

LIE ALGEBRAS, VI

ROOTS

Let L be a semisimple Lie algebra $\neq 0$. A subalgebra T of L is said to be *toral* if any element of T is semisimple in L .

Lemma. *If T is toral then T is abelian.*

Let $x \in T$. Assume that $ad(x) : T \rightarrow T$ has some eigenvalue $a \neq 0$. Thus $[x, y] = ay$ for some $y \in T - \{0\}$. Now $ad(y) : L \rightarrow L$ is semisimple hence $ad(y) : T \rightarrow T$ is semisimple hence $x = \sum_j u_j$ where $u_j \in T$ are eigenvectors of $ad(y) : T \rightarrow T$ with corresponding eigenvalue λ_j . Hence $ad(y)x = \sum_{j; \lambda_j \neq 0} \lambda_j u_j$. But $ad(y)x = -ay$. But y is in the 0-eigenspace of $ad(y)$ and $\sum_{j; \lambda_j \neq 0} \lambda_j u_j = -ay$ is a contradiction. Thus, all eigenvalues of $ad(x) : T \rightarrow T$ are 0. Now $ad(x) : L \rightarrow L$ is semisimple hence $ad(x) : T \rightarrow T$ is semisimple hence $ad(x) : T \rightarrow T$ is 0. The lemma follows.

Let H be a maximal toral subalgebra of L . Now $\{ad(h) : L \rightarrow L | h \in H\}$ is a family of commuting semisimple endomorphisms of L . Hence $L = \bigoplus_{\alpha} L_{\alpha}$ where α runs over the dual space H^* of H and $L_{\alpha} = \{x \in L | [h, x] = \alpha(h)x \forall h \in H\}$. Now $L_0 = \{x \in L | [h, x] = 0 \forall h \in H\}$. and $H \subset L_0$ by the lemma. We say that $\alpha \in H^*$ is a *root* or $\alpha \in R$ if $\alpha \neq 0$ and $L_{\alpha} \neq 0$. We have $L = L_0 \oplus \bigoplus_{\alpha \in R} L_{\alpha}$ (root decomposition or Cartan decomposition).

Lemma. (a) *For any $\alpha, \beta \in H^*$ we have $[L_{\alpha}, L_{\beta}] \subset L_{\alpha+\beta}$.*

(b) *If $x \in L_{\alpha}, \alpha \neq 0$ then $ad(x)$ is nilpotent.*

(c) *If $\alpha, \beta \in H^*, \alpha + \beta \neq 0$ then $\kappa(L_{\alpha}, L_{\beta}) = 0$.*

(a) Let $x \in L_{\alpha}, y \in L_{\beta}$. For $h \in H$ we have

$[h, [x, y]] = [[h, x], y] + [x, [h, y]] = \alpha(h)[x, y] + \beta(h)[x, y] = (\alpha + \beta)(h)[x, y]$ hence $[x, y] \in L_{\alpha+\beta}$.

(b) For any $\beta \in H^*$ we have $n\alpha + \beta \notin R$ for large n hence using (a), $ad(x)^n L_{\beta} = 0$. Now (b) follows.

(c) We can find $h \in H$ with $(\alpha + \beta)(h) \neq 0$. Let $x \in L_{\alpha}, y \in L_{\beta}$. We have $\kappa([h, x], y) = \kappa([y, h], x)$ hence $\alpha(h)\kappa(x, y) = -\beta(h)\kappa(x, y)$. Thus $(\alpha + \beta)(h)\kappa(x, y) = 0$ and $\kappa(x, y) = 0$.

Lemma. *The restriction of κ to L_0 is non-singular.*

Proposition. $L_0 = H$.

We show:

(a) *If $x \in L_0$ and $x = s + n$ is a Jordan decomposition in L then $s \in L_0, n \in L_0$.*

We have $ad(x)H \subset \{0\}$ hence $ad(x)_s H \subset \{0\}, ad(x)_n H \subset \{0\}$, hence $ad(s)H \subset \{0\}, ad(n)H \subset \{0\}$, hence $s \in L_0, n \in L_0$.

(b) *If $x \in L_0$ is semisimple in L then $x \in H$.*

From the assumption, $H + kz$ is a toral algebra hence it is H by the maximality of H . Hence $x \in H$.

