lambda \), \( n_\lambda \) is the element of maximal length of \( \mathcal{W}_\lambda \), \( \nu_\lambda \) is the number of reflections in \( \mathcal{W}_\lambda \), \( \varrho_\lambda = \sum q_\lambda(w) \) (\( q \) is an indeterminate). For \( \lambda = 0 \), we set \( \nu_0 = \nu, \varrho_0 = \varrho \); \( p \in P \) denotes half the sum of all positive roots; \( p \in h \) denotes half the sum of all positive coroots.

The fundamental alcove \( A_0 \) is the open simplex in \( P \otimes \mathbb{R} \) (embedded in \( h^* \)) bounded by the fixed hyperplanes of the various reflections in \( S_a \). An alcove is an open simplex in \( P \otimes \mathbb{R} \) of the form \( (A_0)w \), \( w \in \mathcal{W}_a \) (which is unique). Define a new (left) action of \( \mathcal{W}_a \) on the set of alcoves (denotes \( A \rightarrow \gamma A \)) by the rule \( \gamma (A_0)w = (A_0)\gamma w \). For each \( \lambda \in P \), we denote \( A_\lambda = (A_0)p_\lambda \), \( A_\lambda' = (A_0)p_\lambda \). Let \( \leq \) be the standard partial order on the Coxeter group \( \mathcal{W}_a \). It is generated by the relations \( s_i s_j ... s_n \leq s_i s_j ... s_n \) for any reduced expression \( s_i ... s_n \) \( (s_i \in S_a) \), \( 1 \leq i \leq n \). We extend it to a partial order \( \leq \) on \( \mathcal{W}_a \) by \( \gamma \gamma' \leq \gamma' \Rightarrow \gamma = \gamma' \) and \( w \leq w' \) \( (\gamma, \gamma' \in \Omega, w, w' \in \mathcal{W}_a) \). Let \( \leq \) be the partial order on \( P \) defined by

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\( \lambda \leq \lambda' \iff \lambda' - \lambda \) is a linear combination of positive roots, with \( \geq 0 \) integral coefficients. If \( \lambda, \lambda' \in \mathfrak{h}^+ \), we have \( \lambda \leq \lambda' \) if and only if \( n_{\lambda} \leq n_{\lambda'} \) (in \( \mathcal{W} \)).

For \( \lambda \in \mathfrak{h}^* \), \( M_\lambda \) denotes the Verma module for \( g \) with highest weight \( \lambda \) (with respect to \( \mathfrak{h} \)) and \( L_\lambda \) denotes the unique irreducible quotient \( g \)-module of \( M_\lambda \).

3. We will restrict our attention to the Verma modules \( M_{-\rho w} \) \( (w \in W) \). In the Grothendieck group of \( g \)-modules, \( L_{-\rho w} \) is a linear combination with integral coefficients of the \( g \)-modules \( M_{-\rho y} \) \( (y \leq w) \). The \( g \)-module \( M_{-\rho w} \) appears with coefficient 1, but the other coefficients were rather mysterious. A study of representations of Hecke algebras has led Kazhdan and the author [7] to give a (conjectural) algorithm for these coefficients and to interpret them in terms of singularities of Schubert varieties. Let us define the Schubert varieties. Consider the adjoint group \( G \) of \( g \), and let \( B \) be the Borel subgroup corresponding to \( \mathfrak{h} \), \( G_w \) the \( B-B \) double coset of \( C \) containing a representative of \( w \in W \), \( \mathcal{O}_w = G_w^B \subset G/B \). The Zariski closure \( \overline{\mathcal{O}_w} \) of \( \mathcal{O}_w \) in \( G/B \) is said to be a Schubert variety.

It is the union of the various \( \mathcal{O}_y \) for \( y \leq w \).

The following result was conjectured by D. Kazhdan and the author [7,8] and was proved by J.L. Brylinski and M. Kashiwara [3] and independently by A.A. Beilinson and J.N. Bernstein [1], using the theory of holonomic systems.

Theorem 3.1. In the Grothendieck group of \( g \)-modules, we have, for any \( w \in W \):

\[
L_{-\rho w} = \sum_{y \leq w} (-1)^i \omega_i \dim H^i_{\mathcal{O}_y}(\overline{\mathcal{O}_w})(\mathbb{C} \mathfrak{g}) M_{-\rho y}
\]

where \( \dim H^i_{\mathcal{O}_y}(\overline{\mathcal{O}_w}) \) is the dimension of the stalk of \( H^i_{\mathcal{O}_y}(\overline{\mathcal{O}_w}) \) at a point in \( \mathcal{O}_y \).

4. We shall now describe the integers \( \dim H^i_{\mathcal{O}_y}(\overline{\mathcal{O}_w}) \) following [7,8]. Let us recall the definition of the Hecke algebra \( H \) associated to \( (W, S) \). It consists of all formal linear combinations \( \sum a_w T_w \) with \( a_w \in \mathbb{Z}[q^{1/2}, q^{-1/2}] \) with multiplication defined by the rules \( T_w T_{w'} = T_{ww'} \) if \( T(w) = T(w') \) and \( (T_s^2 - 1)((1 - q) s) = 0 \) if \( s \in S \); here \( q^{1/2} \) is an indeterminate. There is a unique ring involution \( \overline{h} < h \) of \( H \) which takes \( q^{1/2} \) to \( q^{-1/2} \) and \( T_w \) to \( T_{w^{-1}}(\omega \in W) \).
It is semilinear with respect to the ring involution $q^{1/2} \rightarrow q^{-1/2}$ of $\mathbb{Z}[q^{1/2}, q^{-1/2}]$.

According to [7,1.1], for each $w \in W$, there is a unique element $C_w' \in H$ of the form $C_w' = q^{-\ell(w)/2} \sum_{y \leq w} P_{y,w}^T y$, where $P_{y,w}$ are polynomials in $q$ satisfying $P_{y,w} = 1$ and $\deg P_{y,w} \leq 1/2(\ell(w) - \ell(y) - 1)$ for $y \leq w$, and such that $C_w' = C_w'$. The uniqueness of $C_w'$ holds also if $P_{y,w}$ for $y < w$ is only assumed to be a polynomial in $q$ and $q^{-1}$ in which only powers $q^i$ with $i \leq 1/2(\ell(w) - \ell(y) - 1)$ are allowed to occur. It follows automatically that the $P_{y,w}$ are polynomials in $q$.

The proof in [7] applies without change. (The discussion so far in this section, applies to an arbitrary Coxeter group and in particular to $(W_A, S_A)$.) It also applies word by word to $(\tilde{W}_A, S_A)$ which although is not a Coxeter group, possesses the length function and the partial order $\leq$ which give a sense to the previous definitions and results.)

We can now state

**Theorem 4.1.** Let $y < w$ be two elements in the Weyl group $W$. Then

$$\dim H^i_y(O_w) = 0 \text{ if } i \text{ is odd}$$

$$\sum_i \dim H^i_y(O_w) q^i = P_{y,w}$$

Besides the original proof in [8], there is another proof in [12] which has the advantage that it also applies in the case where $O_w$ is replaced by the closure of a $K$-orbit on $G/B$, where $K$ is the centralizer of an involution in $G$. (This plays a role in a character formula for real semisimple Lie groups.) Both proofs make use of reduction to characteristic $> 1$ and of a form of Weil's conjectures.

Combining Theorems 3.1, 4.1, we can rewrite (3.2) in the form

$$L_{\rho \omega}^{-\rho} = \sum_{y \leq w} \left(-q^{\ell(w) - \ell(y)} \cdot P_{y,w}(1) M_{\rho y^{-\rho}} \right)$$

where $P_{y,w}(1)$ is the value of $P_{y,w}$ at $q = 1$. Using the inversion formula [7, 3.1] for the matrix $(P_{y,w})$, this can be also written as

$$M_{\rho w^{-\rho}} = \sum_{w \leq y} P_{w,y}(1) L_{\rho y^{-\rho}}$$
5. Remarks. (a) In the case where \( y, w \in W_\lambda \), the polynomials \( P_{y, w} \) have been interpreted in [7] in terms analogous to (4.3), as intersection cohomology of certain generalized Schubert varieties. (In particular, they have \( \geq 0 \) coefficients).

