LIE ALGEBRAS

Construction of a Lie algebra attached to a root system.

We fix a root system $(E, (\,,\,), R)$ and a set $\Pi = \{\alpha_i\}_{i\in I}$ of simple roots. Let $a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$.

Let F be the free Lie algebra on generators $\{e_i, f_i, h_i | i \in I\}$. Let K be the ideal of F generated by the elements

$[h_i, h_j], [e_i, f_j] - \delta_{ij} h_i, [h_i, e_j] - a_{ji} e_j, [h_i, f_j] + a_{ji} f_j.$
Let $\bar{F} = F/K$. The images of e_i, f_i, h_i in F_0 are denoted by the same letters.

Let T be the tensor algebra on a vector space with basis $\{v_i; i \in I\}$. We write $v_{i_1} v_{i_2} \dots v_{i_s}$ instead of $v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_s}$. (These form a basis of T.) Define linear maps $h_j, e_j, f_j : T \longrightarrow T$ by:

$h_j(v_{i_1} v_{i_2} \dots v_{i_s}) = -(a_{i_1 j} + \dots + a_{i_s j}) v_{i_1} v_{i_2} \dots v_{i_s};$
$f_j(v_{i_1} v_{i_2} \dots v_{i_s}) = v_j v_{i_1} v_{i_2} \dots v_{i_s}$
$e_j(v_{i_1} v_{i_2} \dots v_{i_s}) = -\sum_{k=1}^s \delta_{j,i_k} (a_{i_{k+1} j} + \dots + a_{i_s j}) v_{i_1} v_{i_2} \dots \hat{v}_{i_k} \dots v_{i_s}.$

Lemma. *$[h_i, h_j] = 0, [e_i, f_j] = \delta_{ij} h_i, [h_i, e_j] = a_{ji} e_j,\ [h_i, f_j] = -a_{ji} f_j$ as maps $T \longrightarrow T$. Hence h_j, f_j, e_j define an $\bar{F}$-module structure on T.*

The relation $[h_i, h_j] = 0$ is obvious. We have

$$
\begin{aligned}
&(e_i f_j - f_j e_i)(v_{i_1} v_{i_2} \dots v_{i_s}) \\
&= e_i(v_j v_{i_1} v_{i_2} \dots v_{i_s}) + + \sum_{k=1}^s \delta_{i,i_k} (a_{i_{k+1} i} + \dots + a_{i_s i}) v_j v_{i_1} v_{i_2} \dots \hat{v}_{i_k} \dots v_{i_s} \\
&= -\delta_{ij} (a_{i_1 i} + \dots + a_{i_s i}) v_{i_1} v_{i_2} \dots v_{i_s} - \sum_{k=1}^s \delta_{i,i_k} (a_{i_{k+1} i} + \dots + a_{i_s i}) v_j v_{i_1} v_{i_2} \dots \hat{v}_{i_k} \dots v_{i_s} \\
&+ \sum_{k=1}^s \delta_{i,i_k} (a_{i_{k+1} i} + \dots + a_{i_s i}) v_j v_{i_1} v_{i_2} \dots \hat{v}_{i_k} \dots v_{i_s} \\
&= -\delta_{ij} (a_{i_1 i} + \dots + a_{i_s i}) v_{i_1} v_{i_2} \dots v_{i_s} = \delta_{ij} h_i (v_{i_1} v_{i_2} \dots v_{i_s}). \qquad \blacksquare
\end{aligned}
$$

