TRANSFER MAPS FOR QUANTUM AFFINE $\mathfrak{sl}_n$

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Introduction

This paper is a continuation of [L3], whose notations we shall keep.

Let $n \geq 2$ be an integer and let $\mathbf{U}$ be the quantum affine $\mathfrak{gl}_n$, as defined in [L3, 7.7] (an algebra over $\mathbf{Q}(v)$ where v is an indeterminate). In [L3] we have defined for any $D \geq 0$ a $\mathbf{Q}(v)$-algebra $\mathbf{U}_D$ with a given surjective algebra homomorphism $\mathbf{q}_D : \mathbf{U} \to \mathbf{U}_D$. (An analogous map of $\mathbf{U}$ into an algebra larger than $\mathbf{U}_D$ was also defined in [GV].) The definition of $\mathbf{U}_D$ was an affine analogue of the geometric construction in [BLM] of finite dimensional quotients ("q-Schur-algebras") of the ordinary quantum $\mathfrak{gl}_n$. In [L3, 9.1] it was stated without proof that for $D \geq n$, there exists a (necessarily unique) algebra homomorphism $\phi_{D,D-n} : \mathbf{U}_D \to \mathbf{U}_{D-n}$ with certain prescribed values on generators. One purpose of this paper is to provide a proof of the existence of the "transfer maps" $\phi_{D,D-n}$.

Using these transfer maps we may regard $(\mathbf{U}_D)$ as a projective system of algebras. Let $\hat{\mathbf{U}}$ be its projective limit. Since $\phi_{D,D-n}\mathbf{q}_D, \mathbf{q}_{D-n}$ are not necessarily equal, we cannot map $\mathbf{U}$ naturally into $\hat{\mathbf{U}}$. However, if we replace $\mathbf{U}$ by ${}^1\mathbf{U}$, the quantum affine $\mathfrak{sl}_n$, then ${}^1\mathbf{U}$ maps naturally into $\hat{\mathbf{U}}$. This gives rise to a corresponding homomorphism ${}^1\dot{\mathbf{U}} \to \hat{\mathbf{U}}$, where ${}^1\dot{\mathbf{U}}$ is the modified quantized enveloping algebra, as in [L2, Ch.23]. In this paper we show that the natural homomorphism ${}^1\dot{\mathbf{U}} \to \hat{\mathbf{U}}$ is injective.

In [L3, 9.2, 9.3] it was conjectured that $\phi_{D,D-n}$ behaves in the simplest possible way with respect to the "intersection cohomology" bases of $\mathbf{U}_D, \mathbf{U}_{D-n}$ and that the canonical basis [L2, Ch.25] of ${}^1\dot{\mathbf{U}}$ is related in the simplest possible way with the canonical basis of $\mathbf{U}_D$. These conjectures were recently proved in [SV]. Combining the result of [SV] with the injectivity result mentioned above, we obtain a parametrization of the canonical basis of ${}^1\dot{\mathbf{U}}$ in combinatorial terms involving certain infinite matrices.

Entirely similar results hold for ordinary quantum $\mathfrak{sl}_n$.

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1. Transfer maps: the affine case

1.1. We consider the $\bar{\mathbf{Q}}_l$-algebra $\mathfrak{A}_{D,n,n;q}$ (see [L3, 1.10]) where q is the number of elements of a finite field $\mathbf{k}$. Assume that $D \ge n$. Before defining $\phi_{D,D-n}$, we shall define a homomorphism of $\bar{\mathbf{Q}}_l$-algebras

$$\phi_{D,D-n;q} : \mathfrak{A}_{D,n,n;q} \to \mathfrak{A}_{D-n,n,n;q}$$

as a composition

$$\mathfrak{A}_{D,n,n;q} \xrightarrow{\Delta} \mathfrak{A}_{D-n,n,n;q} \otimes \mathfrak{A}_{n,n,n;q} \xrightarrow{\xi\otimes\chi} \mathfrak{A}_{D-n,n,n;q} \otimes \bar{\mathbf{Q}}_l = \mathfrak{A}_{D-n,n,n;q}$$

where Δ is an algebra homomorphism which will be defined in 1.2 (see 1.5), (with $D' = D-n, D'' = n$), $\xi : \mathfrak{A}_{D-n,n,n;q} \to \mathfrak{A}_{D-n,n,n;q}$ is an algebra homomorphism which will be defined in 1.7 (for $D-n$ instead of D) and $\chi : \mathfrak{A}_{n,n,n;q} \to \bar{\mathbf{Q}}_l$ is an algebra homomorphism which will be defined in 1.8 (see 1.9).

1.2. The map Δ. In the ordinary (non-affine) case, I have found the definition of the analogue of $\phi_{D,D-n;q}$ around 1991, by interpreting $\mathfrak{A}_{D,n,n;q}$ as the commutant of Jimbo's Hecke algebra action on $(\mathbf{C}^n)^{\otimes D}$. In that language, Δ is an obvious map given by restricting invariants on $(\mathbf{C}^n)^{\otimes D}$ under a large Hecke algebra to invariants of a subalgebra (a product of two smaller Hecke algebras.) For the present purposes, it will be more convenient to use a geometric description of Δ, given in the non-affine case in [G], which we will now generalize to the affine case.

Let V be a free $\mathbf{k}[\epsilon,\epsilon^{-1}]$-module of rank D where ϵ is an indeterminate. We write $\mathcal{F}^{n,V}$ instead of $\mathcal{F}^n$ and G_V instead of G (see [L3, 1.3]). Assume that we are given a $\mathbf{k}[\epsilon,\epsilon^{-1}]$-submodule V'' of V such that V'' is a direct summand of V. Let $V' = V/V''$. Then V', V'' are free $\mathbf{k}[\epsilon,\epsilon^{-1}]$-modules of rank, say, D', D'' where $D'+D'' = D$. Let $G_{V'}, G_{V''}$ (resp. $\mathcal{F}^{n,V'}, \mathcal{F}^{n,V''}$) be defined as $\mathcal{F}^{n,V}$ (resp. $\mathcal{F}^{n,V}$) in terms of V', V'' instead of V.

We consider the $\bar{\mathbf{Q}}_l$-vector space of all functions

(a) $\mathcal{F}^{n,V'} \times \mathcal{F}^{n,V'} \times \mathcal{F}^{n,V''} \times \mathcal{F}^{n,V''} \to \bar{\mathbf{Q}}_l$

that are constant on the orbits of the obvious action of $G_{V'} \times G_{V''}$ and are zero outside a union of finitely many orbits. We identify this vector space with $\mathfrak{A}_{D',n,n;q} \otimes \mathfrak{A}_{D'',n,n;q}$ in such a way that $e_{A'} \otimes e_{A''}$ (see [L3, 1.10]) corresponds to the function which is identically 1 on $\mathcal{O}_{A'} \times \mathcal{O}_{A''}$ (see [L3, 1.6]) and is zero on all other $G_{V'} \times G_{V''}$ orbits. (Here $A' \in \mathfrak{S}_{D',n,n}$, $A'' \in \mathfrak{S}_{D'',n,n}$; see [L3, 1.2] or 3.3).

We define $\pi'' : \mathcal{F}^{n,V} \to \mathcal{F}^{n,V''}$ by $\pi''(\mathbf{L}) = \mathbf{L}''$ where $L''_i = L_i \cap V''$ for all $i \in \mathbf{Z}$. We define $\pi' : \mathcal{F}^{n,V} \to \mathcal{F}^{n,V'}$ by $\pi'(\mathbf{L}) = \mathbf{L}'$ where $L'_i = (L_i + V'')/V''$ for all $i \in \mathbf{Z}$.

We define a linear map $\Delta : \mathfrak{A}_{D,n,n;q} \to \mathfrak{A}_{D',n,n;q} \otimes \mathfrak{A}_{D'',n,n;q}$ as follows. Let $f \in \mathfrak{A}_{D,n,n;q}$, regarded as a function $\mathcal{F}^{n,V} \times \mathcal{F}^{n,V} \to \bar{\mathbf{Q}}_l$. Then

(b) $$\Delta(f)(\mathbf{L}', \tilde{\mathbf{L}'}, \mathbf{L}'', \tilde{\mathbf{L}''}) = \sum_{\tilde{\mathbf{L}}} f(\mathbf{L}, \tilde{\mathbf{L}})$$

where $\mathbf{L}$ is a fixed element of $\mathcal{F}^{n,V}$ such that $\pi'(\mathbf{L}) = \mathbf{L}'$, $\pi''(\mathbf{L}) = \mathbf{L}''$ and the sum is taken over all $\tilde{\mathbf{L}} \in \mathcal{F}^{n,V}$ such that $\pi'(\tilde{\mathbf{L}}) = \tilde{\mathbf{L}'}, \pi''(\tilde{\mathbf{L}}) = \tilde{\mathbf{L}''}$. (The sum has only finitely many non-zero terms.)

We must prove that Δ is independent of the choice of $\mathbf{L}$.

Let $\mathcal{U}$ be the subgroup of G_V consisting of the elements $g \in G_V$ such that $g(x) = x$ for all $x \in V''$ and $g(x) - x \in V''$ for all $x \in V$. The fact that the right hand side of (b) is independent of the choice of $\mathbf{L}$ follows from 1.4 below using the fact that f is constant on G_V-orbits (hence on $\mathcal{U}$-orbits) in $\mathcal{F}^{n,V} \times \mathcal{F}^{n,V}$.

Lemma 1.3. *Let $\mathbf{L} \in \mathcal{F}^{n,V}$. There exists a $\mathbf{k}[\epsilon, \epsilon^{-1}]$-submodule T of V such that $V = V'' \oplus T$ and $L_i = (L_i \cap V'') \oplus (L_i \cap T)$ for all $i \in \mathbf{Z}$.*

We omit the proof.