(b) There is a (conjectural) formula analogous to (3.2) for the characters of irreducible rational representations of a semisimple group over an algebraically closed field of characteristic \( \geq 1 \). It involves the polynomials \( P_{y, w} \) for \( y, w \) in an affine Weyl group. (See [9] for a precise statement).

6. If \( \lambda \in P^{++} \), the \( \mathfrak{g} \)-module \( L_\lambda \) is finite dimensional. With respect to the action of \( h \), it decomposes into direct sum of weight spaces parametrized by elements \( \mu \in P \). For \( \mu \in P^{++} \), we denote \( d_\mu (L_\lambda) \) the dimension of the \( \mu \)-weight space in \( L_\lambda \). It is well known that \( d_\mu (L_\lambda) = 0 \) unless \( \mu \leq \lambda \). The remainder of this paper is mainly concerned with the proof of the following result.

**Theorem 6.1.** If \( \mu, \lambda \in P^{++} \), \( \mu \leq \lambda \), then \( d_\mu (L_\lambda) = P_{\mu, \lambda} (1) \).

Here, \( P_{\mu, \lambda} \) is defined in terms of the Hecke algebra of \( \mathfrak{g} \), see section 4.

(This Hecke algebra will be denoted \( \mathcal{H} \); from now on, we shall reserve the letter \( H \) to denote the Hecke algebra of \( W_\lambda \). It is a subalgebra of \( \mathcal{H} \).) Note that

\[
\begin{align*}
P_{y, w} = P_{y, w} (y \in W, y, w \in W_\lambda) & \quad \text{so that the polynomials } P_{y', w'} , \text{ for } y', w' \in W_\lambda, \\
& \text{have } \geq 0 \text{ coefficients. For type } A, \text{ Theorem 6.1 follows from the results of [11],}
\end{align*}
\]

where \( P_{\mu, \lambda} \) are interpreted as Green-Foulkes polynomials. In general, 6.1 would be a consequence of the conjecture 5(b) together with the Steinberg tensor product theorem. The integers \( d_\mu (L_\lambda) \) are given by Weyl's character formula. To state the formula, we consider the elements

\[
(6.2) \quad k_\lambda = \frac{1}{|W|} \sum_{w \in W_\lambda} (\lambda \in P^{++}), \quad j_\lambda = \frac{1}{|W|} \sum_{w \in W} (\lambda \in P^{++}), \quad \gamma_\lambda = (\lambda \in P^{++}, \gamma_\lambda = (\lambda \in P^{++}, \gamma_\lambda = (\lambda \in P^{++},
\]

of the group algebra \( q[W_\lambda] \). Then \( k_\lambda, j_\lambda \) form a \( \mathbb{Z} \)-basis for the subgroup \( K^1 \) and \( j_\lambda \) form a \( \mathbb{Z} \)-basis for the subgroup \( J^1 \).

It follows that, with \( \chi \) a right \( K^{1-\alpha} \) isomorphism of Prop. 2(iii). Theorem 2.3.

(6.3) For \( \lambda \in P^{++} \), the element in \( K^{1-\alpha} \)

This is equivalent as a quotient of the elements

(6.4)

and therefore

(6.5)

Then \( K_\lambda (\lambda \in P^{++}) \) and \( J_\lambda (\lambda \in P^{++}) \)

Note that \( K \) is a \( K \)-module.

In the st for arbitrary

(6.6)

\( J_\lambda = (-1)^\delta (\alpha_\lambda) J_\lambda \).
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It follows that $K^1$ is a subring of $\mathbb{Q}(\mathbb{W})$ with unit element $\frac{1}{[W]} \Sigma \omega$ and that, with respect to the product in $\mathbb{Q}(\mathbb{W})$, we have $J^1 \cdot K^1 \subseteq J^1$, i.e. $J^1$ is a right $K^1$-module. Moreover, the map $\kappa^1 : J^1 \to k$ is an isomorphism of right $K^1$-modules. (This is a reformulation of [2, Ch. VI, 3.3, Prop. 2(iii)].) We can now state Weyl's character formula as follows.

(6.3) For $\lambda \in P^{++}$, let $c_{\lambda}^1 = \sum_{\mu \in P^{++}} d_{\mu}(l_\lambda) k_\mu \in K^1$. Then $c_{\lambda}^1$ is the unique element in $K^1$ such that $j^{1+}_{\lambda} c_{\lambda}^1 = j^{1+}_{\lambda}$.

(This is equivalent to the usual formulation in which the character of $L_\lambda$ appears as a quotient of two alternating expressions.)

We wish to consider a $q$-analog of the multiplicity $d_{\mu}(l_\lambda)$. The $q$-analog of the elements (6.2) are the following elements of the Hecke algebra $\tilde{H}$:

(6.4) $K^{1} = \frac{1}{p} \sum_{w \in W \cap \mathbb{P}} \Sigma T_w = \Sigma_{w \in W \cap \mathbb{P}} (\Sigma T_w) P_{\lambda} \omega \in \mathbb{W}$

(6.5) $J^{1} = \Sigma_{w \in W \cap \mathbb{P}} (-q)^{\ell(w)} T_w^{-1} \omega^{1-q(p_\lambda)}/p^{1-q(p_\lambda)} \in \mathbb{W}$

and therefore $J^{1} = \frac{q^{1/2}}{2} (\Sigma_{w \in W \cap \mathbb{P}} (-q)^{\ell(w)} T_w^{-1} \omega^{1-q(p_\lambda)}/p^{1-q(p_\lambda)} \in \mathbb{W}$ for $\lambda \in P^{++}$.

Then $K^{1}(\lambda \in P^{++})$ form a $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-basis for

$K = \{ x \in \tilde{H} : (\Sigma T_w) x = x (\Sigma T_w) = P \cdot x \} \subset \mathbb{W} \otimes \mathbb{Q}(q^{1/2})$

and $J^{1}(\lambda \in P^{++})$ form a $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-basis for

$J = \{ y \in \tilde{H} : (\Sigma (-q)^{\ell(w)} T_w^{-1} y = y (\Sigma T_w) = P \cdot y \} \subset \mathbb{W} \otimes \mathbb{Q}(q^{1/2})$.

Note that $K$ is a subring of $\mathbb{W} \otimes \mathbb{Q}(q^{1/2})$ with unit element $\frac{1}{[W]} \Sigma T_w$ and that, with respect to the product in $\mathbb{W} \otimes \mathbb{Q}(q^{1/2})$, we have $J \cdot K \subseteq J$, i.e. $J$ is a right $K$-module.

In the statement of the following theorem, we shall give a meaning to $J_\lambda \in J$ for arbitrary $\lambda \in P$ if $(\lambda) \omega \neq \lambda$ for all $w \in W$, $w \neq e$, we set

$J_\lambda = (-1)^{\ell(w)} j_{(\lambda)w}$ where $w$ is the unique element of $W$ such that $(\lambda)w \in P^{++}$.
For the remaining \( \lambda \in P \), we set \( J_\lambda = 0 \).

**Theorem 6.6.** For any \( \lambda \in P^{++} \), we have

\[
(6.7) \quad J_\lambda \cdot (-q(p_\lambda)^{-1} K_\lambda) = \frac{1}{P_\lambda} \sum \prod (-q)^{-|I|} J_{\lambda + \rho - \alpha_I}
\]

(sum over all subsets \( I \) of the set of positive roots); here \( \alpha_I \) denotes the sum of the roots in \( I \).

The proof will be given in Section 7.