(c) *The restriction of κ to H is non-singular.*

Assume that $h \in H$ and $\kappa(h, H) = 0$. Let $x = s + n \in L_0$ be as in (a). Then $s \in L_0, n \in L_0$. By (b) we have $s \in H$ hence $\kappa(h, s) = 0$. Now $ad(n) : L \rightarrow L$ is nilpotent and $ad(n), ad(h)$ commute hence $ad(h)ad(n)$ is nilpotent hence $\text{tr}(ad(h)ad(n), L) = 0$. Thus $\kappa(h, n) = 0$. Hence $\kappa(h, x) = 0$. Thus $\kappa(h, L_0) = 0$. Since $\kappa|_{L_0}$ is non-singular, we have $h = 0$.

(d) *L_0 is nilpotent.*

By Engel it is enough to show that, if $x \in L_0$ then $ad(x) : L_0 \rightarrow L_0$ is nilpotent. Write $x = s + n$ as in (a). Now $ad(s) : L_0 \rightarrow L_0$ is 0 since $s \in H$ (by (b)). Also $ad(n) : L \rightarrow L$ is nilpotent hence $ad(x) = ad(n) : L_0 \rightarrow L_0$ is nilpotent.

(e) $[L_0, L_0] \cap H = 0$.

Let $x \in [L_0, L_0] \cap H$. Write $x = \sum_i [x_i, y_i]$ where $x_i, y_i \in L_0$. If $h \in H$ we have $\kappa(h, x) = \sum_i \kappa(h, [x_i, y_i]) = \sum_i \kappa(x_i, [y_i, h]) = 0$ since $[y_i, h] = 0$. Thus $\kappa(H, x) = 0$. Since $x \in H$ we see from (c) that $x = 0$.

(f) $[L_0, L_0] = 0$.

Otherwise, we have $[L_0, L_0] \neq 0$. Since L_0 is nilpotent and $[L_0, L_0]$ is a non-zero ideal, we then have $[L_0, L_0] \cap \text{centre}(L_0) \neq 0$ (by a corollary of Engel). Let $x \in [L_0, L_0] \cap \text{centre}(L_0), x \neq 0$. Write $x = s + n$ as in (a). Since $ad(x)(L_0) \subset 0$ we have $ad(x)_n(L_0) \subset 0$ hence $ad(n)(L_0) \subset 0$ hence $n \in \text{centre}(L_0)$. Hence for any $x' \in L_0$, $ad(x'), ad(n) : L \rightarrow L$ commute and $ad(n)$ is nilpotent hence $ad(x')ad(n) : L \rightarrow L$ is nilpotent hence $\text{tr}(ad(x')ad(n), L) = 0$ hence $\kappa(x', n) = 0$. Thus $\kappa(L_0, n) = 0$. Since $\kappa|_{L_0}$ is non-singular we have $n = 0$. Thus $x = s \in H$ (see (b)). Hence $x \in [L_0, L_0] \cap H$ which is 0 by (e). Hence $x = 0$ a contradiction.

(g) *If $x \in L_0$ is nilpotent then $x = 0$.*

For all $y \in L_0$, $ad(x), ad(y)$ commute and $ad(x)$ is nilpotent hence $ad(x)ad(y) : L \rightarrow L$ is nilpotent hence $\text{tr}(ad(x)ad(y), L) = 0$. Hence $\kappa(x, y) = 0$. Hence $\kappa(x, L_0) = 0$. Since $\kappa|_{L_0}$ is non-singular we have $x = 0$.

We can now prove the proposition. Let $x \in L_0$. Write $x = s + n$ as in (a). Then $s \in L_0, n \in L_0$. By (g) we have $n = 0$. By (b) we have $s \in H$. Hence $x \in H$. The proposition is proved.

Properties of roots.

Let $\xi \in H^*$. Since $\kappa|_H$ is non-singular there exists a unique element $t_\xi \in H$ such that $\xi(h) = \kappa(t_\xi, h)$ for all $h \in H$. Now $\xi \mapsto t_\xi$ is an isomorphism $H^* \xrightarrow{\sim} H$.