Lemma 1.4. *Let $\mathbf{L}' \in \mathcal{F}^{n,V'}, \mathbf{L}'' \in \mathcal{F}^{n,V''}$. We consider the $\mathcal{U}$-action on $Z = \{\mathbf{L} \in \mathcal{F}^{n,V} | \pi'(\mathbf{L}) = \mathbf{L}', \pi''(\mathbf{L}) = \mathbf{L}''\}$, given by the restriction of the G_V-action on $\mathcal{F}^{n,V}$. This $\mathcal{U}$-action is transitive.*

Let $\tilde{Z}$ be the set of all $\mathbf{k}[\epsilon, \epsilon^{-1}]$-submodules T of V such that $V = V'' \oplus T$. For $T \in \tilde{Z}$ we define $\mathbf{L}^T \in \mathcal{F}^{n,V}$ by $L_i^T = L_i'' \oplus \pi^{-1}(L_i')$ for $i \in \mathbf{Z}$, where $\pi : T \xrightarrow{\sim} V/V''$ is the obvious isomorphism. Note that $\mathbf{L}^T \in Z$. Thus we have defined a map $\tilde{Z} \to Z$. This map is surjective by 1.3 and is obviously compatible with the natural $\mathcal{U}$-actions on $\tilde{Z}, Z$. It remains to note that the $\mathcal{U}$-action on $\tilde{Z}$ is (simply) transitive.

Proposition 1.5. *Δ is an algebra homomorphism.*

Here $\mathfrak{A}_{D',n,n;q} \otimes \mathfrak{A}_{D'',n,n;q}$ is regarded as an algebra (tensor product of two algebras). This algebra structure corresponds to the convolution product on functions 1.2(a) analogous to the convolution product in [L3, 1.10].

Let $f_1, f_2 \in \mathfrak{A}_{D,n,n;q}$, regarded as a functions $\mathcal{F}^{n,V} \times \mathcal{F}^{n,V} \to \bar{\mathbf{Q}}_l$. Let
$(\mathbf{L}', \tilde{\mathbf{L}'}, \mathbf{L}'', \tilde{\mathbf{L}''}) \in \mathcal{F}^{n,V'} \times \mathcal{F}^{n,V'} \times \mathcal{F}^{n,V''} \times \mathcal{F}^{n,V''}$.
From the definitions we have

(a) $\Delta(f_1 \cdot f_2)(\mathbf{L}', \tilde{\mathbf{L}'}, \mathbf{L}'', \tilde{\mathbf{L}''}) = \sum_{\tilde{\mathbf{L}}, \hat{\mathbf{L}}} f_1(\mathbf{L}, \hat{\mathbf{L}}) f_2(\hat{\mathbf{L}}, \tilde{\mathbf{L}})$

where $\mathbf{L}$ is any element of $\mathcal{F}^{n,V}$ such that $\pi'(\mathbf{L}) = \mathbf{L}'$, $\pi''(\mathbf{L}) = \mathbf{L}''$ and the sum is taken over all $\tilde{\mathbf{L}}, \hat{\mathbf{L}} \in \mathcal{F}^{n,V}$ such that $\pi'(\tilde{\mathbf{L}}) = \tilde{\mathbf{L}'}, \pi''(\tilde{\mathbf{L}}) = \tilde{\mathbf{L}''}$. On the other hand,

$$
\begin{aligned}
&(\Delta(f_1) \cdot \Delta(f_2))(\mathbf{L}', \tilde{\mathbf{L}'}, \mathbf{L}'', \tilde{\mathbf{L}''}) \\
&= \sum_{\hat{\mathbf{L}'}, \hat{\mathbf{L}''}} \Delta(f_1)(\mathbf{L}', \hat{\mathbf{L}'}, \mathbf{L}'', \hat{\mathbf{L}''}) \Delta(f_2)(\hat{\mathbf{L}'}, \tilde{\mathbf{L}'}, \hat{\mathbf{L}''}, \tilde{\mathbf{L}''})
\end{aligned}
$$

where the sum is taken over all $\hat{\mathbf{L}'} \in \mathcal{F}^{n,V'}, \hat{\mathbf{L}''} \in \mathcal{F}^{n,V''}$.

Let $\mathbf{L} \in \mathcal{F}^{n,V}$ be as above. For each $\hat{\mathbf{L}'}, \hat{\mathbf{L}''}$ we pick $\hat{\mathbf{L}} \in \mathcal{F}^{n,V}$ such that $\pi'(\hat{\mathbf{L}}) = \hat{\mathbf{L}'}, \pi''(\hat{\mathbf{L}}) = \hat{\mathbf{L}''}$. Then the sum over $\hat{\mathbf{L}'}, \hat{\mathbf{L}''}$ can be replaced by a sum over

$\hat{\mathbf{L}}$ running through a set of representatives R for the $\mathcal{U}$-orbits on $\mathcal{F}^{n,V}$. (See 1.4.) We then have

$$(\Delta(f_1)\cdot\Delta(f_2))(\mathbf{L}',\tilde{\mathbf{L}}',\mathbf{L}'',\tilde{\mathbf{L}''}) = \sum_{\hat{\mathbf{L}},\dot{\mathbf{L}},\tilde{\mathbf{L}}} f_1(\mathbf{L},\dot{\mathbf{L}})f_2(\hat{\mathbf{L}},\tilde{\mathbf{L}})$$

where $\hat{\mathbf{L}}$ runs over R, and $\dot{\mathbf{L}},\tilde{\mathbf{L}}$ run over $\mathcal{F}^{n,V}$ subject to the requirement that $\dot{\mathbf{L}},\hat{\mathbf{L}}$ are in the same $\mathcal{U}$-orbit (see 1.4) and $\pi'(\tilde{\mathbf{L}}) = \tilde{\mathbf{L}}', \pi''(\tilde{\mathbf{L}}) = \tilde{\mathbf{L}''}$. The last sum is clearly equal to the right hand side of (a). The proposition is proved.

For $\mathbf{L}\in\mathcal{F}^{n,V}$ and $i\in\mathbf{Z}$, we write $|\mathbf{L}|_i = \dim L_i/L_{i-1}$.

Lemma 1.6. *We denote the elements of $\mathfrak{A}_{D',n,n;q}$ (resp. $\mathfrak{A}_{D'',n,n;q}$) analogous to $E_i, F_i, K_{\mathbf{b}}$ (see* [L3, 2.4, 2.2]*) by $E'_i, F'_i, K'_{\mathbf{b}}$ (resp. $E''_i, F''_i, K''_{\mathbf{b}}$). For any $i\in\mathbf{Z}, \mathbf{b}\in\tilde{\mathfrak{S}}_n$ (see* [L3, 2.2] *or 3.1) we have*

(a) $\Delta(E_i) = E'_i\otimes K''_{-\mathbf{c}} + K'_{\mathbf{c}}\otimes E''_i$,
(b) $\Delta(F_i) = F'_i\otimes K''_{\mathbf{d}} + K'_{-\mathbf{d}}\otimes F''_i$,
(c) $\Delta(K_{\mathbf{b}}) = K'_{\mathbf{b}}\otimes K''_{\mathbf{b}}$,

where $\mathbf{c}\in\tilde{\mathfrak{S}}_n$ (resp. $\mathbf{d}\in\tilde{\mathfrak{S}}_n$) has j-component equal to 1 if $j = i \mod n$ (resp. if $j = i+1 \mod n$) and the other components equal to 0.

We prove (a). We have

(d) $\Delta(E_i)(\mathbf{L}',\tilde{\mathbf{L}}',\mathbf{L}'',\tilde{\mathbf{L}''}) = q^{-|\tilde{\mathbf{L}}|_i/2}N$

where $\mathbf{L}$ is a fixed element of $\mathcal{F}^{n,V}$ such that $\pi'(\mathbf{L}) = \mathbf{L}'$, $\pi''(\mathbf{L}) = \mathbf{L}''$ and N is the number of $\tilde{\mathbf{L}}\in\mathcal{F}^{n,V}$ such that

(e) $\pi'(\tilde{\mathbf{L}}) = \tilde{\mathbf{L}}', \pi''(\tilde{\mathbf{L}}) = \tilde{\mathbf{L}''}$, $\tilde{L}_i\subset L_i, \dim(L_i/\tilde{L}_i) = 1$, $\tilde{L}_j = L_j$ for $j\ne i \mod n$.

Note that $N = 0$ unless we are in case (A) or (B) below.

(A) $\tilde{L}'_i\subset L'_i$, $\dim(L'_i/\tilde{L}'_i) = 1$; $\tilde{L}'_j = L'_j$ for $j\ne i \mod n$; $\tilde{L}''_j = L''_j$ for all j.

(B) $\tilde{L}''_i\subset L''_i$, $\dim(L''_i/\tilde{L}''_i) = 1$; $\tilde{L}''_j = L''_j$ for $j\ne i \mod n$; $\tilde{L}'_j = L'_j$ for all j.

In case (A), there is exactly one $\tilde{\mathbf{L}}$ satisfying (e), namely the one with

$\tilde{L}_i = \{x\in L_i; pr(x)\in\tilde{L}'_i\}$,

where $pr : L_i\to L'_i$ is the canonical map). Thus, $N = 1$ and the right hand side of (d) is

$$q^{-|\tilde{\mathbf{L}}|_i/2+|\tilde{\mathbf{L}}'|_i/2}(E'_i\otimes 1)(\mathbf{L}',\tilde{\mathbf{L}}',\mathbf{L}'',\tilde{\mathbf{L}''}) = q^{-|\tilde{\mathbf{L}}''|_i/2}(E'_i\otimes 1)(\mathbf{L}',\tilde{\mathbf{L}}',\mathbf{L}'',\tilde{\mathbf{L}''}).$$

In case (B), the set of $\tilde{\mathbf{L}}$ in (e) is in bijection with the set of hyperplanes in the vector space $L_i/(L_{i-1}+\tilde{L}''_i)$ that are complementary to a line (the image of L''_i). Now

$$\dim(L_i/(L_{i-1}+\tilde{L}''_i)) = \dim(L_i/(L_{i-1}+L''_i))+1 = \dim(L_i/L_{i-1})-\dim(L''_i/L''_{i-1})+1.$$

Thus, $N = q^{|\mathbf{L}|_i-|\mathbf{L}''|_i} = q^{|\mathbf{L}'|_i}$ and the right hand side of (d) is

$$q^{-|\tilde{\mathbf{L}}|_i/2+|\tilde{\mathbf{L}}''|_i/2+|\mathbf{L}'|_i}(1\otimes E''_i)(\mathbf{L}',\tilde{\mathbf{L}}',\mathbf{L}'',\tilde{\mathbf{L}''}) = q^{-|\mathbf{L}'|_i/2}(1\otimes E''_i)(\mathbf{L}',\tilde{\mathbf{L}}',\mathbf{L}'',\tilde{\mathbf{L}''});$$

(a) follows.