If \( I \) is as in the previous sum and if \( \nu \in W \) is such that \( \nu \cdot \alpha_I = (\nu \cdot \beta) \nu \cdot \lambda' \in P^{++} \), then \( \lambda - \lambda' = \lambda - (\nu \cdot \beta) \nu^{-1} (\nu \cdot \rho) \nu^{-1} + (\nu \cdot \alpha_I) \nu^{-1} = \lambda - (\nu \cdot \beta) \nu^{-1} + \alpha_I \) where \( J \) is the set of positive roots \( \beta \) such that \((\nu \cdot \beta) \nu \in I \) or such that \(- (\nu \cdot \beta) \nu \) is positive, \( \nu \in I \). Since \( \lambda \geq (\nu \cdot \beta) \nu^{-1} (\nu \cdot \rho) \nu^{-1} \) and \( \alpha_I \geq 0 \), it follows that \( \lambda \geq \lambda' \).

Thus, the right hand side of (6.7) is a linear combination of elements \( J_{\lambda + \rho} \) \( (\lambda' \leq \lambda) \) with formal power series in \( q^{-1} \) without terms of form \( q^i \) \( (i > 0) \) as coefficients; moreover for \( \lambda' < \lambda \), the coefficient doesn't have a constant term. On the other hand, since the left hand side of (6.7) is in \( J \), the coefficients must be polynomials in \( q^{1/2} \), \( q^{-1/2} \). It follows that they are polynomials in \( q^{-1} \) (without constant term if \( \lambda' < \lambda \)). The coefficient of \( J_{\lambda + \rho} \) is equal to \( 1 \); this follows from the identity

\[
(6.10) \quad \frac{1}{P_\lambda} \sum \prod (-q)^{-|I|} = 1.
\]

Since a triangular matrix with 1's on diagonal has an inverse of the same form, we see that for any \( \lambda \in P^{++} \), the element \( J_{\lambda + \rho} \) is a linear combination of elements \( J_{\rho} (q^{-\frac{1}{2}(p_{\lambda'})^2 k_{\lambda'}}), \lambda' \leq \lambda, \) with coefficients polynomials in \( q^{-1} \) (without constant term, if \( \lambda' < \lambda \) and \( \rho = I \), if \( \lambda' = \lambda \)). Hence we have

**Corollary 6.8.** For any \( \lambda \in P^{++} \), there is a unique element \( C_\lambda^i \in K \) such that

\[
(6.9) \quad J_{\rho} \cdot C_\lambda^i = J_{\lambda + \rho}.
\]

It is of the form

\[
(6.11) \quad d_{\mu}(L_{\lambda} \; \sigma)\mu \in \sigma
\]

	over, the powers

\[
(6.12) \quad d_{\lambda}(L_{\lambda} \; \sigma) = \text{right} \; K\text{-modules}
\]

Note that known [5] that, (6.13)

\[
\text{Hence} \quad \frac{1}{2}(L_{\lambda} \; \sigma) = \mu \in \mu
\]

We shall coefficients.

We have

**Theorem 6.12.** we have

\[
(6.14) \quad \text{for the proof of}
\]

\[
(6.15) \quad \text{Lemma 6.14. If}
\]

\[
(6.16) \quad \text{in the case which is proved in the}
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The defir tended to a ring

\[
(6.17) \quad \text{From (6.9) it th}
\]

\[
(6.18) \quad C_\lambda^i - C_\lambda^j \in K, \; \text{we}
\]

\[
(6.19) \quad \text{The element}
\]
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\[ (6.10) \quad C'_\lambda = q^{-\frac{1}{2}c(p_\lambda)/2} \sum_{\mu \subseteq \lambda} d_\mu (L_\lambda \mid q^2) K_\mu \]

where \( d_\mu (L_\lambda \mid q) \) are polynomials in \( q \) and \( q^{-1} \) with integer coefficients; moreover, the powers \( q^\frac{1}{2} \) appearing in \( d_\mu (L_\lambda \mid q) \) satisfy \( \frac{1}{2} (\varepsilon(p_\lambda) - \varepsilon(p_\mu)) \) if \( \mu \subseteq \lambda \)

and \( d_\mu (L_\lambda \mid q) = 1 \). In particular, the map \( h \mapsto J_h \) defines an isomorphism of right \( K \)-modules of \( K \) onto \( J \).

Note that, if \( \mu \subseteq \lambda \), then \( \frac{1}{2} (\varepsilon(p_\lambda) - \varepsilon(p_\mu)) \) is an integer. Indeed, it is known [5] that, for \( \lambda \in P^+ \),

\[ (6.11) \quad \varepsilon(p_\lambda) = \langle \lambda, 2\beta \rangle. \]

Hence \( \frac{1}{2} (\varepsilon(p_\lambda) - \varepsilon(p_\mu)) = \frac{1}{2} \langle \lambda, 2\beta \rangle - \langle \mu, 2\beta \rangle = \langle \lambda - \mu, \beta \rangle \) and this is an integer since \( \lambda - \mu \in Q \).

We shall now show that \( d_\mu (L_\lambda \mid q) \) are actually polynomials in \( q \) with \( \geq 0 \) coefficients.

We have

**Theorem 6.12.** \( C'_\lambda = q^{\nu/2} p^{-1} C'_{\eta_\lambda} \text{ (} \lambda \in P^+ \text{).} \) In particular, for \( \mu \subseteq \lambda \text{ in } P^+ \), we have

\[ (6.13) \quad d_\mu (L_\lambda \mid q) = p_{\mu, \eta_\lambda} \]

hence \( d_\mu (L_\lambda \mid q) \) is a polynomial in \( q \) with \( \geq 0 \) coefficients.

For the proof of 6.12, we need the following result.

**Lemma 6.14.** If \( \lambda \in P^+ \), then \( J_{\lambda^+, \rho} = J_{\lambda^+, \rho} \).

In the case where \( \lambda \in Q \cap P^+ \), this is just Lemma 11.7 of [10]. The general case is proved in the same way.

The definition of \( K \) shows that \( K \) is stable under \( h \mapsto \overline{h} \) (which is extended to a ring involution of \( \mathbb{H} \otimes \mathfrak{g}(q^{1/2}) \)). (Note that \( \overline{p^\tau \sum T_v} = p^{-1} \sum \overline{T_v} \).

From (6.9) it then follows that \( J_{\rho^+, \lambda} = J_{\rho^+, \lambda} \). Thus \( J_{\rho} (C'_{\lambda^+} - C'_{\lambda^+}) = 0 \) and, since \( C'_{\lambda^+} - C'_{\lambda^+} \in K \), we have \( C'_{\lambda^+} = C'_{\lambda^+} \), by the last sentence in Corollary 6.8.

The element \( q^{\nu/2} p_{\lambda} \) is also fixed by \( h \mapsto \overline{h} \), since \( \overline{q^{\nu/2}} p_{\lambda} = q^{-\nu/2} p_{\lambda} \). This ele-
moment is equal to
\[ q^{-\frac{1}{2}\lambda_\nu} \sum_{y \in \mathfrak{C}_\nu} \mathfrak{d}_\nu(y) (L_\nu | q)_T y \]
where \( \mathfrak{d}_\nu(y) \in \mathfrak{P}^+ \) is defined by \( y \in \mathfrak{W} \mathfrak{P}_\nu(y) \).

We now use the bounds on the powers of \( q \) appearing in \( d_\nu(L_\nu | q) \) given in Corollary 6.8. If follows that \( q^{-\frac{1}{2} \mathfrak{C}_\nu} \) satisfies the defining property of \( \mathfrak{C}_\nu \), hence it is equal to it. Thus Theorem 6.12 follows from Theorem 6.6. On the other hand, it implies Theorem 6.1. Indeed, under the specialization \( Z(q^{1/2}, q^{-1/2}) = \mathbb{Z} \), given by \( q^{1/2} = 1 \), \( \mathfrak{N} \) becomes the group ring \( \mathbb{Z}[\mathfrak{C}_\nu] \), \( k_\lambda \) becomes \( k_\lambda(L \in \mathfrak{P}^+) \), \( J_\lambda \) becomes \( j_\lambda \), \( \lambda \in \mathfrak{P}^+ \) and (6.9) becomes (6.3). It follows that for \( \nu, \lambda \in \mathfrak{P}^+, \nu < \lambda \), \( d_\nu(L_\lambda) \) is the value of \( d_\nu(L_\lambda | q) \), at \( q = 1 \) and theorem 6.1 follows.

