LIE ALGEBRAS IV

ROOT SYSTEMS

We fix a finite dimensional $\mathbf{Q}$-vector space E. A *reflection* in E is a linear transformation $r : E \to E$ such that the (-1)-eigenspace of r has dimension 1 and the 1-eigenspace of r has codimension 1. Then $r^2 = 1$.

Assume now that E has a fixed positive definite symmetric bilinear form $(,) : E \times E \to \mathbf{Q}$. An *orthogonal reflection* in E is a reflection r which is an orthogonal transformation. If $l \in E - \{0\}$ is in the (-1) eigenspace of r then $r(e) = e - 2\frac{(e,l)}{(l,l)}l$ for all $e \in E$. Conversely, if $l \in E - \{0\}$, the previous formula defines an orthogonal reflection denoted r_l.

Definition. A finite subset R of E is said to be a *root system* if:

R spans E as a $\mathbf{Q}$-vector space, $0 \notin R$. If $\alpha \in R$ then $-\alpha \in R$ but $c\alpha \notin R$ if $c \in \mathbf{Q} - \{1, -1\}$. If $\alpha \in R$ then $r_\alpha(R) = R$. If $\alpha, \beta \in R$ then $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbf{Z}$.

For example, if L is a semisimple Lie algebra, H is a maximal toral subalgebra and $R \in H^*$ is the corresponding set of roots then R is a root system in the $\mathbf{Q}$-vector subspace of H^* spanned by R with its natural $(,)$.

We return to the general case. Let W be the subgroup of the group of orthogonal transformations $E \to E$ generated by the reflections $r_\alpha : E \to E, \alpha \in R$.

Clearly, if $w \in W$ then $w(R) = R$. Thus we have a natural homomorphism of W in the group of permutations of R. This homomorphism is injective since R spans E. Hence W is finite.

A *set of simple roots* for R is a basis Π of E such that $\Pi \subset R$ and any $\alpha \in R$ is either ≥ 0 or ≤ 0 combination of elements of Π. We will show below that such Π exists.

Let E^+ be a subset of E such that the following hold:

$E^+ + E^+ \subset E^+$;

$\mathbf{Q}_{>0}E^+ \subset E^+$;

$-E^+ = E^-$;

$E = E^+ \cup (-E^+) \cup \{0\}$ (disjoint union).

(Such E^+ exists; for example if $e_1, \ldots, e_n$ is a basis of E we may take E^+ to be the set of all $\sum_{i=1}^n c_i e_i$ in which the first non-zero $c_i \in \mathbf{Q}$ is > 0.)

Fix E^+ as above. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a subset of $R \cap E^+$ such that $R \cap E^+ \subset \sum_{i=1}^n \mathbf{Q}_{\geq 0}\alpha_i$ and Π is minimal with this property. The inclusion above is satisfied by R itself (instead of Π) hence Π exists.

We show that for distinct $\alpha, \alpha' \in \Pi$ we have $(\alpha, \alpha') \leq 0$. Assume that $(\alpha, \alpha') > 0$. Then $r_\alpha(\alpha') = \alpha' - \lambda\alpha$ where $\lambda > 0$.

If $r_\alpha(\alpha') \in R \cap E^+$ then $r_\alpha(\alpha') = \sum_i c_i\alpha_i$ where $c_i \geq 0$. Hence $\alpha' = \lambda\alpha + \sum_i c_i\alpha_i$. The coefficient of α' in the r.h.s. is < 1; otherwise, $0 = \lambda\alpha + \sum_i c_i\alpha_i - \alpha' \in E^+$, absurd. Thus α' can be expressed as a ≥ 0 combination of $\Pi - \{\alpha'\}$, absurd.

If $-r_\alpha(\alpha') \in R \cap E^+$ then $-r_\alpha(\alpha') = \sum_i c_i\alpha_i$ where $c_i \geq 0$. Hence $\lambda\alpha = \alpha' + \sum_i c_i\alpha_i$. The coefficient of α in the r.h.s. is $< \lambda$; otherwise, $0 = \alpha' + \sum_i c_i\alpha_i - \lambda\alpha \in E^+$, absurd. Thus α can be expressed as a ≥ 0 combination of $\Pi - \{\alpha\}$, absurd. Our claim is proved.