We prove (b). We have

(f) $\Delta(F_i)(\mathbf{L}',\tilde{\mathbf{L}'},\mathbf{L}'',\tilde{\mathbf{L}''}) = q^{-|\tilde{\mathbf{L}}|_{i+1}/2}N$

where $\mathbf{L}$ is a fixed element of $\mathcal{F}^{n,V}$ such that $\pi'(\mathbf{L}) = \mathbf{L}'$, $\pi''(\mathbf{L}) = \mathbf{L}''$ and N is the number of $\tilde{\mathbf{L}} \in \mathcal{F}^{n,V}$ such that

(g) $\pi'(\tilde{\mathbf{L}}) = \tilde{\mathbf{L}'}, \pi''(\tilde{\mathbf{L}}) = \tilde{\mathbf{L}''}$, $L_i \subset \tilde{L}_i, \dim(\tilde{L}_i/L_i) = 1$, $\tilde{L}_j = L_j$ for $j \neq i \mod n$.

Note that $N = 0$ unless we are in case (C) or (D) below.

(C) $L'_i \subset \tilde{L}'_i$, $\dim(\tilde{L}'_i/L'_i) = 1$; $\tilde{L}'_j = L'_j$ for $j \neq i \mod n$; $\tilde{L}''_j = L''_j$ for all j.

(D) $L''_i \subset \tilde{L}''_i$, $\dim(\tilde{L}''_i/L''_i) = 1$; $\tilde{L}''_j = L''_j$ for $j \neq i \mod n$; $\tilde{L}'_j = L'_j$ for all j.

In case (C), the set of $\tilde{\mathbf{L}}$ in (g) is in bijection with the set of lines in the vector space $pr^{-1}(\tilde{L}'_i)/L_i)$ that are not contained in a hyperplane, namely $pr^{-1}(L'_i)$; here $pr : L_{i+1} \to L'_{i+1}$ is the canonical map. Now

$$\dim(pr^{-1}(\tilde{L}'_i)/L_i) = \dim(L_{i+1} \cap V'')/L_i \cap V'') + \dim(\tilde{L}'_i/L'_i) = |\mathbf{L}''|_{i+1} + 1.$$

Thus, $N = q^{|\mathbf{L}''|_{i+1}}$ and the right hand side of (f) is

$$\begin{aligned} &q^{-|\tilde{\mathbf{L}}|_{i+1}/2+|\tilde{\mathbf{L}}'|_{i+1}/2+|\mathbf{L}''|_{i+1}}(F'_i \otimes 1)(\mathbf{L}',\tilde{\mathbf{L}'},\mathbf{L}'',\tilde{\mathbf{L}''}) \\ &= q^{|\mathbf{L}''|_{i+1}/2}(F'_i \otimes 1)(\mathbf{L}',\tilde{\mathbf{L}'},\mathbf{L}'',\tilde{\mathbf{L}''}). \end{aligned}$$

In case (D), there is exactly one $\tilde{\mathbf{L}}$ satisfying (g), namely the one with $\tilde{L}_i = L_i + \tilde{L}''_i$. Thus, $N = 1$ and the right hand side of (f) is

$$q^{-|\tilde{\mathbf{L}}|_{i+1}/2+|\tilde{\mathbf{L}}''|_{i+1}/2}(1 \otimes F''_i)(\mathbf{L}',\tilde{\mathbf{L}'},\mathbf{L}'',\tilde{\mathbf{L}''}) = q^{-|\tilde{\mathbf{L}'}|_{i+1}/2}(1 \otimes F''_i)(\mathbf{L}',\tilde{\mathbf{L}'},\mathbf{L}'',\tilde{\mathbf{L}''}).$$

(b) follows. (c) is immediate. The lemma is proved.

1.7. The map ξ. If L, L' are lattices in V, we define

$(L : L') = \dim(L/\tilde{L}) - \dim(L'/\tilde{L}) \in \mathbf{Z}$

where $\tilde{L}$ is any lattice contained in $L \cap L'$. Clearly, $(L : L')$ is independent of the choice of $\tilde{L}$. Moreover, if L, L', L'' are lattices in V, we have

$(L : L') + (L' : L'') + (L'' : L) = 0.$

For $\mathbf{L}, \mathbf{L}'$ in $\mathcal{F}^{n,V}$ we set

$(\mathbf{L} : \mathbf{L}') = \sum_i (L_i : L'_i)$

where i runs over a set of representatives for the residue classes $\mod n$ in $\mathbf{Z}$. Note that, for $\mathbf{L}, \mathbf{L}', \mathbf{L}''$ in $\mathcal{F}^{n,V}$, we have

(a) $(\mathbf{L} : \mathbf{L}') + (\mathbf{L}' : \mathbf{L}'') + (\mathbf{L}'' : \mathbf{L}) = 0.$

We define a $\bar{\mathbf{Q}}_l$-linear isomorphism $\xi : \mathfrak{A}_{D,n,n;q} \to \mathfrak{A}_{D,n,n;q}$ by

$(\xi(f))(\mathbf{L},\mathbf{L}') = q^{-(\mathbf{L}:\mathbf{L}')/2} f(\mathbf{L},\mathbf{L}')$.

Using (a) and the definitions, we see that ξ is an algebra isomorphism.

It is easy to see that, if $(\mathbf{L},\mathbf{L}') \in \mathcal{O}_A$, then

$(\mathbf{L}:\mathbf{L}')=t(A)$
where
$t(A)=\sum a_{i,j}(j-i)$,
sum over a set of representatives for the orbits of the permutation
$(i,j)\mapsto(i-n,j-n)$
of $\mathbf{Z}\times\mathbf{Z}$. (Only finitely many terms of the sum are non-zero.)

1.8. The map χ. Let $\mathbf{b}_0=(\dots,1,1,1,\dots)\in\mathfrak{S}_{n,n}$. (see [L3, 1.1]).

For any $A\in\mathfrak{S}_{n,n,n}$ we define $\operatorname{sgn}_A\in\bar{\mathbf{Q}}_l$ as follows. If $r(A)\ne\mathbf{b}_0$ or if $c(A)\ne\mathbf{b}_0$ (notation of [L3, 1.2] or 3.3), we set $\operatorname{sgn}_A=0$. Assume now that $r(A)=c(A)=\mathbf{b}_0$. Then there is a unique permutation $w:\mathbf{Z}\to\mathbf{Z}$ such that $a_{i,j}=\delta_{w(j),i}$. (We have $w(j-n)=w(j)-n$ for all j.) If the set of all (i,j) in $\mathbf{Z}\times\mathbf{Z}$ (up to simultaneous translation by the same multiple of n) such that $i<j$, $w(i)>w(j)$, has an even number of elements (this set is necessarily finite), we set $\operatorname{sgn}(w)=1$. If it has an odd number of elements, we set $\operatorname{sgn}(w)=-1$. We set $\operatorname{sgn}_A=\operatorname{sgn}(w)$.

We define a linear map $\chi:\mathfrak{A}_{n,n,n;q}\to\bar{\mathbf{Q}}_l$ by $\chi(e_A)=\operatorname{sgn}_A$ for all $A\in\mathfrak{S}_{n,n,n}$. Note that $\chi(E_i)=\chi(F_i)=0$ for all i and $\chi(K_{\mathbf{b}})=q^{\mathbf{b}\cdot\mathbf{b}_0/2}$ for all $\mathbf{b}$.

Lemma 1.9. *χ is an algebra homomorphism.*

Let ${}'\mathfrak{A}_{n,n,n;q}$ be the subspace of $\mathfrak{A}_{n,n,n;q}$ spanned by the elements e_A with $r(A)=c(A)=\mathbf{b}_0$. Note that ${}'\mathfrak{A}_{n,n,n;q}$ is a subalgebra of $\mathfrak{A}_{n,n,n;q}$. In fact, it is an affine Hecke algebra. The fact that the restriction of χ to ${}'\mathfrak{A}_{n,n,n;q}$ is an algebra homomorphism is a well known property of the affine Hecke algebra. Let ${}''\mathfrak{A}_{n,n,n;q}$ be the kernel of $\chi:{}'\mathfrak{A}_{n,n,n;q}\to\bar{\mathbf{Q}}_l$ (a two-sided ideal of ${}'\mathfrak{A}_{n,n,n;q}$).

To prove the lemma it is enough to verify the following statement: if A,A' in $\mathfrak{S}_{n,n,n}$ are such that $r(A)=c(A')=\mathbf{b}_0$ and $c(A)=r(A')\ne\mathbf{b}_0$, then the product $e_A\cdot e_{A'}\in{}'\mathfrak{A}_{n,n,n;q}$ belongs to ${}''\mathfrak{A}_{n,n,n;q}$.

We define $f\in\mathfrak{A}_{n,n,n;q}$ as follows. If $|\mathbf{L}|=|\mathbf{L}'|=\mathbf{b}_0$ (notation of [L3, 1.3], then $f(\mathbf{L},\mathbf{L}')=e_A(\mathbf{L},\bar{\mathbf{L}}')$ where $\bar{\mathbf{L}}'$ is obtained from $\mathbf{L}'$ by omitting some lattices in $\mathbf{L}'$ so that $|\bar{\mathbf{L}}'|=c(A)$; otherwise, $f(\mathbf{L},\mathbf{L}')=0$.