7. For the proof of Theorem 6.6 we shall need several preliminary steps. We shall begin with a definition (due to J. Bernstein) of a large commutative subalgebra of \( \mathfrak{N} \), which is a \( q \)-analogue of the subring \( \mathbb{Z}[\mathfrak{P}] \) of \( \mathbb{Z}[\mathfrak{A}] \). To each \( \lambda \in \mathfrak{P} \), Bernstein associates an element \( \mathfrak{T}_\lambda \in \mathfrak{N} \) defined by \( \mathfrak{T}_\lambda = (q^{1/2} \mathfrak{T}_\lambda)^{1/2} \mathfrak{T}_\lambda^{-1} \) where \( \lambda_1, \lambda_2 \) are elements of \( \mathfrak{P}^+ \) such that \( \lambda = \lambda_1 - \lambda_2 \). This is independent of the choice of \( \lambda_1, \lambda_2 \), since for \( \lambda', \lambda'' \in \mathfrak{P}^+ \) we have the identity \( \mathfrak{T}_{\lambda'} \mathfrak{T}_{\lambda''} = \mathfrak{T}_{\lambda'} \mathfrak{T}_{\lambda''} \mathfrak{T}_{\lambda'} \mathfrak{T}_{\lambda''} \mathfrak{T}_{\lambda'} \mathfrak{T}_{\lambda''} \). (Indeed, we have \( \mathfrak{T}_{\lambda'} \mathfrak{T}_{\lambda''} = \mathfrak{T}_{\lambda'} \mathfrak{T}_{\lambda''} \mathfrak{T}_{\lambda'} \mathfrak{T}_{\lambda''} \mathfrak{T}_{\lambda'} \mathfrak{T}_{\lambda''} \).)

Lemma 7.1. (J. Bernstein) Let \( \lambda \in \mathfrak{P} \) and let \( s \in \mathfrak{S} \). We have
\[ \mathfrak{T}_s \mathfrak{T}_\lambda \mathfrak{T}_s = (\mathfrak{T}_s \mathfrak{T}_\lambda \mathfrak{T}_s) \mathfrak{T}_s. \]

Proof: We may clearly assume that \( \nu, \nu_\lambda \geq 0 \). Assume first that \( \nu, \nu_\lambda = 0 \).
We can write \( \nu = \nu_1 - \nu_2 \), with \( \nu_1, \nu_2 \in \mathfrak{P}^+ \), \( \nu_1, \nu_2 = 0 \). To prove the identity \( \mathfrak{T}_s \mathfrak{T}_\lambda \mathfrak{T}_s = \mathfrak{T}_s \mathfrak{T}_\lambda \mathfrak{T}_s \), we are thus reduced to the case where \( \lambda \in \mathfrak{P}^+ \), \( \nu_1, \nu_2 = 0 \). But then \( \mathfrak{T}(s_\lambda) = \mathfrak{T}(s_\lambda) \mathfrak{T}(s_\lambda) \), hence \( \mathfrak{T}_s \mathfrak{T}_\lambda \mathfrak{T}_s = \mathfrak{T}_s \mathfrak{T}_\lambda \mathfrak{T}_s \), as required.
Next, we consider the case where \( \langle \lambda, \alpha_s^\vee \rangle = 1 \), i.e. \( (\lambda)_s = \lambda - \alpha_s \). In this case, the result follows from Lemma 4.4(b) in (C. Lusztig, Some examples of square integrable representations of semisimple \( p \)-adic groups, preprint IHES, 1982).

Next, we assume that \( \langle \lambda, \alpha_s^\vee \rangle = d \geq 2 \) and that the result is already known when \( d \) is replaced by \( d' \), \( 0 \leq d' < d \). We can write \( \lambda = \lambda_1 + \lambda_2 \) where \( \langle \lambda_1, \alpha_s^\vee \rangle = d - 1 \), \( \langle \lambda_2, \alpha_s^\vee \rangle = 1 \). Then \( \langle \lambda_1 + \lambda_2, \alpha_s^\vee \rangle = d - 2 \). The induction hypothesis is applicable to \( \lambda_1, \lambda_2 \) and to \( \lambda_1 + \lambda_2 \). Hence \( T_{\alpha} \) commutes with \( A = \prod_{\lambda_1}^{\gamma} \prod_{\lambda_2} \gamma \), \( B = \prod_{\lambda_1}^{\gamma} \prod_{\lambda_2} \gamma \), \( C = \prod_{\lambda_1}^{\gamma} \prod_{\lambda_2} \gamma \) for \( \alpha \in \mathbb{P}^+ \), \( \gamma \in \mathbb{P}^+ \). The lemma is proved.

We now define, for any \( \lambda \in \mathbb{P} \), an element \( \tilde{J}_\lambda \in J \) by the formula

\[
\tilde{J}_\lambda = q^{-1/2} \gamma \prod_{\alpha} \psi \theta
\]

where \( \theta = \sum_{\omega} T_{\omega} \), \( \gamma = \sum_{\omega} (-q)^{\omega} T_{\omega}^{-1} \). When \( \lambda \in \mathbb{P}^++_p \), we have clearly \( \tilde{J}_\lambda = J_\lambda \). In general, we have

**Lemma 7.3.** \( \tilde{J}_{\lambda} \in J \) for any \( \lambda \in \mathbb{P} \), \( \omega \in \mathbb{W} \); hence, \( \tilde{J}_\lambda = J_\lambda \) for all \( \lambda \in \mathbb{P} \).

**Proof:** We may assume that \( \omega = s \in S \). Note that \( T_{\alpha} T_{\lambda} = T_{\lambda} T_{\omega} \), \( T_{\omega}^{-1} = T_{\omega}^{-1} \), hence

\[
\tilde{J}_{\lambda} \circ T_{\alpha} = q^{-1/2} \gamma \prod_{\alpha} \psi \theta \tilde{J}_{\lambda} T_{\alpha} \theta
\]

\[
= q^{-1/2} \gamma \prod_{\alpha} \psi \theta (\tilde{J}_{\lambda} \circ T_{\alpha} \tilde{J}_{\lambda}) T_{\alpha} \theta
\]

\[
= -q \cdot q^{-1/2} \gamma \prod_{\alpha} \psi \theta (\tilde{J}_{\lambda} \circ T_{\alpha} \tilde{J}_{\lambda}) T_{\alpha} \theta
\]

by lemma (7.1)

\[
= -q \tilde{J}_{\lambda} \circ T_{\alpha} \tilde{J}_{\lambda} T_{\alpha} \theta
\]

Thus, \( \tilde{J}_{\lambda} \circ T_{\alpha} \tilde{J}_{\lambda} = 0 \), as required.

**Lemma 7.4.** There is a unique function \( f : \mathbb{P}^+ \rightarrow \mathbb{Z}[q, q^{-1}] \) with finite support satisfying properties (i), (ii), (iii) below:

(i) \( f(0) = q^0 \)

(ii) \( f(\lambda) \neq 0 \Rightarrow \lambda \leq \rho \)

(iii) \( f(\lambda + \alpha) = q^{-\langle \alpha, \alpha \rangle} f(\lambda) \)

for \( \alpha \in \mathbb{P}^+ \), \( \lambda \in \mathbb{P}^+ \).
(iii) Let $X \subset Q^+p$ be an $a$-string: $X = \{x \in \mathbb{Q}^+p, n \in \mathbb{Z}\}$, where $x$ is any fixed element of $Q^+p$ and $a$ is any fixed simple root. Let $a > 0$ be an integer such that $<\lambda, \tilde{\alpha}_b> = a \pmod{2}$ for all $\lambda \in X$. Then

$$\frac{1}{\mathcal{X}} \sum_{\lambda \in \mathcal{X}} f(\lambda) = -q^{-(a-1)} \sum_{\lambda \in \mathcal{X}} f(\lambda) \quad \frac{1}{\mathcal{X}} \sum_{\lambda \in \mathcal{X}} <\lambda, \tilde{\alpha}_b> = a$$

This function is given by the formula

$$(7.5) \quad f(\lambda) = (-1)^v \sum_{I \subseteq \mathcal{X}} (-q)^{|I|} q^{-<\lambda, \tilde{\alpha}_b>}$$

where $I$ runs through the subsets of the set of positive roots, and $\alpha_I$ is defined as in 6.6.