Assume that there is a non-trivial linear relation between the elements of Π. This is of the form $\sum_{j \in J} c_j\alpha_j = \sum_{k \in K} d_k\alpha_k$ where J, K are disjoint subsets of $[1, n]$, J or K is nonempty and all c_j, d_k are > 0. Let e be the l.h.s. or r.h.s. of the equality. Then $(e, e) = \sum_{j,k} c_j d_k(\alpha_j, \alpha_k) \leq 0$ so that $e = 0$. But $e \in E^+$. Contradiction. We see that Π linearly independent. Let $\alpha \in R$. If $\alpha \in R \cap E^+$ then $\alpha = \sum_{i=1}^n \mathbf{Q}_{\geq 0}\alpha_i$. If $-\alpha \in R \cap E^+$ then $-\alpha = \sum_{i=1}^n \mathbf{Q}_{\geq 0}\alpha_i$. In any case $R \subset \sum_{i=1}^n \mathbf{Q}\alpha_i$. Since R spans E we see that $E = \sum_{i=1}^n \mathbf{Q}\alpha_i$ so that Π is a basis of E. We see that Π is a set of simple roots of R. We have $\Pi \subset R \cap E^+$.

Assume that $\Pi' = \{\alpha'_1, \ldots, \alpha'_n\}, \Pi'' = \{\alpha''_1, \ldots, \alpha''_n\}$ are two sets of simple roots contained in $R \cap E^+$. We show that $\Pi' = \Pi''$. We have $\alpha'_i = \sum_j a_{ij}\alpha''_j, \alpha''_i = \sum_j b_{ij}\alpha'_j$ where $a_{ij} \geq 0, b_{ij} \geq 0$ and $(a_{ij}), (b_{ij})$ are inverse matrices. This forces $(a_{ij}), (b_{ij})$ to be monomial matrices and after a change of numbering, diagonal matrices. Hence $\alpha'_i = a_{ii}\alpha''_i$. It follows that $a_{ii} = 1$ and our claim is proved.

We see that any E^+ contains a unique set of simple roots. Conversely, any set Π of simple roots of R is contained in some E^+ (since any basis of E is contained in some E^+). It follows that, if Π is a set of simple roots for R then $(\alpha, \alpha') \leq 0$ for $\alpha \neq \alpha'$ in Π.

We fix a set $\Pi = \{\alpha_i | i \in I\}$ of simple roots for R; here I is an indexing set. For $i \in I$ we write r_i instead of r_{α_i}. We have $R = R^+ \cup R^-$ (disjoint union) where R^+ (resp. R^-) is the set of all $\alpha \in R$ which are ≥ 0 (resp. ≤ 0) combinations of elements of Π.

Let $\alpha \in \Pi$. Then $r_\alpha(\alpha) = -\alpha$ but $r_\alpha(\beta) \in R^+$ if $\beta \in R^+ - \{\alpha\}$.

Let β be as above. Then $\beta = \sum_i c_i\alpha_i$ where $c_i \geq 0$. We have $c_i > 0$ for some i with $\alpha_i \neq \alpha$ (otherwise, β is a multiple of α hence it equals α, absurd). Then the coefficient of α_i in $r_\alpha(\beta)$ is > 0 hence $r_\alpha(\beta) \in R^+$.

Any $\alpha \in R$ is a $\mathbf{Z}$-linear combination of elements of Π.

We may assume that $\alpha \in R^+$. Let $h(\alpha) = \sum_i c_i$ where $\alpha = \sum_i c_i\alpha_i, c_i \geq 0$. Then $h(\alpha) > 0$ has only finitely many possible values. We argue by induction on $h(\alpha)$.

If $\alpha \in \Pi$, the result is clear. Assume that $\alpha \notin \Pi$. Then at least two c_i are > 0. If $(\alpha, \alpha_i) \leq 0$ for all i then $(\alpha, \alpha) = \sum_i c_i(\alpha, \alpha_i) \leq 0$, contradiction. Thus there exists i such that $(\alpha, \alpha_i) > 0$. Then $r_i(\alpha) = \alpha - \frac{2(\alpha_i, \alpha)}{(\alpha_i, \alpha_i)}\alpha_i$ differs from α in only one coefficient. Thus at least one coefficient of $r_i(\alpha)$ is > 0 so that $r_i(\alpha) \in R^+$. Now

$h(r_i(\alpha)) < h(\alpha)$. By the induction hypothesis, $r_i(\alpha)$ is a $\mathbf{Z}$-linear combination of elements of Π hence $\alpha = r_i(\alpha) + \frac{2(\alpha_i, \alpha)}{(\alpha_i, \alpha_i)}\alpha_i$ is a $\mathbf{Z}$-linear combination of elements of Π. Thus we are reduced to the case where $h(\alpha)$ is minimal. In this case the argument above shows that $\alpha \in \Pi$ when the result is clear.

