Tilting characters for $SL_3$

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$G$-semisimple alg. group $\sqrt{F_p}$, $p$ - prime number

$T$ - maximal torus

$X = \text{Hom}(T, F_p^*)$, $Y = \text{Hom}(F_p^*, T)$

$\alpha_i \in Y$ simple roots, $i = 1,..., n$

$X^+ = \{ \lambda \in X ; \langle \alpha_i, \lambda \rangle \in \mathbb{N}, \forall i \}$

$\text{Rep} G$ - category of f.d. rational representations of $G$ / $F_p$

$M \in \text{Rep} G$

$[M] = \sum_{\lambda \in X^+} \dim M(\lambda) \lambda \in \mathbb{Z}[X]$  

$\lambda$-weight space

Simple objects of $\text{Rep} G$ $\leftrightarrow X^+$

$L_\lambda$ $\leftrightarrow$ $\lambda$

$V_\lambda$ - Weyl module corrs to $\lambda \in X^+$ ( $L_\lambda$ a quotient of $V_\lambda$)

$V_\lambda'$ - co-Weyl module  

$[V_\lambda] = [V_\lambda']$ given by Weyl character formula
Tilt $G$ - objects $M \in \text{Rep } G$ which admit
- a finite filtration with subquotients Weyl modules
- a finite filtration with co-Weyl modules

"Tilting modules".
Indecomposable objects of $\text{Tilt } G \iff X^+ \leftrightarrow X^-$

(Ringel 1991, Derksen 1993)
\[
[T_\lambda] = \sum_{\mu \in X^+} m_{\mu \lambda} [V_\mu], \quad \lambda \in X^+, \quad m_{\mu \lambda} \in \mathbb{N}.
\]

Problem: compute $m_{\mu \lambda}$!
Actually $m_{\mu \lambda}$ are in principle computable for any fixed $\mu, \lambda, \rho$ but the calculation is extremely complicated.

By work of Riche-Williamson, Achar-Maki-Sumi-Riche-Williamson, Elias-Lusztig, Libedinsky-Williamson this computation can be given to a computer, but the calculation is still extremely long in each case.

So the problem can be restated as: [find closed formula for $m_{\mu \lambda}$], uniform in $\rho$, for $\rho > 0$.

Example, $G = S\mathbb{L}_2$, $X = \mathbb{Z}$, $\dim V_\lambda = 2^{\lambda + 1}$

For $\lambda = 0, 1, \ldots, p-1$, $V_\lambda = V_{\lambda}^{'}, T_\lambda$

For $\lambda = p$, $T_\lambda = \otimes V_1 \otimes V_{p-1}$, $[T_\lambda] = 2[\xi_{p-2}] + [\xi_p]$

\[
= [V_\lambda] + [V_{\lambda-2}]
\]
Replace $\text{Rep } G$ by representations of a quantum group at $q^\lambda$.

$\mathfrak{g}_\lambda^q, \mathcal{V}_\lambda^q, V_\lambda^q, T_\lambda^q$: quantum analogues of $\mathfrak{g}_\lambda, \mathcal{V}_\lambda, V_\lambda, T_\lambda$.

Then $[T_\lambda^q], [V_\lambda^q] \in \mathbb{Z}[X]$ defined, and

$$[T_\lambda^q] = \sum_{m \in \mathbb{X}^+} m_{\mu, \lambda}^q [V_{\mu}^q], \quad m_{\mu, \lambda}^q \in \mathbb{X}^+.$$ 

$m_{\mu, \lambda}^q$ can be expressed in terms of $K-L$ polynomials of an affine Weyl group (Soergel 1997). Can write

$$[T_\lambda] = \sum_{m \in \mathbb{X}^+} \mathcal{m}_{\mu, \lambda} [T_{\mu}] , \quad m \in \mathbb{X}^+, \quad \mathcal{m}_{\mu, \lambda} \in \mathbb{N}.$$

Our problem reduced to

* compute/understand $\mathcal{m}_{\mu, \lambda}$

Now assume $G = SU_3$, $p > 3$.

In the affine space $X_R = \mathbb{R} \otimes X$

consider the $p$-hyperplanes

$$H_k = \{ x \in X_R \mid \langle \xi_x, x \rangle + 1 = kp^\frac{2}{3} \}, \quad k \in \mathbb{Z}.$$

$$H'_k = \{ x \in X_R \mid \langle \xi_{-1}, x \rangle + 1 = kp^\frac{2}{3} \}, \quad -1 -$$

$$H''_k = \{ x \in X_R \mid \langle \xi_{+1}, x \rangle + 2 = kp^\frac{2}{3} \}, \quad -1 -$$
p-alcoves: connected components of $X_R - U p$-hyperplanes.

dominant p-alcoves: those containing some $\lambda \in X^+$. 

\[ \begin{array}{c}
H_2 \\
H_1 \\
H_0 \\
H_0' \end{array} \]

\[ \begin{array}{c}
H_2' \\
H_1' \\
H_0' \end{array} \]

p-boxes: connected components of $X_R - (U H_k) U H_k$

\[ \begin{array}{c}
H_2 \\
H_1 \\
H_0 \\
H_0' \end{array} \]

\[ \begin{array}{c}
H_2' \\
H_1' \\
H_0' \end{array} \]

A matrix $\tilde{m}_{BA}$ indexed by dominant p-alcoves such that:

for $\mu, \lambda$ of any $p$-hyperplane

\[ \tilde{m}_{\mu, \lambda} = \begin{cases} 
\tilde{m}_{BA} & \text{if } \mu \in B, \lambda \in A \text{ are in some affine Weyl group orbit} \\
0 & \text{if } \mu \in B, \lambda \in A \text{ are not in the same orbit} 
\end{cases} \]

Our problem becomes:

* compute/understand $\tilde{m}_{BA}$ (B, A dominant p-alcoves)

Andersen: if $B, A$ are not too far from 0 then $\tilde{m}_{BA} = \delta_{BA}$

Donkin: if $\tilde{m}_{BA}$ known for $A$ of type

\[ \begin{array}{c}
\ldots \\
A_1 \end{array} \]

then $\tilde{m}_{BA}$ known for any $B$. 