(a) *R spans the vector space H^* .*

If not, we can find $h \in H, h \neq 0$ so that $\alpha(h) = 0$ for all $\alpha \in R$. Then $[h, L_\alpha] = 0$ for all $\alpha \in R$. Also $[h, L_0] = 0$ since $L_0 = H$ is abelian. Hence $[h, L] = 0$ so that $h \in Z(L)$. But $Z(L) = 0$ since L is semisimple. Thus $h = 0$, contradiction.

(b) *If $\alpha \in R$ then $-\alpha \in R$.*

Assume that $-\alpha \notin R$. Then $L_{-\alpha} = 0$. Hence $\kappa(L_\alpha, L_\beta) = 0$ for any $\beta \in H^*$ hence $\kappa(L_\alpha, L) = 0$. Since κ is non-singular we have $L_\alpha = 0$, absurd.

(c) *If $\alpha \in R, x \in L_\alpha, y \in L_{-\alpha}$ then $[x, y] = \kappa(x, y)t_\alpha$.*

Let $h \in H$. We have $\kappa(h, [x, y]) = \kappa(y, [h, x]) = \alpha(h)\kappa(y, x) = \kappa(t_\alpha, h)\kappa(x, y)$ hence $\kappa(h, [x, y] - \kappa(x, y)t_\alpha) = 0$. Thus $\kappa([x, y] - \kappa(x, y)t_\alpha, H) = 0$. Since $[x, y] - \kappa(x, y)t_\alpha \in H$ and κ_H is non-singular, we have $[x, y] - \kappa(x, y)t_\alpha = 0$.

(d) *Let $\alpha \in R$ and let $x \in L_\alpha - \{0\} \neq 0$. There exists $y \in L_{-\alpha}$ such that $\kappa(x, y) \neq 0$.*

Assume that $\kappa(x, L_{-\alpha}) = 0$. Then $\kappa(x, L_\beta) = 0$ for any $\beta \in H^*$ hence $\kappa(x, L) = 0$ hence $x = 0$ absurd.

(e) *Let $\alpha \in R$. We have $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$.*

The equality comes from the definition of t_α . Assume that $\alpha(t_\alpha) = 0$. Then $[t_\alpha, L_\alpha] = 0, [t_\alpha, L_{-\alpha}] = 0$. Let x, y be as in (d). We can assume that $\kappa(x, y) = 1$. Then $[x, y] = t_\alpha$. Let $S = kx + ky + kt_\alpha$, a Lie subalgebra of L . We have $[S, S] = kt_\alpha, [kt_\alpha, kt_\alpha] = 0$ hence S is solvable. By Lie's theorem for $ad : S \rightarrow \text{End}(L)$ we see that $ad(x') : L \rightarrow L$ is nilpotent for any $x' \in [S, S]$. In particular $ad(t_\alpha) : L \rightarrow L$ is nilpotent. Since $t_\alpha \in H$ and all elements of H are semisimple, we see that $ad(t_\alpha) : L \rightarrow L$ is also semisimple hence is 0. Thus $t_\alpha \in Z(L) = 0$. This contradicts $t_\alpha \neq 0$.

(f) *Let $\alpha \in R$. Let $x \in L_\alpha, x \neq 0$. We can find $y \in L_{-\alpha}$ such that, setting $h = [x, y] \in H$ we have $[h, x] = 2x, [h, y] = -2y$.*

By (d),(e) we can find $y \in L_{-\alpha}$ such that $\kappa(x, y) = 2/\alpha(t_\alpha)$. Then $h = 2t_\alpha/\alpha(t_\alpha)$. Hence

$$[h, x] = (2/\alpha(t_\alpha))[t_\alpha, x] = (2/\alpha(t_\alpha))\alpha(t_\alpha)x = 2x,$$

$$[h, y] = (2/\alpha(t_\alpha))[t_\alpha, y] = (2/\alpha(t_\alpha))(-\alpha(t_\alpha)y) = -2y.$$

(g) *Let $\alpha \in R$. Let x, y, h be as in (f). Then $S = kx + ky + kh$ is a Lie subalgebra of L and $e \rightarrow x, f \rightarrow y, h \rightarrow h$ is an isomorphism of Lie algebras $\mathfrak{sl}_2(k) \xrightarrow{\sim} S$.*

This is clear.