We define $f'\in\mathfrak{A}_{n,n,n;q}$ as follows. If $|\mathbf{L}|=|\mathbf{L}'|=\mathbf{b}_0$, then $f'(\mathbf{L},\mathbf{L}')=e_{A'}(\bar{\mathbf{L}},\mathbf{L}')$ where $\bar{\mathbf{L}}$ is obtained from $\mathbf{L}$ by omitting some lattices in $\mathbf{L}$ so that $|\bar{\mathbf{L}}|=r(A')$; otherwise, $f'(\mathbf{L},\mathbf{L}')=0$.

From the definition, we have $e_A\cdot e_{A'}=N^{-1}f\cdot f'$ for some integer $N>1$. Since $c(A)\ne\mathbf{b}_0$, we have $f\in{}''\mathfrak{A}_{n,n,n;q}$. Since ${}''\mathfrak{A}_{n,n,n;q}$ is a two-sided ideal of ${}'\mathfrak{A}_{n,n,n;q}$ and $f'\in{}'\mathfrak{A}_{n,n,n;q}$, we have $f\cdot f'\in{}''\mathfrak{A}_{n,n,n;q}$. Hence $e_A\cdot e_{A'}\in{}''\mathfrak{A}_{n,n,n;q}$. The lemma is proved.

Lemma 1.10. *Let $i\in\mathbf{Z},\mathbf{b}\in\tilde{\mathfrak{S}}_n$. The transfer homomorphism $\phi_{D,D-n;q}:\mathfrak{A}_{D,n,n;q}\to\mathfrak{A}_{D-n,n,n;q}$ (see 1.1) carries E_i to E'_i, F_i to F'_i and $K_{\mathbf{b}}$ to $q^{\mathbf{b}\cdot\mathbf{b}_0/2}K'_{\mathbf{b}}$. (Notation of* [L3, 2.2]*.)*

By 1.6, we have
$\phi_{D,D-n;q}(E_i)=\chi(K''_{-\mathbf{c}})\xi(E'_i)+\xi(K'_{\mathbf{c}})\chi(E''_i)=q^{-1/2}q^{1/2}E'_i=E'_i$,
$\phi_{D,D-n;q}(F_i)=\chi(K''_{\mathbf{d}})\xi(F'_i)+\xi(K'_{-\mathbf{d}})\chi(F''_i)=q^{1/2}q^{-1/2}F'_i=F'_i$,

$\phi_{D,D-n;q}(K_{\mathbf{b}}) = \xi(K'_{\mathbf{b}})\chi(K''_{\mathbf{b}}) = q^{\mathbf{b}\cdot\mathbf{b}_0/2}K'_{\mathbf{b}}$.
The lemma is proved.

1.11. For $A \in \mathfrak{S}_{D,n,n}$ we have

$$\phi_{D,D-n;q}(q^{-d_A/2}e_A) = \sum_{A'} \sigma_{A,A';q}q^{-d_{A'}/2}e_{A'}$$

where A' runs over $\mathfrak{S}_{D-n,n,n}$, $d_A, d_{A'}$ are as in [L3, 1.2] and $\sigma_{A,A';q} \in \bar{\mathbf{Q}}_l$ are zero for all but finitely many A'. It is easy to see that $\sigma_{A,A';q}$ is a polynomial with integer coefficients in $q^{\pm 1/2}$. (An analogous statement holds for the matrix entries of the maps Δ, ξ, χ from which $\phi_{D,D-n;q}$ is built.) Hence there are unique polynomials $\sigma_{A,A'}$ with integer coefficients in $v^{\pm 1}$ which specialize to $\sigma_{A,A';q}$ for $v = q^{1/2}$ where q is any prime power. We define a $\mathbf{Q}(v)$-linear map $\phi_{D,D-n}$: $\mathfrak{A}_{D,n,n} \to \mathfrak{A}_{D-n,n,n}$ (notation of [L3, 1.9] or 3.4) by

$$\phi_{D,D-n}([A]) = \sum_{A'} \sigma_{A,A'}[A']$$

for any $A \in \mathfrak{S}_{D,n,n}$. Here $[A] = v^{-d_A}e_A$.

The fact that $\phi_{D,D-n;q}$ is an algebra homomorphism for any prime power q implies that $\phi_{D,D-n}$ is an algebra homomorphism. This homomorphism has a property analogous to that of $\phi_{D,D-n;q}$ given in 1.10. Since $\mathbf{U}_D$ is by definition the subalgebra of $\mathfrak{A}_{D,n,n}$ generated by the elements $E_i, F_i, K_{\mathbf{b}}$ (as in 1.10), it follows that $\phi_{D,D-n}$ restricts to an algebra homomorphism $\mathbf{U}_D \to \mathbf{U}_{D-n}$ which will be denoted again by $\phi_{D,D-n}$. Thus, the assertion in [L3, 9.1] is verified.

Note that $\phi_{D,D-n}$ carries $[\mathbf{i}_{\mathbf{a}}] \in \mathbf{U}_D$ (see [L3, 1.9]) to $[\mathbf{i}_{\mathbf{a}-\mathbf{b}_0}] \in \mathbf{U}_{D-n}$. This follows from the equality $K_{\mathbf{b}} = \sum_{\mathbf{a}} v^{\mathbf{a}\cdot\mathbf{b}}[\mathbf{i}_{\mathbf{a}}]$, see [L3, 2.1].

1.12. In the special case where $D = n$, the transfer homomorphism $\phi_{n,0;q}$: $\mathfrak{A}_{n,n,n;q} \to \mathfrak{A}_{0,n,n;q} = \bar{\mathbf{Q}}_l$ is equal to χ and conjecture [L3, 9.2] can be easily verified. Note however that, if in [L3, 9.2] we drop the condition that A is aperiodic, the resulting statement would be false. Indeed, take for example A given by $a_{ij} = 1$ for $j = i+1$ and $a_{ij} = 0$ for $j \neq i+1$. (This A is not aperiodic.) Then $d_A = 0$ and $\chi(e_A) = 1$ hence $\chi(q^{-d_A/2}e_A) = 1$, in spite of the fact that $A \neq I$.

2. Transfer maps: non-affine case

2.1. All constructions in Section 1 can be done also in the (easier) non-affine case considered in [BLM]. The results in the non-affine case are proved in the same way as in the affine case; they can also be deduced from the results in the affine case.

We assume that $\mathbf{k}$ is a finite field with q elements. Let V be a $\mathbf{k}$-vector space of dimension D. Let $\mathfrak{T}_{D,n,n}$ be the set of all matrices $A = (a_{ij})_{i,j\in[1,n]}$ with entries in $\mathbf{N}$ such that $\sum_{i,j\in[1,n]} a_{ij} = D$. Let $P = P^V$ be the space of all n-step filtrations in V. Given

$\mathbf{L} = (0 = L_0 \subset L_1 \subset \dots \subset L_n = V)$, $\mathbf{L}' = (0 = L'_0 \subset L'_1 \subset \dots \subset L'_n = V)$
in P^V, we define $A \in \mathfrak{T}_{D,n,n}$ by

$$a_{ij} = \dim(L_i \cap L'_j)/(L_{i-1} \cap L'_j + L_i \cap L'_{j-1}).$$

We then write $(\mathbf{L}, \mathbf{L}') \in \mathcal{O}_A$.

Let $\mathfrak{B}_{D,n,n;q}$ be the $\bar{\mathbf{Q}}_l$-vector space of all functions $P^V \times P^V \to \bar{\mathbf{Q}}_l$ which are constant on the orbits of the natural $GL(V)$-action. The orbits are the sets $\mathcal{O}_A$ above; they are indexed by the elements $A \in \mathfrak{T}_{D,n,n}$. The corresponding characteristic functions are denoted by e_A. For $A \in \mathfrak{T}_{D,n,n}$, let

$$d_A = \dim(X_A) = \sum_{i \geq k; j < l} a_{ij} a_{kl}.$$

We set $[A] = q^{-d_A/2} e_A$.

2.2. Assume that we are given a subspace V'' of V. Let $V' = V/V''$. Let $p : V \to V'$ be the projection. Let $D' = \dim V', D'' = \dim V''$. We define, as in [G], a linear map
$\Delta : \mathfrak{B}_{D,n,n;q} \to \mathfrak{B}_{D',n,n;q} \otimes \mathfrak{B}_{D'',n,n;q}$
by
$\Delta(e_A) = \sum_{A',A''} n_{A,A',A''} e_{A'} \otimes e_{A''}$
for $A \in \mathfrak{T}_{D,n,n}$, where $n_{A,A',A''}$ is defined as follows.

Pick $(\mathbf{L}', \tilde{\mathbf{L}'}) \in \mathcal{O}_{A'}, (\mathbf{L}'', \tilde{\mathbf{L}''}) \in \mathcal{O}_{A''}$ and pick $\mathbf{L} \in P^V$ such that $L_i \cap V'' = L''_i, p(L_i) = L'_i$ for all i. Then $n_{A,A',A''}$ is the number of elements of

$$\{\tilde{\mathbf{L}} | (\mathbf{L}, \tilde{\mathbf{L}}) \in \mathcal{O}_A, \tilde{L}_i \cap V'' = \tilde{L}''_i, p(\tilde{L}_i) = \tilde{L}'_i\}.$$

As in 1.5, we see that Δ is an algebra homomorphism for the natural algebra structures given by convolution. Moreover, formulas like the ones in 1.6 hold.

2.3. We define a $\bar{\mathbf{Q}}_l$-linear isomorphism $\xi : \mathfrak{B}_{D,n,n;q} \to \mathfrak{B}_{D,n,n;q}$ by
$(\xi(f))(\mathbf{L}, \mathbf{L}') = q^{-t'(A)/2} f(\mathbf{L}, \mathbf{L}')$.
for $(\mathbf{L}, \mathbf{L}') \in \mathcal{O}_A$, where
$t'(A) = \sum_{i,j} a_{i,j}(j - i)$,
As in 1.7, we see that ξ is an algebra isomorphism.