Denote:
Set \( \mathcal{X} = \{ \lambda \in \mathbb{R}^n : \lambda_1 + 1 \leq 0, \lambda_2 + 2 \lambda_3 \leq 0, \lambda_2 \leq 0, \lambda_3 \leq 0 \} \)

\( \mathcal{X}_0 = \{ \lambda \in \mathcal{X} : \lambda_1 + 1 = 0 \} \)

\( x_{>0} = \mathcal{X} - \mathcal{X}_0 \)

For each \( \lambda \in \mathcal{X}_{>0} \), consider the two \( p \)-alcoves \( \mathcal{A}_\lambda, \mathcal{A}'_\lambda \) as follows:
We will define explicitly a set $Z$ and a function
\[ Z \rightarrow \mathbb{R}_0 \times \mathbb{N}_0 \times \{1, 2\} \]
\[ Z \rightarrow (\lambda_2, n_2, s_2) \]
so that, setting
\[ \hat{m}_\lambda, i = \sum_{z \subseteq Z} s_z n_z \]
for $\lambda \in \mathbb{R}_0$, $n_0 > 0$
we have
\[ \sum_{B \in \text{dom. dom.}} \left( \sum_{i \geq 1} m_{B, i} \right) B = \sum_{i \geq 1} (i) A_i + \sum_{i \geq 1} (m_{\lambda, i + 2} + [i + 1]) \lambda \]
\[ + \sum_{\lambda \in \mathbb{R}_0} \left( \sum_{i \geq 1} m_{\lambda, i} [i] + [i + 2] \right) \lambda \]
equality of formal sums.
We view $X$ as a set of vertices of an oriented graph as follows.

We now consider a subgraph $X^*$ of $X$ with induced orientation.

$\{H_p \mid p \geq 2\}$ hyperplanes

$\{H_0, H_1, H_2, H_3, H_4\}$ algebras

$(p = 5)$
A sequence in $Z$ has three parts:

- \(x_0, x_1, \ldots, x_k, x_{k+1}, \ldots, x_L, x_{L+1}, \ldots, x_{n}\)
  - **Dull part**
  - "Brownian motion"
  - "Billiards"

The dull part consists of \(k=p^r \geq p\) small steps starting with \(x_0=0\) on the hyperplane \(H_0 = \{x | \langle x, x \rangle + 1 = 0\}\).

(Example: \(p=5, r=2, k=10\))

The Brownian motion part \(x_{k+1}, \ldots, x_L\) takes place in \(\mathbb{P}\); each \(x_i\) is on a \(p^2\)-hyperplane but not on a \(p\)-corner (intersection of two \(p\)-hyperplanes).
It is a succession of small steps followed by big steps, followed by small steps, followed by big steps, etc, with rest in between.

The first \( p-1 \) steps are as shown. We cannot have \( p \) steps without hitting a corner point.

After \( p-1 \) steps and a rest we are at a distance \( 1 \) from a corner point. There are one or two available long steps from there and whichever we get either which are distance \( 2 \) from a corner point.

From these points we can do one or two long steps. After \( p-2 \) long steps we cannot do long steps any more (we would hit a corner).

Instead we start doing small steps. They are uniquely determined and we can do at most \( p-2 \) of them without hitting a corner. If we do all \( p-2 \) small steps we rest and continue with long steps, etc. We can stop this process at any time but \( k \) must be \( > k \).
In the case where $\mathcal{U}$ could be a start of a big step we have the option of stopping completely or continuing the sequence by leaving the $p$-hyperplanes and getting inside (and remaining inside) a $p$-cube. The sequence moves as a ball on a billiard table shaped as an equilateral triangle.

This defines the set $\mathcal{Z}$.

The function $\mathcal{Z} \to \mathcal{X}_0$ is $(x_0, \ldots, x_n) \mapsto x_n$

The function $\mathcal{Z} \to \mathcal{N}_\mathcal{X}_0$ is $[x_0, \ldots, x_n] \mapsto |x_n|$

where $|x_i|$ is defined by induction on $i$ as follows

$|x_0| = 0$

if $x_i = x_{i+1}$ then $|x_{i+1}| = |x_i| + 3$

if $x_i \not= x_{i+1}$ then $|x_{i+1}| = |x_i| + 2$

if $x_i, x_{i+1}$ is a big step then $|x_{i+1}| = |x_i| + 2p + 1$

Thus $|x_n|$ is approximately $2^n$

The function $\mathcal{Z} \to \{1, 2\}$ is $(x_0, \ldots, x_n) \mapsto \begin{cases} 1 & \text{if } x_0 \not= x_1 \ldots \not= x_n \\
2 & \text{otherwise} \end{cases}$

Remarks: 1) The dimension of the tilting module $\overline{T}_\lambda$ $(\lambda \in \mathcal{A}_i)$ grows at least exponentially in $i$ (assuming the conjecture).

2) The conjecture implies a statement about the decomposition numbers of the symmetric group $S(n)$ partitions with $\leq 3$ parts. (Erdmann)