**Proof:** The function $f$ defined by (7.5) clearly satisfies (i) and (ii). We now verify that it satisfies (iii). We shall set $a = \alpha, \tilde{\alpha}_b = \tilde{\alpha}$. We have, with the notations of (iii):

$$(7.6) \quad \mathcal{X} \sum_{\lambda \in \mathcal{X}} f(\lambda) = (-1)^v \mathcal{X} \sum_{\lambda \in \mathcal{X}} (-q)^{|I|} q^{-<\lambda, \tilde{\alpha}_b>}$$

where

$$(7.7) \quad \sum_{\lambda \in \mathcal{X}} (-q)^{|I|} q^{-<\lambda, \tilde{\alpha}_b>} = (1) (7.8) \quad \sum_{\lambda \in \mathcal{X}} (-q)^{|I|+1} q^{-<\lambda, \tilde{\alpha}_b>} =$$

Comparing $\mathcal{X}$ with finite $\mathfrak{g}$, with finite $\mathfrak{sl}_2$.

To $g$ with finite $\mathfrak{g}$, with finite $\mathfrak{sl}_2$.

An element invariant through $X$.

Let $a$ be a
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Hence

\[(7.6) \quad \sum_{\lambda \in \mathfrak{X}} f(\lambda) = (-1)^{\nu} \sum_{\lambda \in \mathfrak{X}} (-q)^{\left| \mathfrak{I}_{q} \right|-\langle \lambda \rangle_{p, \delta}} \cdot \]

A similar computation shows that

\[\sum_{\lambda \in \mathfrak{X}} f(\lambda) = (-1)^{\nu} \sum_{\lambda \in \mathfrak{X}} (-q)^{\left| \mathfrak{I}_{q} \right|-\langle \lambda \rangle_{p, \delta}} \cdot \]

Now the simple reflections maps the set of positive roots +\(a\) onto itself. Hence the last sum is equal to

\[\sum_{\lambda \in \mathfrak{X}} (-q)^{\left| \mathfrak{I}_{q} \right|-\langle \lambda \rangle_{p, \delta}} \cdot \]

Comparing with the right hand side of (7.6), we conclude that \(f\) satisfies (iii).

To prove the converse it is enough to show that if a function \(g: Q_{+} \to \mathbb{Z}_{[q, q^{-1}]}\) with finite support satisfies \(g(\rho) = 0\), \(g(\lambda) \neq 0 \Rightarrow \lambda \leq \rho\) and the identity (iii) with \(f\) replaced by \(g\), then \(g = 0\). Assume that \(g \neq 0\), and let \(x \in Q_{+}\) be an element of maximal possible length (with respect to some positive definite, \(W\)-invariant scalar product on \(P \otimes \mathbb{R}\) such that \(g(x) \neq 0\). Let \(X\) be the string through \(x\) corresponding to the simple root \(a_{\alpha}\). Then \(x' = (x)s\) is also in \(X\). Let \(a\) be the absolute value of \(\langle x, a_{\alpha} \rangle = -\langle x', a_{\alpha} \rangle\). If \(y \in X\) satisfies...
\[ \langle y, \delta_a \rangle > a \] then clearly the length of \( y \) is strictly bigger than that of \( x \)
hence \( g(y) = 0 \). Hence the identity (iii) for \( g \), and \( X \), \( a \), as above, reduces
to \( g(x) = -q^{k+1}g(x') \). It follows that \( g(x') \neq 0 \). Note also that \( x, x' \) have the
same length. Iterating this, we see that \( g(wx) \neq 0 \) for all \( w \in W \); moreover,
\((x)w \) has the same length as \( x \). For suitable \( w \in W \), we have \( \langle (x)w, \delta_a \rangle > 0 \) for
all simple roots \( \alpha \). Replacing \( y \) by \( (x)w \), we may thus assume that \( \langle x, \delta_a \rangle > 0 \)
for all simple roots \( \alpha \). If we had \( \langle x-\rho, \delta_a \rangle > 0 \) for all simple roots \( \alpha \), then
it would follow that \( \langle x, \delta_a \rangle > 0 \); since \( g(x) \neq 0 \), we would have \( \rho - x \geq 0 \), hence
\( \rho - x = \sum a_\alpha \delta_\alpha \) (\( a_\alpha \) simple, \( a_\alpha \geq 0 \) integers), hence \( \langle -\sum a_\alpha \delta_\alpha, \rho \rangle > 0 \). Thus
\( \sum a_\alpha = 0 \), hence \( a_\alpha = 0 \) for all simple roots \( \alpha \), hence \( \rho = \delta_\alpha \). But \( g(\rho) = 0 \)
and this is a contradiction with \( g(x) \neq 0 \). Thus, there exists a simple root \( \alpha \) such that
\( \langle x, \delta_\alpha \rangle < 0 \); since \( \langle x, \delta_\alpha \rangle > 0 \), it follows that \( \langle x, \delta_\alpha \rangle = 0 \).
Consider the string \( X \) through \( x \) corresponding to the simple root \( \alpha \). The equality
\( \langle x, \delta_\alpha \rangle = 0 \) shows that among the elements of \( X \), the element \( x \) has minimal length.
It follows that \( g(y) = 0 \) for all \( y \in X \), \( y \neq x \). Let us now write the identity
(iii) for \( g \), this \( X \), and \( a = 0 \). We get \( g(x) = -q^{k}g(x) \) hence \( g(x) = 0 \).
This contradiction shows that \( g = 0 \) and the Lemma is proved.

We shall now introduce as in [10] an \( H \)-module \( M \) as follows. \( M \) is the
free \( \mathbb{Z}[q^{1/2}, q^{-1/2}] \) module with basis \( (A) \) where \( A \) are the various alcoves in
\( P \otimes \mathbb{R} \). For each \( s \in S_a \), we define an endomorphism \( T_s \) of this \( \mathbb{Z}[q^{1/2}, q^{-1/2}] \)-
module by

\[
T_s (A) = \begin{cases} 
q^aA, & \text{if there exists a root } \alpha \text{ with } \langle x, \delta_\alpha \rangle > n \\
q^aA, & \text{for } x \in sA, \langle x, \delta_\alpha \rangle < n \\
q^aA, & \text{for } x \in A
\end{cases}
\]

These endomorphisms make \( M \) into an \( H \)-module.

Let \( W' \) be the subgroup of \( W \) that is generated by those \( s \in S_a \) for which
\( s(A \delta_\alpha) \) contains \( a \) in its closure. (This is a parabolic subgroup of \( W \) conjugate
to \( W \) under an element in \( \Omega \).)
Lemma 7.7. Let $y \in W_a$. We define a function $f : Q^+ \rightarrow \mathbb{Z} [q^{1/2}, q^{-1/2}]$ as follows:

$$f(\lambda) \text{ is the coefficient with which } A_\lambda^{-} \text{ appears in}$$

$$\left( \sum_{w \in W} (-q)^{l(w)}\mu_{T_w}^{-1} \right)_{T_w} (\sum_{w \in W} \mu_{T_w}) A_o^{-} \in M.$$ 

Then

(i) If $y(\Lambda_\alpha^+) = A_\alpha^{-}$, $\alpha \in \Pi^+$, $\lambda \in Q^+$, then $f(\lambda) = q^\lambda$; moreover $\lambda' \in Q^+$ implies $\lambda' \leq \lambda$.