2.4. For any $A \in \mathfrak{T}_{n,n,n}$ we set $\mathrm{sgn}_A = \det(A)$. Thus, $\mathrm{sgn}(A) = 0$ if A is not a permutation matrix and $\mathrm{sgn}(A) = \pm 1$ if A is a permutation matrix. We define a linear map $\chi : \mathfrak{B}_{n,n,n;q} \to \bar{\mathbf{Q}}_l$ by $\chi(e_A) = \mathrm{sgn}_A$ for all $A \in \mathfrak{T}_{n,n,n}$. As in 1.9, we see that χ is an algebra homomorphism.

2.5. Assuming that $D \geq n$, we define as in 1.1 a homomorphism of $\bar{\mathbf{Q}}_l$-algebras

$$\phi_{D,D-n;q} : \mathfrak{B}_{D,n,n;q} \to \mathfrak{B}_{D-n,n,n;q}$$

as a composition

$$\mathfrak{B}_{D,n,n;q} \xrightarrow{\Delta} \mathfrak{B}_{D-n,n,n;q} \otimes \mathfrak{B}_{n,n,n;q} \xrightarrow{\xi\otimes\chi} \mathfrak{B}_{D-n,n,n;q} \otimes \bar{\mathbf{Q}}_l = \mathfrak{B}_{D-n,n,n;q}$$

where Δ is the algebra homomorphism defined in 2.2 (with $D' = D-n, D'' = n$), $\xi : \mathfrak{B}_{D-n,n,n;q} \to \mathfrak{B}_{D-n,n,n;q}$ is the algebra homomorphism defined in 2.3 (for $D-n$ instead of D) and $\chi : \mathfrak{B}_{n,n,n;q} \to \bar{\mathbf{Q}}_l$ is the algebra homomorphism defined in 2.4. This is called the transfer homomorphism. It has a property like that in 1.10.

2.6. For $A \in \mathfrak{T}_{D,n,n}$ we have

$$\phi_{D,D-n;q}(q^{-d_A/2}e_A) = \sum_{A'} \sigma_{A,A';q} q^{-d_{A'}/2} e_{A'}$$

where A' runs over $\mathfrak{T}_{D-n,n,n}$ and $d_A, d_{A'}$ are as in 2.1. It is easy to see that $\sigma_{A,A';q}$ is a polynomial with integer coefficients in $q^{\pm 1/2}$.

2.7. We define $\tilde{\sigma}_{A,A';q}$ by

$$\sigma_{A,A';q} = q^{t'(A')/2 - d_A/2 + d_{A'}/2} \tilde{\sigma}_{A,A';q}.$$

From the definitions, we have

$$\tilde{\sigma}_{A,A';q} = \sum_w \mathrm{sgn}(w) n_{A,A',I_w}$$

where w runs over the permutations of $[1,n]$, I_w is the corresponding permutation matrix and n_{A,A',I_w} is computed as follows.

Recall that $V'' \subset V$ has dimension n and $p : V \to V' = V/V''$ is the projection. We fix $(\mathbf{L}', \tilde{\mathbf{L}}') \in \mathcal{O}_{A'}$ and a direct sum decomposition $V'' = \oplus_{i\in[1,n]} D_i$ where D_i are lines. For each w let $\mathbf{D}^w$ be the n-step filtration of V'' given by
$D^w_i = D_{w(1)} + D_{w(2)} + \cdots + D_{w(i)}$.
We pick $\mathbf{L} \in P^V$ such that $L_i \cap V'' = D^1_i, p(L_i) = L'_i$. We have

$$n_{A,A',I_w} = \sharp\{\tilde{\mathbf{L}} | (\mathbf{L}, \tilde{\mathbf{L}}) \in \mathcal{O}_A, \tilde{L}_i \cap V'' = D^w_i, p(\tilde{L}_i) = \tilde{L}'_i\}.$$

2.8. In the case where $n = 2$, we have

$$\phi_{D,D-2;q}(q^{-d_A/2}e_A) = q^{-d_{A-I}/2}e_{A-I} - q^{-(a_{11}+a_{22}+1)/2} q^{-d_{A-J}/2} e_{A-J}$$

where one omits the terms which are not defined (that is, such that $A-I$ or $A-J$ does not have entries in $\mathbf{N}$). From this one can easily deduce that the analogue of conjecture [L3, 9.2] holds in this case.

3. An injectivity result

3.1. Let $\tilde{\mathfrak{S}}_n$ be the set of all $\mathbf{a} = (a_i)_{i \in \mathbf{Z}}$ with $a_i \in \mathbf{Z}$ such that $a_i = a_{i-n}$ for all $i \in \mathbf{Z}$. We regard $\tilde{\mathfrak{S}}_n$ as an abelian group with addition component by component. For $\mathbf{a} \in \tilde{\mathfrak{S}}_n$, the sum $a_{i_0} + a_{i_0+1} + \dots + a_{i_0-n+1}$ is independent of $i_0 \in \mathbf{Z}$; we denote it by $\nabla_{\mathbf{a}}$. Let $Y = \{\mathbf{a} \in \tilde{\mathfrak{S}}_n | \nabla_{\mathbf{a}} = 0\}$. Let X be the quotient of $\tilde{\mathfrak{S}}_n$ by the subgroup generated by the element $\mathbf{b}_0$ with all components equal to 1. For $\mathbf{a}, \mathbf{b} \in \tilde{\mathfrak{S}}_n$ we set $\mathbf{a} \cdot \mathbf{b} = \sum a_i b_i \in \mathbf{Z}$ where the sum is taken over a set of representatives for the residue classes modulo n in $\mathbf{Z}$. This induces a non-singular pairing

$Y \times X \to \mathbf{Z}$.

Consider the root datum (cf. [L2, 2.2]) consisting of the free abelian groups Y, X above, with the pairing above and with maps $I = \mathbf{Z}/n\mathbf{Z} \to Y, I = \mathbf{Z}/n\mathbf{Z} \to X$ which takes i to the element $\mathbf{a} \in \tilde{\mathfrak{S}}_n$ given by $a_i = 1, a_{i+1} = -1, a_j = 0$ for $j \ne i, i+1 \mod n$ or to its image in X. This is a root datum of affine type $\tilde{A}_{n-1}$.

Let ${}^1\mathbf{U}$ be the (Drinfeld-Jimbo) quantized enveloping algebra associated to this root datum as in [L2, Ch. 3]. This is an algebra over $\mathbf{Q}(v)$ where v is an indeterminate. Let ${}^1\dot{\mathbf{U}}$ be the corresponding modified algebra and let ${}^1\dot{\mathbf{B}}$ be the canonical basis of ${}^1\dot{\mathbf{U}}$ (see [L2, Ch. 23. Ch. 25]). For any $\lambda \in X$, let $1_\lambda \in {}^1\dot{\mathbf{U}}$ be the idempotent defined in [L2, 23.1.1]. By [L2, 23.1.3], ${}^1\dot{\mathbf{U}}$ is naturally an ${}^1\mathbf{U}$-bimodule.

Let $\mathbf{f}$ be the $\mathbf{Q}(v)$-algebra attached to the root datum above (or rather to the associated Cartan datum) in [L2, Ch. 3]. Let $x \to x^+, x \to x^-$ be the algebra homomorphisms $\mathbf{f} \to {}^1\mathbf{U}$ defined in [L2, 3.1.1]. Let $x \mapsto \tilde{x}$ be the canonical antiautomorphism [L2, 1.2.9] of $\mathbf{f}$. By [L2, 23.2.1],

(a) the elements $x^+ 1_\lambda x'^-$ (for various $x, x' \in \mathbf{f}$, $\lambda \in X$), generate ${}^1\dot{\mathbf{U}}$ as a $\mathbf{Q}(v)$-vector space.

3.2. Let $D \in \mathbf{N}$. Let $\mathfrak{S}_{D,n}$ be the (finite) subset of $\tilde{\mathfrak{S}}_n$ consisting of the $\mathbf{a}$ such that $a_i \ge 0$ for all i and $\nabla_{\mathbf{a}} = D$.

3.3. Let $\mathcal{S}_n$ be the set of all matrices $A = (a_{i,j})_{(i,j) \in \mathbf{Z} \times \mathbf{Z}}$ with entries $a_{i,j} \in \mathbf{Z}$ such that

(a) $a_{i,j} \ge 0$ for all $i \ne j$;

(b) $a_{i,j} = a_{i-n,j-n}$ for all $i, j \in \mathbf{Z}$;

(c1) for any $i \in \mathbf{Z}$, the set $\{j \in \mathbf{Z} | a_{i,j} \ne 0\}$ is finite;

(c2) for any $j \in \mathbf{Z}$, the set $\{i \in \mathbf{Z} | a_{i,j} \ne 0\}$ is finite.

For $A \in \mathcal{S}_n$ we set

$a_{i,*} = \sum_{j \in \mathbf{Z}} a_{ij}, \quad a_{*,j} = \sum_{i \in \mathbf{Z}} a_{ij}$.

The sum $a_{i_0,*} + a_{i_0+1,*} + \dots + a_{i_0-n+1,*}$ is independent of i_0 and is equal to $a_{*,j_0} + a_{*,j_0+1} + \dots + a_{*,j_0-n+1}$ (which is independent of j_0); we denote it by ∇_A.

Let $\mathfrak{S}_{D,n,n}$ be the set of all $A \in \mathcal{S}_n$ such that $a_{i,i} \ge 0$ for all $i \in \mathbf{Z}$ and $\nabla_A = D$.

For any $\mathbf{a} \in \mathfrak{S}_{D,n}$ let $A = \mathbf{i_a} \in \mathfrak{S}_{D,n,n}$ be defined by $a_{i,j} = a_i \delta_{i,j}$.