For any $\alpha \in R$ there exists $w \in W$ and $i \in I$ such that $\alpha = w(\alpha_i)$.

Let W' be the subgroup of W generated by $\{r_i | i \in I\}$. We show that for any $\alpha \in R$ there exists $w \in W'$ and $i \in I$ such that $\alpha = w(\alpha_i)$.

Assume first that $\alpha \in R^+$. We argue by induction on $h(\alpha)$. If $h(\alpha) = 1$ then $\alpha \in \Pi$ and the result is clear. Assume now that $h(\alpha) > 1$. As in the previous proof there exists i such that $(\alpha, \alpha_i) > 0$ and $r_i(\alpha) \in R^+$ has $h()$ strictly smaller than $h(\alpha)$. By the induction hypothesis, $r_i(\alpha) = w'(\alpha_j)$ for some $w' \in W', j \in I$. Hence $\alpha = r_i w'(\alpha_j)$ as desired.

Assume next that $\alpha \in R^+$. Then by the earlier part, $-\alpha = w(\alpha_i)$ for some $w \in W', i \in I$ hence $\alpha = wr_i(\alpha_i)$ as desired.

We have $W = W'$.

Let $\alpha \in R$. We must show that $r_\alpha \in W'$. Now $\alpha = w(\alpha_i)$ for some $w \in W', i \in I$. Hence $r_\alpha = wr_i w^{-1}$ (we use the fact that w is an isometry). Hence $r_\alpha \in W'$.

Length. For $w \in W$ let $X(w) = \{\alpha \in R^+ | w(\alpha) \in R^-\}$. Let $n(w) = \sharp X(w)$.

We have a bijection $X(w^{-1}) \xrightarrow{\sim} X(w)$ given by $\alpha \mapsto -w^{-1}(\alpha)$. Hence $n(w) = n(w^{-1})$.

Lemma. *Let $w \in W, i \in I$.*
  (a) If $w^{-1}(\alpha_i) \in R^+$ then $n(r_i w) = n(w) + 1$.
  (b) If $w^{-1}(\alpha_i) \in R^-$ then $n(r_i w) = n(w) - 1$.
  (c) If $w(\alpha_i) \in R^+$ then $n(wr_i) = n(w) + 1$.
  (d) If $w(\alpha_i) \in R^-$ then $n(wr_i) = n(w) - 1$.

We prove (a). Let $\alpha \in X(w)$. We show that $\alpha \in X(r_i w)$. We have $\alpha \in R^+, w(\alpha) \in R^-$. We must show that $r_i w(\alpha) \in R^-$. If not, then $w(\alpha) = -\alpha_i$, contradicting $w^{-1}(\alpha_i) \in R^+$. Thus, we have $X(w) \subset X(r_i w)$.

Now let $\alpha' \in X(r_i w)$. We show that $\alpha' \in X(w)$ except when $\alpha' = w^{-1}(\alpha_i)$ which is in $X(r_i w)$ but not in $X(w)$. We have $\alpha' \in R^+, r_i w(\alpha') \in R^-$. We must show that $w(\alpha') \in R^-$. If not, then $r_i w(\alpha') = -\alpha_i$ hence $w(\alpha') = \alpha_i$ which has been excluded. We see that $\sharp(X(r_i w) - X(w)) = 1$ and (a) follows.

We prove (b). Set $w' = r_i w$. Then $w'^{-1}(\alpha_i) \in R^+$ hence by (a), $n(r_i w') = n(w') + 1$ and (b) follows.

We prove (c). Set $w' = w^{-1}$. Then $w'^{-1}(\alpha_i) \in R^+$ hence by (a), $n(r_i w') = n(w') + 1$ hence $n(r_i w^{-1}) = n(w^{-1}) + 1$ hence $n(wr_i) = n(w) + 1$.

Similarly, (d) follows by applying (b) to w^{-1} instead of w.

Lemma. *For $w \in W$ let $l(w)$ be the minimum number k such that w is a product of k factors r_i with $i \in I$. We have $l(w) = n(w)$.*

Let $k = l(w)$. We argue by induction on k. When $k = 0$, the result is clear. Assume now that $k \geq 1$. We can write $w = r_i w'$ where $l(w') = k - 1$. By the

induction hypothesis, $n(w') = l(w')$. By the previous lemma, $n(w) \le n(w') + 1$. Hence $n(w) \le l(w') + 1 = l(w)$.