(h) *Let $\alpha \in R$. Let $h_\alpha = 2t_\alpha/\alpha(t_\alpha)$ (see (f)). We have $h_\alpha = -h_{-\alpha}$.*

It suffices to show that $t_\alpha = -t_{-\alpha}$. Since $\kappa|_H$ is non-singular it suffices to show that, for any $h \in H$ we have $\kappa(h, t_\alpha) = -\kappa(h, t_{-\alpha})$ or that $\alpha(h) = -(-\alpha(h))$. This is clear.

(i) *Let $\alpha \in R$. Then $2\alpha \notin R$.*

Let x, y, h, S be as in (g). Let $M = \bigoplus_{c \in k} L_{c\alpha}$ is an S -module under ad and h acts on $L_{c\alpha}$ as multiplication by $c\alpha(h) = c\alpha(2t_\alpha)/\alpha(t_\alpha) = 2c$. By representation theory of $\mathfrak{sl}_2(k)$ (see below) the eigenvalues of $h : M \rightarrow M$ are integers. Hence $M = \bigoplus_{c \in \mathbf{Z}/2} L_{c\alpha}$. Now $S + H$ is an S -submodule of M . By Weyl, there exists an S -submodule M' of M such that $M = (S + H) \oplus M'$. Now the 0-eigenspace of $h : M \rightarrow M$ is $L_0 = H$ hence it is contained in $S + H$. Thus the 0-eigenspace of

$h : M' \rightarrow M'$ is 0. Hence $h : M' \rightarrow M'$ does not have eigenvalues in $2\mathbf{Z}$. The eigenvalues of $h : S + H \rightarrow S + H$ are $0, 2, -2$. We see that 4 is not an eigenvalue of $h : M \rightarrow M$.

If we had $2\alpha \in R$ then a non-zero-vector in $L_{2\alpha}$ would be an eigenvector of $h : M \rightarrow M$ with eigenvalue 4, contradiction.

(j) *Let $\alpha \in R$. Then $\alpha/2 \notin R$.*

If we had $\alpha/2 \in R$ then applying (i) to $\alpha/2$ we would deduce that $\alpha \notin R$, contradiction.

(k) *In (i) we have $M' = 0$.*

From (j) we see that $L_{\alpha/2} = 0$ hence the 1-eigenspace of $h : M \rightarrow M$ is 0. Thus $h : M' \rightarrow M'$ has no eigenvalue 1 (nor 0, see (i)). Hence $h : M' \rightarrow M'$ has no odd or even eigenvalues. Hence $M' = 0$.

(l) *Let $\alpha \in R$. We have $\dim L_\alpha = 1$. Moreover $c\alpha \in R, c \in k$ implies $c \in \{1, -1\}$.*

Let x, y, h, S be as in (g). By (k) we have $\bigoplus_{c \in k} L_{c\alpha} = S + H$. The result follows.

(m) *Let $\alpha, \beta \in R, \beta \neq \pm\alpha$. Let $h_\alpha = 2t_\alpha/\alpha(t_\alpha)$. Then $\beta(h_\alpha) \in \mathbf{Z}$ and $\{n \in \mathbf{Z} | \beta + n\alpha \in R\}$ is of the form $\{-r, -r+1, \dots, 0, \dots, q-1, q\}$ where $-r \leq 0 \leq q$.*

Let x, y, h, S be as in (g). Then $h = h_\alpha$. Let $K = \bigoplus_{n \in \mathbf{Z}} L_{\beta+n\alpha} \subset L$. This is an $S = \mathfrak{sl}_2(k)$ -module under ad such that any eigenvalue of h on $L_{\beta+n\alpha}$ is $\beta(h_\alpha) + 2n$. For $n = 0$ the eigenvalue is $\beta(h_\alpha)$ and it has multiplicity > 0 hence $\beta(h_\alpha) \in \mathbf{Z}$. We see that all eigenvalues have multiplicity one and they all have the same parity. It follows that the S -module K is simple. (See below.) The result follows.

(n) *Let $\alpha, \beta \in R, \beta \neq \pm\alpha$. Let $h_\alpha = 2t_\alpha/\alpha(t_\alpha)$. We have $\beta - \beta(h_\alpha)\alpha \in R$.*

By (m), we have $\beta(h_\alpha) - 2r = -(\beta(h_\alpha) + 2q)$ that is $\beta(h_\alpha) = r - q$ and we must show that $-r \leq -\beta(h_\alpha) \leq q$ that is $-r \leq -r + q \leq q$. This is clear.