Thus, $[e_i, f_j] = \delta_{ij} h_i$ holds. We have

$$
\begin{aligned}
&(h_i e_j - e_j h_i)(v_{i_1} v_{i_2} \dots v_{i_s}) \\
&= -\sum_{k=1}^{s} \delta_{j,i_k} (a_{i_{k+1}j} + \dots + a_{i_s j}) h_i (v_{i_1} v_{i_2} \dots \hat{v}_{i_k} \dots v_{i_s}) \\
&- (a_{i_1 i} + \dots + a_{i_s i}) \sum_{k=1}^{s} \delta_{j,i_k} (a_{i_{k+1}j} + \dots + a_{i_s j}) v_{i_1} v_{i_2} \dots \hat{v}_{i_k} \dots v_{i_s} \\
&= \sum_{k=1}^{s} \delta_{j,i_k} (a_{i_{k+1}j} + \dots + a_{i_s j})(a_{i_1 i} + \dots + \hat{a}_{i_k i} + \dots + a_{i_s i})(v_{i_1} v_{i_2} \dots \hat{v}_{i_k} \dots v_{i_s}) \\
&- (a_{i_1 i} + \dots + a_{i_s i}) \sum_{k=1}^{s} \delta_{j,i_k} (a_{i_{k+1}j} + \dots + a_{i_s j}) v_{i_1} v_{i_2} \dots \hat{v}_{i_k} \dots v_{i_s} \\
&= \sum_{k=1}^{s} \delta_{j,i_k} (a_{i_{k+1}j} + \dots + a_{i_s j}) - a_{i_k i} (v_{i_1} v_{i_2} \dots \hat{v}_{i_k} \dots v_{i_s}) \\
&= -a_{ji} \sum_{k=1}^{s} \delta_{j,i_k} (a_{i_{k+1}j} + \dots + a_{i_s j})(v_{i_1} v_{i_2} \dots \hat{v}_{i_k} \dots v_{i_s}) = a_{ji} e_j (v_{i_1} v_{i_2} \dots v_{i_s}).
\end{aligned}
$$

Thus, $[h_i, e_j] = a_{ji} e_j$ holds. We have

$$
\begin{aligned}
&(h_i f_j - f_j h_i)(v_{i_1} v_{i_2} \dots v_{i_s}) \\
&= h_i (v_j v_{i_1} v_{i_2} \dots v_{i_s}) + (a_{i_1 i} + \dots + a_{i_s i}) f_j (v_{i_1} v_{i_2} \dots v_{i_s}) \\
&= -(a_{ji} + a_{i_1 i} + \dots + a_{i_s i}) v_j v_{i_1} v_{i_2} \dots v_{i_s} + (a_{i_1 i} + \dots + a_{i_s i}) v_j v_{i_1} v_{i_2} \dots v_{i_s} \\
&= -a_{ji} v_j v_{i_1} v_{i_2} \dots v_{i_s} = -a_{ji} f_j (v_{i_1} v_{i_2} \dots v_{i_s}).
\end{aligned}
$$

Thus, $[h_i, f_j] = -a_{ji} f_j$ holds. The lemma is proved.

Lemma. *Let $\bar{F}^+$ be the Lie subalgebra of $\bar{F}$ generated by $\{e_i | i \in I\}$. Let $\bar{F}^-$ be the Lie subalgebra of $\bar{F}$ generated by $\{f_i | i \in I\}$. Let $\bar{F}^0$ be the subspace of $\bar{F}$ spanned by $\{h_i | i \in I\}$. Then $\bar{F} = \bar{F}^- \oplus \bar{F}^0 \oplus \bar{F}^+$ and $\{h_i | i \in I\}$ is a basis of $\bar{F}^0$.*

By the previous lemma there is a unique Lie algebra homomorphism $\phi : \bar{F} \longrightarrow End(T)$ which sends e_i, f_i, h_i to the endomorphisms described above.

(a) *The elements $\{h_i | i \in I\}$ are linearly independent in $\bar{F}$.*

Assume that $\sum_i c_i h_i = 0$ in $\bar{F}$. Applying this to v_j gives $\sum_i c_i a_{ji} v_j = 0$ in T hence $\sum_i c_i a_{ij} = 0$. As (a_{ij}) is nonsingular, we have $c_j = 0$.

(b) *Let $e(s)$ be an iterated bracket of s elements $e_{i_1}, \dots, e_{i_s}$. Then $[h_j, e(s)] = (a_{i_1 j} + \dots + a_{i_s j}) e(s)$ in $\bar{F}$.*

We argue by induction on s. For $s = 1$ this follows from the definition. Assume that $s \geq 1$. It is enough to show: if $[h_j, x] = ax$ and $[h_j, x'] = a'x'$ then $[h_j, [x, x']] = (a + a')[x, x']$ (use Jacobi).