Assume now that $n(w) < k$. We have $w = r_{i_1} r_{i_2} \ldots r_{i_k}$ for some sequence $i_1, \ldots, i_k$ in I.

If $\alpha_{i_1}, r_{i_1}(\alpha_{i_2}), \ldots, r_{i_1} r_{i_2} \ldots r_{i_{k-1}}(\alpha_{i_k})$ are all in R^+, then, by the previous lemma,

$n(r_{i_1} r_{i_2} \ldots r_{i_{k-1}} r_{i_k}) > n(r_{i_1} r_{i_2} \ldots r_{i_{k-1}}) > n(r_{i_1} r_{i_2} \ldots r_{i_{k-2}}) > \cdots > n(r_{i_1})$,

hence $n(w) \ge k$, contradiction. Thus, for some $m \in [1, k]$, $r_{i_1} r_{i_2} \ldots r_{i_{m-1}}(\alpha_{i_m}) \in$ ▮ R^-.

Hence in the sequence $\alpha_{i_m}, r_{i_{m-1}}(\alpha_{i_m}), \ldots, r_{i_1} r_{i_2} \ldots r_{i_{m-1}}(\alpha_{i_m})$, which begins in R^+ and ends in R^-, there exist two consecutive terms, the first in R^+, the second in R^-. Thus we can find $n \in [1, m-1]$ such that

$r_{i_{n+1}} r_{i_{n+2}} \ldots r_{i_{m-1}}(\alpha_{i_m}) \in R^+$, $r_{i_n} r_{i_{n+1}} \ldots r_{i_{m-1}}(\alpha_{i_m}) \in R^-$.

Since r_{i_n} changes the sign of only $\alpha_{i_n}, -\alpha_{i_n}$ we have $r_{i_{n+1}} \ldots r_{i_{m-1}}(\alpha_{i_m}) = \alpha_{i_n}$. It follows that

$r_{i_n} = r_{i_{n+1}} \ldots r_{i_{m-1}} r_{i_m} r_{i_{m-1}} \ldots r_{i_{n+1}}$.

Hence

$r_{i_{n+1}} \ldots r_{i_{m-1}} r_{i_m} = r_{i_n} r_{i_{n+1}} \ldots r_{i_{m-1}}$.

We use this to write

$$w = r_{i_1} r_{i_2} \ldots r_{i_k} = r_{i_1} r_{i_2} \ldots r_{i_n} r_{i_n} r_{i_{n+1}} \ldots r_{i_{m-1}} r_{i_{m+1}} \ldots r_{i_k}$$

$$= r_{i_1} r_{i_2} \ldots r_{i_{n-1}} r_{i_{n+1}} \ldots r_{i_{m-1}} r_{i_{m+1}} \ldots r_{i_k}$$

which shows that $l(w) \le k - 2$, contradiction. The lemma is proved.

Lemma. *Let $w \in W$ be such that $w(\Pi) = \Pi$. Then $w = 1$.*

Indeed, $n(w) = 0$ hence $l(w) = 0$ hence $w = 1$.

Lemma. *(a) If Π is a set of simple roots and $w \in W$ then $w(\Pi)$ is a set of simple roots.*
(b) If Π_1, Π_2 are two sets of simple roots then there exists $w \in W$ such that $w(\Pi_1) = \Pi_2$.
(c) w in (b) is unique.

(a) is obvious. We prove (b). Let R_j^+ (resp. R_j^-) be the set of roots that are ≥ 0 (resp. ≤ 0) combinations of vectors in $\Pi_j, j = 1, 2$. We argue by induction on $n = |R_1^+ \cap R_2^-|$. If $n = 0$ we have $R_1^+ = R_2^+$ hence $\Pi_1 = \Pi_2$. We now assume $n > 0$. If every root in Π_1 is in R_2^+ then $R_1^+ \subset R_2^+$ hence $R_1^+ = R_2^+$ and $n = 0$. Thus there exists $\alpha \in \Pi_1 \cap R_2^-$. Then $r_\alpha(R_1^+) = (R_1^+ - \{\alpha\}) \cup \{-\alpha\}$. Hence $|r_\alpha(R_1^+) \cap R_2^-| = n - 1$. Now $r_\alpha(\Pi_1)$ is the set of simple roots contained in $r_\alpha(R_1^+)$ hence by the induction hypothesis there exists $w' \in W$ such that $w' r_\alpha(\Pi_1) = \Pi_2$. This proves (b). Now (c) follows from the previous lemma.