For $A \in \mathfrak{S}_{D,n,n}$ we set

$r(A) = (a_{i,*})_{i \in \mathbf{Z}} \in \mathfrak{S}_{D,n}, c(A) = (a_{*,j})_{j \in \mathbf{Z}} \in \mathfrak{S}_{D,n}$.

3.4. Let $\mathfrak{A}_{D,n,n}$ be the $\mathbf{Q}$-vector space spanned by the elements $[B]$ indexed by $B \in \mathfrak{S}_{D,n,n}$. (See [L3, 1.9].) It has a second basis $\{B\}$ again indexed by $B \in \mathfrak{S}_{D,n,n}$. (See [L3, 4.1].) We will regard $\mathfrak{A}_{D,n,n}$ as a $\mathbf{Q}(v)$-algebra as in [L3]. The multiplication can be interpreted geometrically as convolution of certain "perverse sheaves" on a space of pairs of periodic chains of lattices in a free module of rank D over the ring of Laurent polynomials over $\mathbf{C}$. Let $\Psi : \mathfrak{A}_{D,n,n} \to \mathfrak{A}_{D,n,n}$ be the antiautomorphism defined in [L3, 1.11, 4.12].

Let $\mathbf{U} \to \mathfrak{A}_{D,n,n}$ be the algebra homomorphism defined in [L3, 7.7]. Here $\mathbf{U}$ contains naturally ${}^1\mathbf{U}$ as a subalgebra. Restricting the homomorphism above to ${}^1\mathbf{U}$ gives us an algebra homomorphism $\rho'_D : {}^1\mathbf{U} \to \mathfrak{A}_{D,n,n}$.

There is a unique algebra homomorphism $\rho_D : {}^1\dot{\mathbf{U}} \to \mathfrak{A}_{D,n,n}$ such that

$\rho_D(u_1 u_2 u_3) = \rho'_D(u_1)\rho_D(u_2)\rho'_D(u_3)$

for $u_1, u_3 \in {}^1\mathbf{U}, u_2 \in {}^1\dot{\mathbf{U}}$ and $\rho_D(1_\lambda) = \{\mathbf{i_a}\} = [\mathbf{i_a}]$ if $\lambda \in X$ can be represented by $\mathbf{a} \in \mathfrak{S}_{D,n}$ (necessarily unique), $\rho_D(1_\lambda) = 0$ if $\lambda \in X$ cannot be represented by an element in $\mathfrak{S}_{D,n}$.

Let $\phi_{D+n,D} : \mathfrak{A}_{D+n,n,n} \to \mathfrak{A}_{D,n,n}$ be the algebra homomorphism defined in 1.11. From the definitions we have

(a) $\phi_{D+n,D}\rho_{D+n} = \rho_D$ for all $D \geq 0$.

This equality would not be true if we used $\mathbf{U}$ instead of ${}^1\mathbf{U}$, due to the exponent $\mathbf{b} \cdot \mathbf{b}_0$ in 1.10. In the case of ${}^1\mathbf{U}$, we have $\mathbf{b} \cdot \mathbf{b}_0 = 0$ since $\mathbf{b} \in Y$ and this makes (a) hold.

Let $\hat{\mathbf{U}} = \varprojlim_D \mathfrak{A}_{D+n,n,n}$ where the limit is taken over the projective system given by the maps $\phi_{D+n,D}$ for $D = 0, 1, 2, \dots$. Then (a) shows that there is a unique algebra homomorphism

(b) $\rho : {}^1\dot{\mathbf{U}} \to \hat{\mathbf{U}}$

such that for any D, ρ_D is the composition of ρ with the canonical map $\hat{\mathbf{U}} \to \mathfrak{A}_{D,n,n}$. We can now state the following result.

Proposition 3.5. *The homomorphism ρ is injective.*

The proof will be given in 3.13.

3.6. Let $\mathfrak{S}^-_{D,n,n}$ be the set of all $A \in \mathfrak{S}_{D,n,n}$ such that $a_{i,j} = 0$ for all $i > j$. Let $\mathfrak{S}^+_{D,n,n}$ be the set of all $A \in \mathfrak{S}_{D,n,n}$ such that $a_{i,j} = 0$ for all $i < j$.

Given $\mathbf{a} \in \mathfrak{S}_{D,n}$, let $\mathfrak{S}^-_{D,n,n;\mathbf{a}}$ be the set of all $B \in \mathfrak{S}^-_{D,n,n}$ such that $r(B) = \mathbf{a}$; let $\mathfrak{S}^+_{D,n,n;\mathbf{a}}$ be the set of all $B \in \mathfrak{S}^+_{D,n,n}$ such that $c(B) = \mathbf{a}$. Let $\mathfrak{A}^\pm_{D,n,n;\mathbf{a}}$ be the subspace of $\mathfrak{A}^\pm_{D,n,n}$ spanned by $\{\{B\} | B \in \mathfrak{S}^\pm_{D,n,n;\mathbf{a}}\}$.

Given $\mathbf{a} \in \mathfrak{S}_{D,n}$ and $s \in \mathbf{N}$, let $\mathfrak{S}^-_{D,n,n;\mathbf{a};s}$ (resp. $\mathfrak{S}^+_{D,n,n;\mathbf{a};s}$) be the set of all $B \in \mathfrak{S}^-_{D,n,n;\mathbf{a}}$ (resp. $B \in \mathfrak{S}^+_{D,n,n;\mathbf{a}}$) such that

$\sum_{j;i<j} b_{ij} \leq s$ (resp. $\sum_{k;k>i} b_{ki} \leq s$) for all i.

Let $\mathfrak{A}^\pm_{D,n,n;\mathbf{a};s}$ be the subspace of $\mathfrak{A}^\pm_{D,n,n}$ spanned by $\{\{B\} | B \in \mathfrak{S}^\pm_{D,n,n;\mathbf{a};s}\}$.

3.7. Let $\nu = (\nu_k)$ be a collection of natural numbers indexed by $k \in \mathbf{Z}$ such that $\nu_{k-n} = \nu_k$ for all k. Let Σ_ν be the set of all tableaux $\mu = (\mu_{t,p})_{t,p \in \mathbf{Z}, p>0}$ with entries in $\mathbf{N}$ such that

$\mu_{t,p} = \mu_{t-n,p}$ for all t, p, $\sum_{t,p;t\le k<t+p} \mu_{t,p} = \nu_k$ for all $k \in \mathbf{Z}$.

Let $\mathfrak{E}_\nu$ be the $\mathbf{Q}(v)$-vector space with basis $\{\mu\}$ indexed by $\mu \in \Sigma_\nu$. Let $\mathfrak{E} = \oplus \mathfrak{E}_\nu$ (direct sum over the various ν as above). Consider the cyclic quiver with vertices $k \in \mathbf{Z}/n\mathbf{Z}$ and arrows $1 \to 2 \to \cdots \to n \to 1$. Let (W_k) be a collection of finite dimensional $\mathbf{C}$-vector spaces indexed by $k \in \mathbf{Z}/n\mathbf{Z}$ and let E_W^{nil} be the space of all nilpotent representation of the cyclic quiver on (W_k). There is an obvious action of $G_W = \prod_k GL(W_k)$ on E_W^{nil}. Its orbits are in natural bijection with Σ_ν where $\dim W_k = \nu_k$. (See [L3, 5.6].) A basis element $\{\mu\}$ of $\mathfrak{E}_\nu$ corresponds then to an irreducible G_W-equivariant perverse sheaf on E_W^{nil}. Using convolution of perverse sheaves in a similar way to [L1], we can define an algebra structure on $\mathfrak{E}$ such that $\mathbf{f}$ can be naturally identified with a subalgebra of $\mathfrak{E}$. (See [L1, 5.8].) Let $\iota : \mathbf{f} \to \mathfrak{E}$ be the imbedding.

Now let $\mathbf{a} \in \mathfrak{S}_{D,n}$. We define a linear map $\mathfrak{E} \xrightarrow{\sigma} \mathfrak{A}^-_{D,n,n;\mathbf{a}}$ as follows. With the notation of [L3, 5.10] we consider the diagram $\mathcal{X} \xleftarrow{\alpha} \tilde{\mathcal{X}} \xrightarrow{\gamma\beta} E_W^{nil}$. In tis diagram α is a principal G_W-bundle, γ is an open imbedding and β is smooth with connected fibres, see [L3, 5.11]. It follows that, if P is a simple G_W-equivariant perverse sheaf on E_W^{nil} then $(\gamma\beta)^*P$ is 0 or is (up to shift) of the form α^*P' where P' is a simple perverse sheaf on $\mathcal{X}$. Now P corresponds to a basis element $\{\mu\}$ of $\mathfrak{E}$ and P', if defined, corresponds to a basis element $\{B\}$ of $\mathfrak{A}^-_{D,n,n;\mathbf{a}}$. We set $\sigma(\{\mu\}) = \{B\}$, if P' is defined and $\sigma(\{\mu\}) = 0$ otherwise. From the definitions one can check that

(a) the composition $\mathbf{f} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\sigma} \mathfrak{A}^-_{D,n,n;\mathbf{a}}$ is the map $x \to [\mathbf{i_a}]\rho'_D(x^-)$.

Note that the map σ is described on the basis by a map σ' from $\cup_\nu \Sigma_\nu$ to the union of $\mathfrak{S}^-_{D,n,n;\mathbf{a}}$ with 0. The map σ' is described combinatorially in [L3, 5.7] as follows.

Let $\mu = (\mu_{t,p}) \in \Sigma_\nu$. If $\mu_{t,1} + \mu_{t,2} + \cdots \le a_t$ for all t, then $\sigma'(\mu) = B$, where

$b_{ij} = \mu_{i,j-i}$ for $i < j$,

$b_{tt} = a_t - (\mu_{t,1} + \mu_{t,2} + \dots)$,

$b_{ij} = 0$ for $i > j$.

Otherwise, $\sigma'(\mu) = 0$.