(o) *If $\alpha, \beta, \alpha + \beta \in R$ then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.*

Since $2\alpha \notin R$ we have $\beta \neq \pm\alpha$. Consider the irreducible $S = \mathfrak{sl}_2(k)$ -module K in (m). With notations in (m) we have $L_\beta = (r - q)$ -eigenspace of $h : K \rightarrow K$ and $L_{\alpha+\beta} = (r - q + 2)$ -eigenspace of $h : K \rightarrow K$. It is enough to show that $e \in \mathfrak{sl}_2(k)$ maps the j -eigenspace of $h : K \rightarrow K$ onto the $(j + 2)$ -eigenspace (if both these eigenspaces are 1-dimensional). This follows from the explicit description of simple $\mathfrak{sl}_2(k)$ -modules (see below).

(p) *The smallest Lie subalgebra L' of L that contains L_α for all $\alpha \in R$ is L itself.*

It suffices to show that L' contains H . From (a) it follows that $\{t_\alpha | \alpha \in R\}$ spans H as a vector space. Hence it is enough to show that for $\alpha \in R$ we have $t_\alpha \in L'$. But by (c),(d) we have $t_\alpha \in [L_\alpha, L_{-\alpha}]$.

Rationality.

Define $(,) : H^* \times H^* \rightarrow k$ to be the symmetric bilinear form $(\xi, \xi') = \kappa(t_\xi, t_{\xi'}) = \sum_{\alpha \in R} \alpha(t_\xi)\alpha(t_{\xi'})$. This form is non-singular. For $\alpha \in R$ we have $(\xi, \alpha) = \kappa(t_\xi, t_\alpha) = \alpha(t_\xi)$. Hence $(\xi, \xi') = \sum_{\alpha \in R} (\xi, \alpha)(\xi', \alpha)$.

For $\alpha \in R$ we have $(\alpha, \alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$. For $\alpha, \beta \in R$ we have

$$2(\alpha, \beta)/(\alpha, \alpha) = \kappa(2t_\alpha/\kappa(t_\alpha, t_\alpha), t_\beta) = \kappa(h_\alpha, t_\beta) = \beta(h_\alpha) \in \mathbf{Z}.$$

Now from $(\beta, \beta) = \sum_{\alpha \in R} (\beta, \alpha)^2$ we deduce $4(\beta, \beta)^{-1} = \sum_{\alpha \in R} (2(\beta, \alpha)/(\beta, \beta))^2 \in \mathbf{Z}$. Thus $(\beta, \beta) \in \mathbf{Q}$ hence $(\alpha, \beta) \in \mathbf{Q}$ for any $\alpha, \beta \in R$.

Let E be the \mathbf{Q} -subspace of H^* spanned by R . Let $\alpha_1, \dots, \alpha_n$ be a k -basis of H^* contained in R . Let $\alpha \in R$. We have $\alpha = \sum_{i=1}^n c_i \alpha_i$ with $c_i \in k$. It suffices to show that $c_i \in \mathbf{Q}$ for all i . For any $j \in [1, n]$ we have

$$2(\alpha, \alpha_j)/(\alpha_j, \alpha_j) = \sum_{i=1}^n c_i 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j).$$

This is a linear system of n equations with n unknowns c_i with non-zero determinant and integer coefficients. Hence $c_i \in \mathbf{Q}$ for all i . Hence E coincides with the \mathbf{Q} -subspace of H^* spanned by $\alpha_1, \dots, \alpha_n$.

Let $\xi \in E, \xi \neq 0$. We have $(\xi, \xi) = \sum_{\alpha \in R} (\xi, \alpha)^2$. This is a rational number ≥ 0 . If it 0 then $(\xi, \alpha) = 0$ for all $\alpha \in R$ hence $\xi = 0$. Thus $(,)|_E$ has rational values and is positive definite.

We may summarize the properties of $R \subset E$ and $(,)|_E$ as follows:

R spans E as a \mathbf{Q} -vector space, $0 \notin R$. If $\alpha \in R$ then $-\alpha \in R$ but $c\alpha \notin R$ if $c \in \mathbf{Q} - \{1, -1\}$. If $\alpha, \beta \in R$ then $\frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbf{Z}$ and $\beta - 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha \in R$.