(ii) In general, let $X = Q^+$ be an $a_s$-string $(a_s$ a simple root) and let $a \geq 0$ be an integer such that $<\lambda, \alpha_s^{-}> = a (\text{mod } 2)$ for all $\lambda \in X$. Then

$$\sum_{\lambda \in X} f(\lambda) = q^{-a-1} \sum_{\lambda \in X} f(\lambda).$$

Proof: (i) Follows from [10, 4.2 (a)] and (ii) is a consequence of [10, 9.2] applied to the element $\sum_{w \in W} \mu_{T_w}^{-1} A_o^{-}$.

Corollary 7.8. If $y$ in the previous lemma is such that $y(\Lambda_\alpha^+) = A_\alpha^{-}$, then

$$(7.9) \quad \left( \sum_{w \in W} (-q)^{l(w)}\mu_{T_w}^{-1} \right)_{T_w} (\sum_{w \in W} \mu_{T_w}) A_o^{-} = q^{-\alpha} \left( \sum_{w \in W} (-q)^{l(w)}\mu_{T_w}^{-1} \right)_{T_w} f(\lambda) h_{\lambda} A_o^{-},$$

where, for $\lambda \in Q^+$, $f(\lambda)$ is given by (7.5), and $h_{\lambda}$ is an element of $H$ such that

$$h_{\lambda} A_o^{-} = A_{\lambda}^{-}.$$  

Proof: In our case, the function $f$ of Lemma 7.7 satisfies the conditions (i), (ii), (iii) of Lemma 7.4, hence is given by (7.5). It follows that for any $\lambda \in Q^+$, $A_{\lambda}^{-}$ appears with the same coefficient in the two sides of (7.9) and the corollary follows.

Since the $H$-module $M$ is faithful, we can erase $A_o^{-}$ from the two sides of (7.9) and we obtain an identity in $H$. We can rewrite this identity as follows. Let $y \in \Pi$ be such that $y \gamma \gamma^{-1} = \gamma$. We multiply both sides of this identity on the left by $T_{\gamma}^{-}$. Note that $T_{\gamma}^{-} T_{\gamma} = T_{\gamma}^{-} T_{\gamma} = T_{\gamma}$. Moreover $T_{\gamma} h_{\lambda} = q^{l(\lambda)}/2q^\gamma$. Thus, we have
\[
\theta^T \theta = (\sum_{\mu}q^{\ell}(\omega)_{T_{\mu}}^{-1})T_{\mu} T = \sum_{\mu}q^{\ell}(\omega)_{T_{\mu}}^{-1} \sum_{\mu}q^{\ell}(\omega)_{T_{\mu}}^{-1} \sum_{\mu}f(\lambda)q^\ell(p_\lambda)/Z_{\mu}T_{\lambda}.
\]

We can now compute for \( \lambda \in P^{++} \):

\[
J_\theta(q^{-\ell}(p_\lambda)/Z_{\lambda}) = q^{-\ell}(m_\lambda)/Z_{\mu} \sum_{\mu}q^{-\ell}(p_\lambda)/Z_{\mu}T_{\mu} T_{\lambda} = \sum_{\mu}q^{-\ell}(m_\lambda)/Z_{\mu}q^{-\ell}(p_\lambda)/Z_{\mu}T_{\mu} T_{\lambda}.
\]

(8.2)

It is clear that

(8.3)

where the \( \geq \) of \( g \)-modules

(8.4)

By Weyl's char.

It follows that

(8.5)

The following result describes the centre \( Z \) of \( \tilde{\mathfrak{h}} \).

8. The following result describes the centre \( Z \) of \( \tilde{\mathfrak{h}} \).

Theorem 8.1. (J. Bernstein). Let \( \lambda \in P^{++} \) and let \( (\lambda)W \) be its \( W \)-orbit in \( P \). Then \( z_\lambda = \sum_{\lambda' \in (\lambda)W} q_{\lambda'} \) is in \( Z \). Moreover, \( Z \) is the free \( \mathbb{Z}[q^{1/2}, q^{-1/2}] \)-module with basis \( z_\lambda \) \( (\lambda \in P^{++}) \).

Proof: Let \( s \in S \). Then \( T_s z_\lambda = z_{\lambda s} \) by 7.1. It follows that \( T_w z_\lambda = z_{\lambda w} \) for all \( w \in W \). It is obvious that, for any \( \mu \in P^{++} \), \( T_{\mu} \) commutes with \( z_\lambda \). But the elements \( T_w \) \( (w \in W) \) and \( T_{\mu} \) \( (\mu \in P^{++}) \) generate \( H \) as an algebra. Hence \( z_\lambda \in Z \).

Let \( z^1_\lambda \) be the specializations of \( z_\lambda \) under the homomorphism \( H \to \mathbb{Z} [\tilde{\mathfrak{h}}] \) given by \( q^{1/2} \to 1 \). Then clearly \( z^1_\lambda \) form a set of \( \mathbb{Z} \)-generators for the centre of \( \mathbb{Z} [\tilde{\mathfrak{h}}] \) : the elements of \( P \) are the only elements of \( \tilde{\mathfrak{h}} \) whose conjugacy class is finite. Using a version of Nakayama's lemma it follows that any element \( z \) of \( Z \)
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is a linear combination of the elements $z_\lambda$ with coefficients being allowed to be in the localization of $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ at the ideal generated by $q^{1/2} - 1$. Since $z \in H$, these coefficients must automatically be in $\mathbb{Z}[q^{1/2}, q^{-1/2}]$. The fact that the elements $z_\lambda$ are linearly independent is obvious. The Theorem is proved.

Let us now define, for $\lambda \in P^{**}$, an element

$$S_\lambda = \sum_{\mu \in P^{**}} d_{\mu} (L_\mu) z_\mu \in \mathbb{Z}$$

It is clear that for $\lambda, \lambda' \in P^{**}$, we have

$$S_\lambda S_{\lambda'} = \sum_{\mu \in P^{**}} m(\lambda, \lambda'; \mu) S_{\mu}$$

where the $\geq 0$ integers $m(\lambda, \lambda'; \mu)$ are the multiplicities in the tensor product of $g$-modules:

$$L_\lambda \otimes L_{\lambda'} = \sum_{\mu \in P^{**}} m(\lambda, \lambda'; \mu) L_\mu$$

By Weyl's character formula (6.3) we have

$$\sum_{\omega \in W} (-1)^{\ell(\omega)} \tau(\omega)_{(\lambda', \lambda)} S_\lambda = \sum_{\omega \in W} (-1)^{\ell(\omega)} \tau(\omega)_{(\lambda, \lambda')} S_{\lambda'}$$

It follows that

$$J_\rho S_\lambda = |W|^{-1} \sum_{\omega \in W} (-1)^{\ell(\omega)} \tau(\omega)_{(\rho, \lambda)} S_\lambda$$

$$= |W|^{-1} \sum_{\omega \in W} q^{-\nu/2} (-1)^{\ell(\omega)} \tau(\omega)_{(\rho, \lambda)}$$

$$= |W|^{-1} \sum_{\omega \in W} q^{-\nu/2} (-1)^{\ell(\omega)} \tau(\omega)_{(\rho, \lambda)}$$

$$= |W|^{-1} \sum_{\omega \in W} (-1)^{\ell(\omega)} \tau(\omega)_{(\lambda', \lambda)} S_{\lambda'}$$

$$= J_{\lambda', \lambda}$$

The identity

$$J_\rho S_\lambda = J_{\lambda', \lambda} \quad (\lambda' \in P^{**})$$

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shows that the map $Z \to J$ given by $x \to J_\rho^x z$ is an isomorphism of $Z[q^{1/2}, q^{-1/2}]$-modules. From this we shall deduce

Proposition 8.6: The map $Z \to K$ given by $z \to (\frac{1}{p} \sum_{\mu \in W} z) = p^{-1/2}z$ is an isomorphism of $Z[q^{1/2}, q^{-1/2}]$-algebras preserving the unit element. Under this isomorphism $S_\lambda \in Z$ correspond to $C_\lambda' \in K$, i.e., $C_\lambda' = p^{-1/2}S_\lambda$.