3.8. For any $s \in \mathbf{N}$, let $\mathfrak{E}^s$ be the subspace of $\mathfrak{E}$ spanned by the $\{\mu\}$ such that $\mu_{t,1} + \mu_{t,2} + \cdots \le s$ for all t. We have $\mathfrak{E}^0 \subset \mathfrak{E}^1 \subset \mathfrak{E}^2 \subset \dots$ and $\mathfrak{E} = \cup_{s\ge 0} \mathfrak{E}^s$.

Lemma 3.9. *If* $\mathbf{a}$ *in 3.7 and* $s \in \mathbf{N}$ *are such that* $s \le a_k$ *for any* $k \in \mathbf{Z}$*, then the map* $\sigma : \mathfrak{E} \to \mathfrak{A}^-_{D,n,n;\mathbf{a}}$ *restricts to an isomorphism* $\mathfrak{E}^s \xrightarrow{\sim} \mathfrak{A}^-_{D,n,n;\mathbf{a};s}$.

If μ satisfies $\mu_{t,1} + \mu_{t,2} + \cdots \le s$ for all t, then it satisfies $\mu_{t,1} + \mu_{t,2} + \cdots \le a_t$ for all t. This shows that the restriction of our map is injective. The fact that it is surjective is obvious. The lemma is proved.

3.10. For any $A \in \mathfrak{S}_{D,n,n}$, we define ${}^-A \in \mathfrak{S}^-_{D,n,n}$ and ${}^+A \in \mathfrak{S}^+_{D,n,n}$ by

${}^-a_{ij} = a_{ij}$ if $i < j$; ${}^-a_{ij} = 0$ if $i > j$; ${}^-a_{ii} = \sum_{k;k\ge i} a_{ki}$;

${}^+a_{ij} = a_{ij}$ if $i > j$; ${}^+a_{ij} = 0$ if $i < j$; ${}^+a_{ii} = \sum_{k;k\ge i} a_{ik}$.

We have $c(A) = c({}^-A), r(A) = r({}^+A), r({}^-A) = v({}^+A)$.

Lemma 3.11. *For any $A \in \mathfrak{S}_{D,n,n}$, we have $\{^+A\} \cdot \{^-A\} = \{A\}$ plus an $\mathcal{A}$-linear combination of elements $\{A_1\}$ with $A_1 < A$.*

The proof is similar to that of [L3, 4.11].

Lemma 3.12. *Let F be a finite subset of $\mathfrak{S}_{D,n}$ and let $s \in \mathbf{N}$. Assume that for any $\mathbf{a} \in F$ and any i we have $a_i \geq 2s$. Then the $\mathbf{Q}(v)$-linear map*

$m : \oplus_{\mathbf{a} \in F} \mathfrak{A}^+_{D,n,n;\mathbf{a};s} \otimes \mathfrak{A}^-_{D,n,n;\mathbf{a};s} \to \mathfrak{A}_{D,n,n}$

defined by multiplication in the algebra $\mathfrak{A}_{D,n,n}$, is injective.

Let T be the set of all pairs (A', A'') such that for some $\mathbf{a} \in F$ we have $A' \in \mathfrak{S}^+_{D,n,n;\mathbf{a};s}$, $A'' \in \mathfrak{S}^-_{D,n,n;\mathbf{a};s}$. Let T' be the set of all $A \in \mathfrak{S}_{D,n,n}$ such that

$\mathbf{a} \in F$, where $a_i = a_{ii} + \sum_{j;i<j} a_{ij} + \sum_{k;k>i} a_{ki}$ for all i;

$\sum_{j;i<j} a_{ij} \leq s$ for all i;

$\sum_{k;k>i} a_{ki} \leq s$ for all i.

We define a bijection

(a) $T' \xrightarrow{\sim} T$.

The map $T' \to T$ is given by $A \mapsto (^+A, {}^-A)$. A map in the opposite direction is given by $(A', A'') \mapsto A$, where

$a_{ij} = a'_{ij}$ for $i > j$, $a_{ij} = a''_{ij}$ for $i < j$,

$a_{ii} = a_i - \sum_{j;i<j} a''_{ij} - \sum_{k;k>i} a'_{ki}$,

where $\mathbf{a} = c(A') = r(A'')$. The inequality $a_{ii} \geq 0$ follows from

$a_i - \sum_{j;i<j} a''_{ij} - \sum_{k;k>i} a'_{ki} \geq a_i - s - s \geq 0$.

This gives the required bijection.

Assume that $\sum_{(A',A'') \in T} c_{A',A''} \{A'\} \otimes \{A''\} \in \ker(m)$; here $c_{A',A''} \in \mathbf{Q}(v)$ are 0 for all but finitely many $(A', A'') \in T$. Assume also that $c_{A',A''} \neq 0$ for some $(A', A'') \in T$. By our assumption, $\sum_{(A',A'') \in T} c_{A',A''} \{A'\} \cdot \{A''\} = 0$ in $\mathfrak{A}_{D,n,n}$. Equivalently, (by (a)),

(b) $\sum_{A \in T'} c_{^+A,{}^-A} \{^+A\} \cdot \{^-A\} = 0$ in $\mathfrak{A}_{D,n,n}$.

We can find $A \in T'$ such that $c_{^+A,{}^-A} \neq 0$ and such that, whenever $\tilde{A} \in T'$, $\tilde{A} > A$, we have $c_{^+\tilde{A},{}^-\tilde{A}} = 0$. Using this and Lemma 3.11, we see that the left hand side of (b) equals $c_{^+A,{}^-A}\{A\}$ plus a $\mathbf{Q}(v)$-linear combination of elements $\{A_1\}$, with $A_1 \in \mathfrak{S}_{D,n,n}$, $A_1 < A$. This contradicts (b). The lemma is proved.

3.13. Proof of Proposition 3.5. We fix an element $u \in \dot{\mathbf{U}}$ in the kernel of ρ. We must prove that $u = 0$. Let $\mathcal{B}$ be a vector space basis of $\mathbf{f}$. Using 3.1(a), we can write

$u = \sum_{b_1, b_2 \in \mathcal{B}; \lambda \in X} c_{b_1,b_2,\lambda} \tilde{b}_1^+ 1_\lambda b_2^-$

where $c_{b_1,b_2,\lambda} \in \mathbf{Q}(v)$ are 0 for all (b_1, b_2, λ) in the complement of some finite subset Z of $\mathcal{B} \times \mathcal{B} \times X$. We may assume that $Z = \mathcal{B}_1 \times \mathcal{B}_1 \times X_1$ where $\mathcal{B}_1$ is a finite subset of $\mathcal{B}$ and X_1 is a finite subset of X. By assumption, we have

$\sum_{(b_1,b_2,\lambda) \in Z} c_{b_1,b_2,\lambda} \rho_D(\tilde{b}_1^+ 1_\lambda b_2^-) = 0$

or equivalently

(a) $\sum_{(b_1,b_2,\lambda)\in Z} c_{b_1,b_2,\lambda}\Psi(\rho_D(1_\lambda)\rho'_D(b_1^-))\rho_D(1_\lambda)\rho'_D(b_2^-)=0$
for all $D\ge 0$. It is enough to prove that $c_{b_1,b_2,\lambda}=0$ for all $(b_1,b_2,\lambda)\in Z$.

Since $\mathcal{B}_1$ is finite, we can find $s\in\mathbf{N}$ such that

(b) $\iota(b_1)\in\mathfrak{E}^s$ for any $b_1\in\mathcal{B}_1$.
We fix such an s. Since X_1 is finite, if D is large enough, then

(c) any $\lambda\in X_1$ is represented by an element of $\mathfrak{S}_{D,n}$ with coordinates $\ge 2s$.
We fix such a D. Let F be the finite subset of $\mathfrak{S}_{D,n}$ which maps bijectively onto X_1 under the obvious map $\zeta:\mathfrak{S}_{D,n}\to X$. From (a) we deduce

(d) $\sum_{(b_1,b_2,\mathbf{a})\in\mathcal{B}_1\times\mathcal{B}_1\times F} c_{b_1,b_2,\zeta(\mathbf{a})}\Psi([\mathbf{i_a}]\rho'_D(b_1^-))[\mathbf{i_a}]\rho'_D(b_2^-)=0.$
By the choice of s, we have $[\mathbf{i_a}]\rho'_D(b_2^-)\in\mathfrak{A}^-_{D,n,n;\mathbf{a};s}$ for $b_2\in\mathcal{B}_1$, hence $\Psi([\mathbf{i_a}]\rho'_D(b_1^-))\in\mathfrak{A}^+_{D,n,n;\mathbf{a};s}$ for $b_1\in\mathcal{B}_1$. Using now Lemma 3.12, we deduce from (d) that

$\sum_{(b_1,b_2,\mathbf{a})\in\mathcal{B}_1\times\mathcal{B}_1\times F} c_{b_1,b_2,\zeta(\mathbf{a})}\Psi([\mathbf{i_a}]\rho'_D(b_1^-))\otimes([\mathbf{i_a}]\rho'_D(b_2^-))=0$
in $\oplus_{\mathbf{a}\in F}\mathfrak{A}^+_{D,n,n;\mathbf{a};s}\otimes\mathfrak{A}^-_{D,n,n;\mathbf{a};s}$. Hence for any $\mathbf{a}\in F$ we have

$\sum_{(b_1,b_2)\in\mathcal{B}_1\times\mathcal{B}_1} c_{b_1,b_2,\zeta(\mathbf{a})}([\mathbf{i_a}]\rho'_D(b_1^-))\otimes([\mathbf{i_a}]\rho'_D(b_2^-))=0$
in $\mathfrak{A}^-_{D,n,n;\mathbf{a};s}\otimes\mathfrak{A}^-_{D,n,n;\mathbf{a};s}$. Using Lemma 3.9, it follows that, for any $\mathbf{a}\in F$ we have

$\sum_{(b_1,b_2)\in\mathcal{B}_1\times\mathcal{B}_1} c_{b_1,b_2,\zeta(\mathbf{a})}\iota(b_1)\otimes\iota(b_2)=0$
in $\mathfrak{E}^s\otimes\mathfrak{E}^s$; since $\iota:\mathbf{f}\to\mathfrak{E}$ is injective, it follows that, for any $\mathbf{a}\in F$ we have

$\sum_{(b_1,b_2)\in\mathcal{B}_1\times\mathcal{B}_1} c_{b_1,b_2,\zeta(\mathbf{a})}b_1\otimes b_2=0$
in $\mathbf{f}\otimes\mathbf{f}$. Since $\mathcal{B}_1$ is part of a basis of $\mathbf{f}$, we deduce that $c_{b_1,b_2,\zeta(\mathbf{a})}=0$ for all $(b_1,b_2,\mathbf{a})\in\mathcal{B}_1\times\mathcal{B}_1\times F$. Since $\zeta:F\to X_1$ is bijective, we have $c_{b_1,b_2,\lambda}=0$ for all $(b_1,b_2,\lambda)\in\mathcal{B}_1\times\mathcal{B}_1\times X_1$. The proposition is proved.