Pairs of roots. Let $\alpha, \beta \in R, \beta \ne \pm\alpha$; the angle θ angle between α, β is given by $\cos\theta = (\alpha, \beta)/\sqrt{(\alpha, \alpha)}\sqrt{(\beta, \beta)}$. Hence $4\cos^2\theta = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)} 2\frac{(\alpha, \beta)}{(\beta, \beta)}$. The right hand

side is integer and the left hand side is in $[0, 4]$. Hence the right hand side is $N \in \{0, 1, 2, 3, 4\}$. Also if $N = 4$ then θ is 0 or π and $\beta = \pm\alpha$ which we have excluded. Thus $N \in \{0, 1, 2, 3\}$. The possibilities are:

$2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$	$2\frac{(\alpha,\beta)}{(\beta,\beta)}$	θ
0	0	$\pi/2$
1	1	$\pi/3$
−1	−1	$2\pi/3$
1	2	$\pi/4$
2	1	$\pi/4$
−1	−2	$3\pi/4$
−2	−1	$3\pi/4$
1	3	$\pi/6$
3	1	$\pi/6$
−1	−3	$5\pi/6$
−3	−1	$5\pi/6$

Let α, β be as above. If $(\alpha, \beta) > 0$ then $\alpha - \beta \in R$. If $(\alpha, \beta) < 0$ then $\alpha + \beta \in R$.

It suffices to prove the first statement. In that case $2\frac{(\beta,\alpha)}{(\alpha,\alpha)}, 2\frac{(\alpha,\beta)}{(\beta,\beta)}$ are integers > 0 hence from the table one of them is 1. If $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} = 1$ then $r_\alpha(\beta) = \beta - \alpha \in R$. If $2\frac{(\alpha,\beta)}{(\beta,\beta)} = 1$ then $r_\beta(\alpha) = \alpha - \beta \in R$. The claim follows.

Remark. Assume that $(\alpha, \alpha) \leq (\beta, \beta)$. From the table above we see that:

$$\frac{(\beta,\beta)}{(\alpha,\alpha)} = \frac{2\frac{(\beta,\alpha)}{(\alpha,\alpha)}}{2\frac{(\alpha,\beta)}{(\beta,\beta)}} = 4\frac{(\beta,\alpha)^2}{(\alpha,\alpha)(\beta,\beta)} \in \{1, 2, 3\}.$$

Dual root system. Let $E, (,)$ be as above and let $R \subset E$ be a root system. For any $\alpha \in R$ we set $\check{\alpha} = \frac{2\alpha}{(\alpha,\alpha)}$. The vectors $\{\check{\alpha} | \alpha \in R\}$ are distinct: if $\alpha, \beta \in R$ and $\frac{2\alpha}{(\alpha,\alpha)} = \frac{2\beta}{(\beta,\beta)}$ then $\beta = c\alpha$ with $c > 0$ hence $c = 1$ and $\beta = \alpha$. Let $\check{R} = \{\check{\alpha} | \alpha \in R\}$. We show that $\check{R}$ is a root system in E. Clearly $\check{R}$ spans E and does not contain 0. If $\alpha \in R$ then $(-\alpha)\check{} = -\check{\alpha}$. If $c\check{\alpha} = \check{\beta}$ where $\alpha, \beta \in R, c \in \mathbf{Q}$ then $c\frac{2\alpha}{(\alpha,\alpha)} = \frac{2\beta}{(\beta,\beta)}$ hence $c\frac{(\beta,\beta)}{(\alpha,\alpha)} = \pm 1$ hence $\alpha = \pm\beta$ hence $\check{\alpha} = \pm\check{\beta}$ hence $c = \pm 1$. Let $\alpha, \beta \in R$. We

have

$$r_{\check{\alpha}}(\check{\beta}) = \check{\beta} - \frac{2(\check{\beta}, \check{\alpha})}{(\check{\alpha}, \check{\alpha})}\check{\alpha} = \frac{2\beta}{(\beta, \beta)} - \frac{2(\frac{2\beta}{(\beta,\beta)}, \frac{2\alpha}{(\alpha,\alpha)})}{(\frac{2\alpha}{(\alpha,\alpha)}, \frac{2\alpha}{(\alpha,\alpha)})}\frac{2\alpha}{(\alpha, \alpha)}$$

$$= \frac{2\beta}{(\beta, \beta)} - \frac{4(\beta, \alpha)}{(\beta, \beta)(\alpha, \alpha)}\alpha = 2\frac{\beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha}{(\beta, \beta)} = \frac{2r_\alpha(\beta)}{(r_\alpha(\beta), r_\alpha(\beta))} = (r_\alpha(\beta))\check{} \in \check{R}.$$