(c) *Let $f(s)$ be an iterated bracket of s elements $f_{i_1},\dots,f_{i_s}$. Then $[h_j, f(s)] = -(a_{i_1 j}+\dots+a_{i_s j})f(s)$ in $\bar F$.*

As in (b).

(d) *If $s'>s>0$ then $[f(s), e(s')]$ is a linear combination of $e(s'-s)$.*

Assume first that $s=1$. We argue by induction on s'. For $s'=2$ we have

$[f_j,[e_i,e_{i'}]] = [[f_j,e_i],e_{i'}]+[e_i,[f_j,e_{i'}]] = [-\delta_{ij}h_i,e_{i'}]-[e_i,\delta_{i',j}h_j] \in \mathbf{k}e_{i'}+\mathbf{k}e_i$.

Hence the result holds in this case. Assume now that $s'>2$. Then $e(s') = [e(s''),e(\tilde s'')]$ where $s''+\tilde s''=s'$, $s''<s'$, $\tilde s''=s'$. If $s''>1, \tilde s''>1$ the result follows from the induction hypothesis using Jacobi. If $s''=1$ we have $e(s')=[e_k,e(s'-1)]$ hence

$[f_j,[e_k,e(s'-1)]] = [[f_j,e_k],e(s'-1)]+[e_k,[f_j,e(s'-1)]] = const[h_k,e(s'-1)]+[e_k,\text{lin.comb}e(s'-2)] = \text{lin.comb}e(s'-1)$.

We now assume that s' is fixed and use induction on s.

(e) *If $s>s'>0$ then $[f(s),e(s')]$ is a linear combination of $f(s-s')$.*

Same proof as (d).

(f) *For $s\ge 1$, $[f(s),e(s)]\in\sum_i \mathbf{k}h_i$.*

Induction on s. If $s=1$ this is clear. Assume that $s\ge 2$. Then $f(s) = [f(s'),f(s-s')]$ where $0<s'<s$. Then

$[f(s),e(s)] = [[f(s'),e(s)],f(s-s')]+[f(s'),[f(s-s'),e(s)] = \text{lin.comb.}[e(s-s'),f(s-s')]+\text{lin.comb.}[f(s'),e(s')] = \text{lin.comb.}h_i$.

(g) *$\bar F^- + \bar F^0 + \bar F^+$ is a Lie subalgebra of $\bar F$ hence it is $\bar F$.*

Follows from the previous points.

(h) *The sum $\bar F^- + \bar F^0 + \bar F^+$ is direct.*

For $\lambda\in E$ let

$\bar F_\lambda = \{x\in\bar F; [h_i,x] = \frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}x \forall i\in I\}$.

We have $e(s)\in\bar F_{\alpha_{i_1}+\dots+\alpha_{i_s}}$, $f(s)\in\bar F_{-\alpha_{i_1}-\dots-\alpha_{i_s}}$, $h_j\in\bar F_0$. It remains to observe that $\alpha_{i_1}+\dots+\alpha_{i_s}\ne 0$ if $s>0$.

Lemma. *Let $i\ne j$ in I. Let $k\in I$. We have $(ade_k)(adf_i)^{-a_{ji}+1}(f_j)=0$ in $\bar F$.*

Assume first that $k\ne i$. Then $[e_k,f_i]=0$ hence ade_k, adf_i commute and $(ade_k)(adf_i)^{-a_{ji}+1}(f_j) = (adf_i)^{-a_{ji}+1}(ade_k)(f_j) = (adf_i)^{-a_{ji}+1}[e_k,f_j]$. If $k\ne j$ this is 0 since $[e_k,f_j]=0$. If $k=j$ this is $(adf_i)^{-a_{ji}+1}h_j = (adf_i)^{-a_{ji}}a_{ij}f_i$. If $a_{ji}>0$ this is 0 since $[f_i,f_i]=0$. If $a_{ji}=0$ this is also 0.

Assume next that $k=i$. Let $m=-a_{ji}\ge 0$. One shows by induction on $t\ge 1$ that $(ade_i)(adf_i)^t(f_j) = t(m-t+1)(adf_i)^{t-1}(f_j)$. Taking $t=m+1$ gives $(ade_i)(adf_i)^t(f_j)=0$.