Indeed, we have a commutative diagram

\[\begin{array}{ccc}
Z & \xrightarrow{F} & J \\
\downarrow{p^{-1}J_0} & & \downarrow{J_0} \\
K & & K
\end{array}\]

(since $p^{-1}J_0 = J_0$) and the maps $Z \to J, K \to J$ given by multiplication by $J_0$ are known to be isomorphisms (see 6.8). Our map $Z \to K$ preserves multiplication: $p^{-1}g_z p^{-1}g_z' = p^{-2}g_z g_z' = p^{-1}g_z g_z'$. Finally $S_\lambda \in Z$ corresponds to $C_\lambda' \in K$, since both correspond to $J_{\lambda', \rho} \in J$ (see (6.9), (8.5)). The isomorphism $Z \to K$ is a version of the Satake isomorphism. It shows in particular that $K$ is a commutative algebra.

Corollary 8.7. If $\lambda, \lambda' \in P^{++}$, we have

$$C_\lambda' \cdot C_{\lambda''} = \sum_{\lambda'''} m(\lambda, \lambda'; \lambda'') C_{\lambda'''}$$

where $m(\lambda, \lambda'; \lambda'')$ are defined by (8.4).

(The remarkable fact in (8.7) is that the coefficients with which $C_{\lambda''}$ appears in the decomposition of $C_\lambda' \cdot C_{\lambda''}$ are independent of $q$.)

Corollary 8.8. For any $\lambda \in P^{++}$, we have $\overline{z}_\lambda = z_\lambda$.

Indeed, the isomorphism given in 8.6 is compatible with $h \to h$ (since $p^{-1}h = p^{-1}b$).

Since $C_\lambda' = C_{\lambda''}$, it follows that $\overline{S}_\lambda = S_\lambda$. But $z_\lambda$ is a $\mathbb{Z}$-linear combination of element $S_{\lambda'}$ ($\lambda' \leq \lambda$) hence $\overline{z}_\lambda = z_\lambda$.

Corollary 8.9. If $\lambda \in P^{++}$, we have

(8.10)

\[\text{(product over)}\]

Proof: The le \[x : \overline{h} \to Z_i \forall_{\lambda} \in \overline{W}_a\] Note

and this is kn Weyl's charact

9. Let $\mu \leq \overline{\gamma, \lambda} \Rightarrow 0$ for the polynomial only depends on well defined f

such that for

(9.1)

for any $\tau \in P$

Proposition 9.:
of $\mathbb{Z}[q^{1/2}, q^{-1/2}]$.

$P^{-1}b_0$ is

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Combination

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(8.10) \[ \sum_{\mu \in \mathcal{P}_+} \frac{q^{\mu} \langle \mu, \lambda \rangle \mu_\lambda (L; q)}{\prod_{\mu \leq \lambda} (q^{\langle \mu, \lambda \rangle} - 1)} = \frac{\prod_{\alpha > 0} (q^{<\alpha, a^\lambda> - 1})}{\prod_{\alpha > 0} (q^{<\alpha, a^\lambda> - 1})} \]

(product over all positive roots $\alpha$)

Proof: The left hand side of (8.10) is $\chi(q^{\langle \rho, \lambda \rangle} / C^\lambda)$ (see 6.10) where $\chi : \mathcal{H} \rightarrow \mathbb{Z}[q^{1/2}, q^{-1/2}]$ is the algebra homomorphism defined by $\chi(T_\mu) = q^{\langle \mu, \lambda \rangle}$, $\forall \mu \in \mathcal{H}$. Note that $\chi(T_{\mu \mu}) = q^{<\mu, \lambda>}$ for any $\mu \in \mathcal{P}_+$, (see (6.11)). We have

$\chi(q^{\langle \rho, \lambda \rangle} / C^\lambda) = \chi(q^{\langle \rho, \lambda \rangle} / 2p^{1/2} \delta^1 \lambda)$

\[ = q^{\langle \rho, \lambda \rangle} / \chi(S^1 \lambda) \]

\[ = q^{<\rho, \lambda>} \sum_{\mu \in \mathcal{P}_+} \mu_\lambda (L; q) \sum_{\mu \leq \lambda} q^{<\mu, \lambda>} \]

and this is known to be equal to the right hand side of (8.10). (See the proof of Weyl's character formula in [6]).

9. Let $\mu \leq \lambda$ be two elements of $\mathcal{P}$. According to [10] if $\tau \in \mathcal{P}$ is such that $<\tau, s_\lambda> = 0$ for all $s \in S$ (so that, in particular, $\mu \tau \in \mathcal{P}^+$, $L \lambda \tau \in \mathcal{P}^+$), the polynomial $P_{\mu \tau \lambda \tau}$ is independent of the choice of $\tau$. In particular, it only depends on the difference $\lambda - \mu$. Using now (6.13), we see that there exists a well defined function

$\hat{\rho} : \{\kappa \in \mathcal{Q} \mid \kappa < 0\} \rightarrow \mathbb{Z}[q^{-1}]$ such that for any $\mu \leq \lambda$ in $\mathcal{P}$, with $\lambda - \mu = \kappa$, we have

(9.1) \[ q^{-<\kappa, \rho> \mu_\tau (L; q)} = \hat{\rho}(\kappa) \]

for any $\tau \in \mathcal{P}$ such that $<\tau, s_\lambda> > 0$ for all $s \in S$.

Proposition 9.2.

(9.3) \[ \hat{\rho}(\kappa) = \sum_{n_1, \ldots, n_\lambda \geq 0} q^{-(n_1 + \ldots + n_\lambda)} \]

\[ \sum_{n_1, \ldots, n_\lambda \geq 0} q^{n_1 + \ldots + n_\lambda} = \kappa \]

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Here $a_1, \ldots, a_r$ is the list of all positive roots and $n_1, \ldots, n_r$ are required to be integers. In particular for $q = 1$, $\hat{P}(\kappa)$ reduces to the Kostant partition function.

Proof: The formulas (8.7), (8.9), (8.10) show that $\hat{P}(\kappa)$ satisfies the recurrence relation

$$\sum_{I} (-q)^{|I|} \hat{P}(\kappa-a_1) = \begin{cases} 1 & \text{if } \kappa = 0 \\ 0 & \text{if } \kappa > 0 \end{cases}$$

(sum over all subsets $I$ of the set of positive roots), with the convention that $\hat{P}(\kappa) = 0$ if $\kappa \neq 0$. From this, the required formula for $\hat{P}(\kappa)$ follows immediately.

It may be conjectured that, for any $\mu \leq \lambda$ in $P^{++}$, we have

$$q^{-\lambda-\mu} \hat{P}(\lambda; q) = \sum_{w \in W} (-1)^{L(w)} \hat{P}(\lambda+\mu-w-(\mu+\mu))$$

(9.4)

For $q = 1$ this reduces to a well known formula of Kostant.

(Note added May 1982: Conjecture (9.4) has been recently proved by S. Kato, to appear in Inventiones Math.)

For type $A$, formula (9.4) follows from a statement in [13, p. 131]; indeed, in that case, the left hand side of (9.4) is a Green-Foulkes polynomial (cf. [11]).

The right hand side of (9.4), in the special case $\mu = 0$, appears also in the work of D. Peterson, in connection with the $g$-module structure of the (graded) coordinate ring of the nilpotent variety of $g$.

10. If $\lambda$ is the highest root, we have $d_{\mu}(L_{\lambda}; q) = 1$ for any $\mu \leq \lambda$. Indeed, the multiplicity $d_{\mu}(L_{\lambda})$ is 1 in this case (it is a dimension of a root space in the adjoint representation of $g$). Since $d_{\mu}(L_{\lambda}; q)$ has 0 coefficients and constant term 1, it must be identically 1. If we write the formula (8.10) for $\lambda$, the only unknown term is, therefore, $d_{\mu}(L_{\lambda}; q)$. We can compute it from (8.10) and we find $d_{0}(L_{\lambda}; q) = \sum q^{e_i-1}$ where $e_i$ (i = 1, ..., rk(g)) are the exponents of $g$.