3.14. An injectivity statement for the (modified) ordinary quantum $\mathfrak{sl}_n$ entirely analogous to that of 3.5 can be proved by exactly the same method.

4. A parametrization of the canonical basis of the (modified) quantum affine $\mathfrak{sl}_n$

4.1. Let $\mathfrak{S}^{\mathrm{ap}}_{D,n,n}$ be the subset of $\mathfrak{S}_{D,n,n}$ consisting of the elements $B\in\mathfrak{S}_{D,n,n}$ such that for any $p\in\mathbf{Z}-\{0\}$ there exists $k\in\mathbf{Z}$ such that $b_{k,k+p}=0$.

Recall from [L3, 2.7, 2.8] that $\mathbf{U}_D$ is the subalgebra of $\mathfrak{A}_{D,n,n}$ generated by certain elements E_i,F_i and by $[\mathbf{i_a}]$ with $\mathbf{a}\in\mathfrak{S}_{D,n}$. By [L3, 8.2], the elements $\{B\}$ with $B\in\mathfrak{S}^{\mathrm{ap}}_{D,n,n}$ form a $\mathbf{Q}(v)$-basis of $\mathbf{U}_D$.

By the argument in [L3, 7.7], there is a unique algebra homomorphism

(a) ${}^1\dot{\mathbf{U}}\to\mathbf{U}_D$
which carries E_i1_λ to $E_i[\mathbf{i_a}]$,F_i1_λ to $F_i[\mathbf{i_a}]$, 1_λ to $[\mathbf{i_a}]$ for any i,λ, provided that $\lambda\in X$ can be represented by an element $\mathbf{a}\in\mathfrak{S}_{D,n}$; if $\lambda\in X$ cannot be represented by an element of $\mathfrak{S}_{D,n}$ then (a) carries $E_i1_\lambda,F_i1_\lambda,1_\lambda$ to 0. The map (a) is clearly surjective.

Let ${}^1\dot{\mathbf{B}}$ be the canonical basis of ${}^1\dot{\mathbf{U}}$ (see [L2, Ch.25]). According to a conjecture in [L3, 9.3] (proved in [SV]),

(b) the map (a) carries any element of ${}^1\dot{\mathbf{B}}$ to zero or to an element of the form $\{B\}$ where $B\in\mathfrak{S}_{D,n,n}$.

Note that in (b) we have automatically $B \in \mathfrak{S}^{\mathrm{ap}}_{D,n,n}$. According to a conjecture in [L3, 9.3] (proved in [SV]),

(c) the homomorphism $\phi_{D+n,n} : \mathfrak{A}_{D+n,n,n} \to \mathfrak{A}_{D,n,n}$ carries an element of the form $\{B'\}$ with $B' \in \mathfrak{S}^{\mathrm{ap}}_{D+n,n,n}$ to $\{B\}$ with $B \in \mathfrak{S}^{\mathrm{ap}}_{D,n,n}$, $b_{ij} = b'_{ij} - \delta_{ij}$, provided that $b'_{ii} > 0$ for all i; otherwise, $\{B'\}$ is carried to 0.

4.2. As in [L3, 9.3], we define $\mathcal{S}^{\mathrm{ap}}_n$ to be the set of all $A \in \mathcal{S}_n$ (see 3.3) such that for any $p \in \mathbf{Z} - \{0\}$ there exists $k \in \mathbf{Z}$ such that $a_{k,k+p} = 0$. If $A, A' \in \mathcal{S}^{\mathrm{ap}}_n$ we write $A \sim A'$ if there exists $l \in \mathbf{Z}$ such that $a_{ij} = a'_{ij} + l\delta_{ij}$ for all i, j. This is an equivalence relation. Let $\mathcal{S}^{\mathrm{ap}}_n/\sim$ be the set of equivalence classes.

Theorem 4.3. *There is a canonical bijection $\kappa : {}^1\dot{\mathbf{B}} \to \mathcal{S}^{\mathrm{ap}}_n/\sim$.*

Let $b \in {}^1\dot{\mathbf{B}}$. By 3.5, there exists $D \geq 0$ such that the canonical map ${}^1\dot{\mathbf{U}} \to \mathbf{U}_D$ takes b to a non-zero element. This non-zero element is, by 4.1(b), of the form $\{B\}$ for some $B \in \mathfrak{S}_{D,n,n}$ (which is automatically in $\mathfrak{S}^{\mathrm{ap}}_{D,n,n}$, by the sentence following 4.1(b)). We define $\kappa(b)$ to be the image of B under the composition

(a) $\mathfrak{S}^{\mathrm{ap}}_{D,n,n} \to \mathcal{S}^{\mathrm{ap}}_n \to \mathcal{S}^{\mathrm{ap}}_n/\sim$

(the obvious imbedding followed by the canonical map). We show that $\kappa(b)$ is independent of the choice of D. Indeed, let $D' \geq 0$ be such that the canonical map ${}^1\dot{\mathbf{U}} \to \mathbf{U}_{D'}$ takes b to a non-zero element, hence to $\{B'\}$ where $B' \in \mathfrak{S}^{\mathrm{ap}}_{D,n,n}$. By [L2, 25.2.1(a)] we have $b = b1_\lambda$ for some $\lambda \in X$. Then $\{B\} = \{B\}[\mathbf{i_a}]$, $\{B'\} = \{B'\}[\mathbf{i_{a'}}]$, where $\mathbf{a} \in \mathfrak{S}_{D,n}, \mathbf{a}' \in \mathfrak{S}_{D',n}$ are representatives of λ. It follows that $\mathbf{a}' - \mathbf{a}$ is an integer multiple of $\mathbf{b}_0$, and in particular, $D' - D = mn$ for some $m \in \mathbf{Z}$. We may assume that $m \geq 0$. Using several times 3.4(a), we see that

$\phi_{D+n,D}\phi_{D+2n,D+n} \cdots \phi_{D+mn,D+(m-1)n}$

carries $\{B'\}$ to $\{B\}$. Using 4.1(c), we deduce that $b'_{ij} - b_{ij} = m\delta_{ij}$ for all i, j. Hence B, B' have the same image under the composition (a). This shows that $\kappa(b)$ is well defined.

We show that κ is injective. Let $b, b' \in {}^1\dot{\mathbf{B}}$ be such that $\kappa(b) = \kappa(b')$. We can find $D, D' \geq 0$ such that the canonical map ${}^1\dot{\mathbf{U}} \to \mathbf{U}_D$ takes b to $\{B\}$ where $B \in \mathfrak{S}^{\mathrm{ap}}_{D,n,n}$ and the canonical map ${}^1\dot{\mathbf{U}} \to \mathbf{U}_{D'}$ takes b' to $\{B'\}$ where $B' \in \mathfrak{S}^{\mathrm{ap}}_{D',n,n}$. By assumption, there exists $m \in \mathbf{Z}$ such that $b'_{ij} - b_{ij} = m\delta_{ij}$ for all i, j. We may assume that $m \geq 0$. Using 3.4(a) and 4.1(c), we see that the canonical map ${}^1\dot{\mathbf{U}} \to \mathbf{U}_{D'}$ takes b to $\{B'\}$. Using again 3.4(a) and 4.1(c) we see that, for any $m' \geq 0$, the canonical map ${}^1\dot{\mathbf{U}} \to \mathbf{U}_{D'+m'n}$ takes b and b' to the same element, namely to $\{B''\}$ where $b''_{ij} = b'_{ij} + m'\delta_{ij}$. But from the proof of 3.5 we know that an element of ${}^1\dot{\mathbf{U}}$ whose image in $\mathfrak{A}_{D'',n,n}$ is zero for infinitely many values of D'', must be zero. Thus, $b = b'$, as required.

We show that κ is surjective. Consider an element of $\mathcal{S}^{\mathrm{ap}}_n/\sim$. We can represent it by an element $B \in \mathfrak{S}^{\mathrm{ap}}_{D,n,n}$ for some $D \in \mathbf{N}$. Since ${}^1\dot{\mathbf{U}} \to \mathbf{U}_D$ is surjective, we see from 4.1(b) that $\{B\}$ is the image of some element in ${}^1\dot{\mathbf{B}}$. Thus κ is surjective. The theorem is proved.

4.4. Let $\mathcal{T}_n$ be the set of all matrices $A = (a_{i,j})_{(i,j)\in[1,n]}$ with entries $a_{i,j} \in \mathbf{Z}$ such that $a_{i,j} \geq 0$ for all $i \neq j$. If $A, A' \in \mathcal{T}_n$ we write $A \sim A'$ if there exists $l \in \mathbf{Z}$ such that $a_{ij} = a'_{ij} + l\delta_{ij}$ for all i, j. This is an equivalence relation. Let $\mathcal{T}_n/\sim$ be the set of equivalence classes. By arguments entirely similar to those in the proof of 4.3, we see that the canonical basis of the (modified) ordinary quantum $\mathfrak{sl}_n$ is naturally in bijection with $\mathcal{T}_n/\sim$.

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