For $i\ne j$ let $X_{ij}=(ade_i)^{-a_{ji}+1}(e_j), Y_{ij}=(adf_i)^{-a_{ji}+1}(f_j)$ (in $\bar F$).

Let J^+ be the ideal of $\bar F^+$ generated by the X_{ij}. Let J^- be the ideal of $\bar F^-$ generated by the Y_{ij}. Let J be the ideal of $\bar F^-$ generated by the X_{ij},Y_{ij}. Let $L^-=\bar F^-/J^-, L^+=\bar F^+/J^+, L^0=\oplus_i\mathbf{k}h_i, L=\bar F/J$.

J^+, J^- are ideals of $\bar F$.

We prove this for J^-. It suffices to show that $[h_k, J^-] \subset J^-$, $[f_k, J^-] \subset J^-$, $[e_k, J^-] \subset J^-$. The first two inclusions are easy. The third follows from the previous lemma.

We have $J = J^- + J^+$.

Clearly, $J^- + J^+ \subset J$. We have seen that $J^- + J^+$ is an ideal of $\bar{F}$. It contains X_{ij}, Y_{ij} hence it contains J.

The obvious map $L^- \oplus L^0 \oplus L^+ \longrightarrow L$ is an isomorphism of vector spaces.

We have $L = \bar{F}/J = (\bar{F}^- \oplus \bar{F}^0 \oplus \bar{F}^+)/(J^- + J^+) = \bar{F}^-/J^- \oplus \bar{F}^0 \oplus \bar{F}^+/J^+$.

For any $\lambda \in E$ let $L_\lambda = \{x \in L; [h_i, x] = \frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)} x \forall i \in I\}$. Then

$L^- = \oplus_{\lambda;\lambda=-\alpha_{i_1}-\cdots-\alpha_{i_n}, n>0} L_\lambda$, $L^+ = \oplus_{\lambda;\lambda=\alpha_{i_1}+\cdots+\alpha_{i_n}, n>0} L_\lambda$, $L^0 = L_0$. Also each L_λ is finite dimensional.

For any i, ade_i and adf_i are locally nilpotent endomorphisms of L.

Let $M = \{x \in L; (ade_i)^n x = 0 \text{ for some } n \geq 1\}$. Then M is a Lie subalgebra of L. It contains e_k (since $X_{ij} = 0$ in L) and f_k. Hence $M = L$.

Let $\tau_i = \exp(ade_i)\exp(ad(-f_i))\exp(ade_i) : L \longrightarrow L$. This is a Lie algebra automorphism of L.

Let $\lambda, \mu \in E$ and let $w \in W$ (Weyl group) such that $w(\lambda) = \mu$. Then $\dim L_\lambda = \dim L_\mu$.

We may assume that $w = r_i$ (simple reflection). Then $\mu = \lambda + n\alpha_i$ for some n. Consider $V = \oplus_{m \in \mathbf{Z}} L_{\lambda + n\alpha_i}$. Then V is an sl_2 submodule of L (where sl_2 acts through e_i, f_i, h_i). By an earlier result, in this sl_2 module $\tau_i : V \longrightarrow V$ interchanges L_λ, L_μ.

For any i we have $e_i \neq 0, f_i \neq 0$ in L.

We already know that $h_i \neq 0$ in L. Now $\{e_i, f_i, h_i\}$ span a Lie subalgebra of L isomorphic to a quotient of sl_2 which is simple. Since $h_i \neq 0$ in L, the three elements e_i, f_i, h_i are linearly independent in L.

For any i, $\dim L_{\alpha_i} = 1$*; $L_{k\alpha_i} = 0$ for any integer $k \notin \{1, 0, -1\}$.*

Follows from the previous claim.

If $\alpha \in R$ then $\dim L_\alpha = 1$*; $L_{k\alpha} = 0$ for any integer $k \notin \{1, 0, -1\}$.*

Follows from the previous claim since $\alpha = w(\alpha_i)$ for some i, w.

Assume $\mu \in E$ satisfies $(\mu, \alpha) \neq 0$ for all $\alpha \in R$. Then there exists $w \in W$ such that $(w(\mu), \alpha_i) > 0$ for all i.