11. We shall prove the following
   a) $\bar{\delta}_{\lambda}$ is a
   b) If $\mu, \lambda \in P^{++}$, the stalks
   c) Let $x \in$

Letting $\lambda$ be an element of $g'$ with we denote by each $\lambda \in P^{++}$ a vector $t^{\lambda}$ submodule of under the Li
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11. We shall now describe the (generalized) Schubert varieties \( \overline{\mathcal{O}}_\lambda \) \((\lambda \in \mathfrak{p}^{++})\) with the following properties:

a) \( \overline{\mathcal{O}}_\lambda \) is an irreducible, projective complex variety of dimension \( <\lambda, 2\gamma > \).

b) If \( \mu, \lambda \in \mathfrak{p}^{++} \), are such that \( \mu \leq \lambda \) then \( \overline{\mathcal{O}}_\mu \leq \overline{\mathcal{O}}_\lambda \).

c) Let \( x \in \overline{\mathcal{O}}_\lambda \) be such that \( x \in \overline{\mathcal{O}}_\mu \) (\( \mu \leq \lambda \)) but \( x \notin \overline{\mathcal{O}}_\nu \), for any \( \nu \neq \mu \). Then the stalks \( H^i_x(\overline{\mathcal{O}}_\lambda) \) are zero if \( i \) is odd and \( \dim H^i_x(\overline{\mathcal{O}}_\lambda) = d_i(\lambda; q) \).

Let \( \mathfrak{g}' \) be a simple complex Lie algebra which is dual to \( \mathfrak{g} \) in the following sense. There is a Cartan subalgebra \( \mathfrak{h}' \subset \mathfrak{g}' \) with a given isomorphism onto \( \mathfrak{h}^* \) which carries the set of coroots of \( \mathfrak{g}' \) with respect to \( \mathfrak{h}' \) onto the set of roots of \( \mathfrak{g}' \) with respect to \( \mathfrak{h} \). Let \( \mathfrak{g}' = \mathfrak{g}' \otimes \mathfrak{c}(\{t\}) \). For each coroot \( \alpha \in \mathfrak{h}^* \) of \( \mathfrak{g} \) we denote by \( X_\alpha \) a non-zero vector in the corresponding root space of \( \mathfrak{g}' \). For each \( \lambda \in \mathfrak{p}^{++} \), we denote by \( l^\lambda \) the \( \mathfrak{c}(\{t\}) \)-submodule of \( \mathfrak{g}' \) generated by the vectors \( t^{\langle \alpha, l^\lambda \rangle} X_\alpha \) and by \( \mathfrak{h} \otimes \mathfrak{c}(\{t\}) \). This is a lattice in \( \mathfrak{g}' \) (i.e., a \( \mathfrak{c}(\{t\}) \)-submodule of maximal rank.) It is moreover an order in \( \mathfrak{g}' \) (i.e., a lattice closed under the Lie bracket). Let \( (\cdot, \cdot) \) be the Killing form on \( \mathfrak{g}' \); we extend it to a symmetric bilinear form on \( \mathfrak{g}' \) with values in \( \mathfrak{c}(\{t\}) \). Then \( l^\lambda = l^\lambda \) where for any lattice \( L \) we denote by \( L^\vee \) the dual lattice \( \{x \in \mathfrak{g}'^* | (x, y) \in \mathfrak{c}(\{t\}) \} \) for all \( y \in L \). It is easy to check that if \( L \) is any order in \( \mathfrak{g}' \), then \( L \subset L^\vee \). It follows that any self-dual order is a maximal order, hence, by a theorem of Bruhat-Tits, it is a "maximal parahoric" order. It moreover, must correspond to a special vertex of the extended diagram of \( \mathfrak{g}' \). Indeed, if \( L \) is a maximal parahoric order corresponding to a non-special vertex \( v \), then \( \dim(L^\vee/L) \) is equal to the number of roots of \( \mathfrak{g}' \) minus the number of roots in a proper semisimple subalgebra of \( \mathfrak{g}' \) (whose Coxeter diagram is obtained by removing \( v \) from the extended diagram of \( \mathfrak{g}' \)); hence \( L \) is not self-dual. It follows that the group \( G' \) of automorphisms of the Lie algebra \( \mathfrak{g}' \) inducing identity on the Weyl group, acts transitively on the set \( X \) of all self-dual orders in \( \mathfrak{g}' \). Let \( G'_0 = G'_0 \) be the stabilizer of \( \lambda \) in \( G' \). It is known that the sets \( \mathcal{O}_\lambda \) (\( \mathcal{O}_\lambda = G'_0 \)-orbit of \( l^\lambda \) in \( X \)) \((\lambda \in \mathfrak{p}^{++})\) are disjoint and cover the whole of \( X \). For any integer \( n \geq 0 \), we consider the subset.
$X_n \subset X$ defined by $X_n = \{ L \in X \mid t^nL_0 \subset L \subset t^{-n}L_0 \}$. Then $X_0 \subset X_1 \subset X_2 \subset \ldots$ and their union is $X$; indeed for any lattice $L$ we can find $n > 0$ such that $t^nL_0 \subset L$ and we then have by duality $L' \subset t^{-n}L_0$.

We will show that $X_n$ is in a natural way a projective algebraic variety. To give a self-dual lattice $L$, $t^nL_0 \subset L \subset t^{-n}L_0$, is the same as to give a subspace $\mathfrak{L}$ of $t^{-n}L_0/t^nL_0$ which is $t$-stable and is maximal isotropic for the symmetric $\mathbb{F}$-bilinear form on $t^{-n}L_0/t^nL_0$ defined by $\text{Res}(x,y)$. Moreover, $L$ gives rise to a subspace $\mathfrak{L} \subset t^{-n}L_0/t^{2n}L_0$ of codimension $= \dim L_0/t^nL_0$. Now $t^{-n}L_0/t^{2n}L_0$ carries a canonical alternating 3-form with values in $\mathbb{F}$, defined by $\text{Res}([x,y],z)$. The condition that $L$ is an order (if we assume that $L$ is already known to be a self-dual lattice) is that this 3-form is identically zero on $\mathfrak{L}$.

Thus, we have a 1-1 correspondence $L \leftrightarrow \mathfrak{L}$ between $X_n$ and the set of maximal isotropic subspaces of $t^{-n}L_0/t^nL_0$, stable under the nilpotent endomorphism $t$, and whose inverse image in $t^{-n}L_0/t^{2n}L_0$ is such that the canonical alternating 3-form vanishes identically on it.

This is a subset of a Grassmannian, defined by algebraic equations, hence is a projective algebraic variety. Thus $X$ can be regarded as an increasing union of projective varieties. If $\lambda \in \mathfrak{p}^{++}$ satisfies $<\lambda,\vartheta > \leq n$ for all roots then $\overline{G}_\lambda' \subset X_n$. It is then a locally closed subset of $X_n$, since it can be regarded as an orbit of the algebraic group $G_\lambda'/([g' \in G_0' \mid g' = 1)$ on $L_0/t^nL_0$ acting on $X_n$.

We then define $\overline{G}_\lambda$ to be the Zariski closure of $\overline{G}_\lambda$ in $X_n$. One could define similarly the varieties $\overline{G}_\lambda$ over a finite field $\mathbb{F}_p$ (instead of over $\mathbb{F}$).

The number of rational points (over $\mathbb{F}_p$) of $\overline{G}_\lambda$ (in the sense of intersection cohomology) i.e., with each rational point $x$ counted with a multiplicity equal to the trace of the Frobenius map on $\mathbb{Z}(1)^{iH^l_x(\overline{G}_\lambda)}$ is the left hand side of (8.10), hence it is given by the right hand side of (8.10), with $q$ replaced by $p^s$.

In particular, the Euler characteristic of $\overline{G}_\lambda$ (in the sense of intersection cohomology) is equal to $\dim(L_\lambda')$. 
SINGULARITIES, CHARACTER FORMULAS, WEIGHT MULTIPlicITIES

REFERENCES