We pick any linear order on $P_\mu = \{e \in E | (e, \mu) = 0\}$, say $P_\mu = P_\mu^- \cup P_\mu^+ \cup \{0\}$. We define a linear order on E by $E = E^- \cup E^+ \cup \{0\}$ where $E^+ = \{e \in E | (e, \mu) > 0\} \cup P_\mu^+$, $E^- = \{e \in E | (e, \mu) < 0\} \cup P_\mu^-$. Then $R \cap E^+$ is a set of positive roots for R. It is $\{\alpha \in R; (\alpha, \mu) > 0\}$. For some $w \in W$ we have $\{\alpha \in R; (\alpha, \mu) > 0\} = w^{-1}(R^+)$. Thus, for any $\alpha \in R^+$ we have $(w^{-1}(\alpha), \mu) > 0$ that is $(w(\mu), \alpha) > 0$.

Assume that $\lambda = \sum_i k_i \alpha_i$ where $k_i \in \mathbf{Z}$ are all ≥ 0 or all ≤ 0. Assume that $\lambda \neq 0$ and λ is not a multiple of a root. Then there exists $w \in W$ such that $w(\lambda) = \sum_i k'_i \alpha_i$ where $k'_i \in \mathbf{Z}$ and some $k'_i > 0$, some $k'_i < 0$.

For each $e \in E$ let $P_e = \{e' \in E; (e', e) = 0\}$. Now P_λ is not contained in $\cup_\alpha P_\alpha$. Pick $\mu \in P_\lambda - \cup_\alpha P_\alpha$. Since $\mu \notin \cup_\alpha P_\alpha$ we can find $w \in W$ such that $(w(\mu), \alpha_i) > 0$

for all i. Then $0 = (\lambda, \mu) = (w(\lambda), w(\mu)) = \sum_i k'_i(\alpha_i, w(\mu))$. Hence some $k'_i > 0$, some $k'_i < 0$.

If $L_\lambda \neq 0$ then either $\lambda \in R$ or $\lambda = 0$.

We may assume that $\lambda \neq 0$. Then $\pm\lambda = \alpha_{i_1} + \cdots + \alpha_{i_n}$, $n > 0$. Assume that λ is not a multiple of a root. Let w be as in the previous claim. We must have $L_{w(\lambda)} = 0$. Hence $L_\lambda = 0$.

We have $\dim L = \sharp(I) + \sharp(R)$.

L is semisimple.

Let A be an abelian ideal of L. Since A is stable under adh_i we have $A = (A \cap L^0) + \sum_{\alpha\in R}(A \cap L_\alpha)$. Assume that $L_\alpha \subset A$ for some α. If $i \in I$ we have $\tau_i(A) \subset A$ hence $L_{r_i\alpha} \subset A$. It follows that $L_{w(\alpha)} \subset A$ for any $w \in W$. Hence $L_{\alpha_i} \subset A$ for some i hence $e_i \in A$. Since $[f_i, e_i] = h_i$ we have $h_i \in A$. Now $[h_i, e_i] = 2e_i \neq 0$ contradicting the fact that $[A, A] = 0$. Thus we have $A \cap L_\alpha = 0$ for any α. Hence $A = A \cap L^0$. Then $[L_{\alpha_i}, A] \subset L_{\alpha_i} \cap L^0 = 0$ for all i. Let $a = \sum_i c_i h_i \in A$. For any j we have $[\sum_i c_i h_i, e_j] = 0$ hence $\sum_i c_i a_{ji} = 0$. By the non-singularity of (a_{ij}) we have $c_i = 0$ for all i. Thus $a = 0$. We see that $A = 0$ hence L is semisimple.

L^0 is a maximal toral subalgebra of L.

Clearly $adx : L \longrightarrow L$ is semisimple for any $x \in L^0$ hence L is toral. Assume that L' is a toral subalgebra of L containing strictly L^0. Since L' contains L^0, it is a sum of L^0 and some L_α. Thus L' contains L_α for some α. Now any element of L_α is nilpotent. (We can reduce this to the case where $\alpha = \alpha_i$ using a succession of τ_j.) Since any element of L' is semisimple we see that $L_\alpha = 0$, contradiction.

Theorem. *L is a semisimple Lie algebra whose root system with respect to L^0 is the given one.*