The previous computation shows that
$\frac{2(\check{\beta}, \check{\alpha})}{(\check{\beta}, \check{\alpha})} = \frac{2(\beta, \alpha)}{(\beta, \beta)} \in \mathbf{Z}.$

We see that $\check{R}$ is a root system in E. It is called the root system *dual* to R.

Remark. Let $E, (,)$ be as above. Let $\mathbf{E} = \mathbf{R} \otimes_\mathbf{Q} E$. There is an induced symmetric bilinear form on $\mathbf{E}$ with value in $\mathbf{R}$ denoted again by $(,)$. It is defined by $(\sum_i c_i e_i, \sum_j c'_j e_j) = \sum_{i,j} c_i c'_j (e_i, e_j)$ where (e_i) is a $\mathbf{Q}$-basis of E, $c_i, c'_i \in \mathbf{R}$ and $(e_i, e_j) \in \mathbf{Q}$ is the original $(,)$ on E. This is independent of the choice of (e_i). We show that $(,) : \mathbf{E} \times \mathbf{E} \to \mathbf{R}$ *is positive definite* that is, $(e, e) > 0$ for any $e \in \mathbf{E} - \{0\}$. Indeed, we can find a $\mathbf{Q}$-basis (e_i) of E such that $(e_i, e_j) = a_i \delta_{ij}$ where $a_i \in \mathbf{Q}_{>0}$. If $e \in \mathbf{R}$ is $e = \sum_i c_i e_i$ with $c_i \in \mathbf{R}$ (not all 0) then $(e, e) = \sum_i c_i^2 > 0$.

Admissible sets. Let $\mathbf{E}$ be a finite dimensional $\mathbf{R}$ vector space with a positive definite symmetric bilinear form $(,)$ and with an *admissible set*, that is a linearly independent set $(e_i)_{i \in I}$ such that

$(e_i, e_i) = 1$ for all i,

$(e_i, e_j) \leq 0$ for all $i \neq j$,

$4(e_i, e_j)^2 \in \{0, 1, 2, 3\}$ for all $i \neq j$.

(Such a set is obtained by taking a set of simple roots for a root system and dividing the vectors in this set by their length.)

We define a graph: the set of vertices is I and we join $i \neq j$ by $4(e_i, e_j)^2$ edges.

(1) *If $I = I' \sqcup J$ then $(e_i)_{i \in I'}$ is again an admissible set and the corresponding graph is obtained by removing from the original graph the vertices in J and the edges with one end in J.*

(2) *Let X be the set of unordered pairs (i, j) of joined vertices. We have $|X| < |I|$.*

Let $e = \sum_i e_i$. Then $e = 0$ hence $0 < (e, e) = |I| + 2\sum_X (e_i, e_j)$. In the sum we have $4(e_i, e_j)^2 \in \{1, 2, 3\}$ hence $2(e_i, e_j) \leq -1$. Hence $0 < |I| - |X|$.

(3) *The graph contains no cycles.*

If it does, we may remove the vertices outside that cycle (by (1)) and we get a graph with $|X| \geq |I|$; absurd by (2).

(4) *Let $i \in I$. There are at most three edges touching i.*

Let $\{j_1, j_2, \ldots, j_s\}$ be the vertices joined with i. Then $(e_i, e_{j_p}) < 0$ for $p \in [1, s]$. By (3), no two vertices in $\{j_1, j_2, \ldots, j_s\}$ are connected. Hence $(e_{j_p}, e_{j_{p'}}) = 0$ for $p \neq p'$. Let $f \in \mathbf{R}e_i + \mathbf{R}e_{j_1} + \cdots + \mathbf{R}e_{j_s}$ be such that $(f, e_{j_p}) = 0$ for all p and $(f, f) = 1$. Clearly $(e_i, f) \neq 0$. We have $e_i = (e_i, f)f + (e_i, e_{j_1})e_{j_1} + \cdots + (e_i, e_{j_s})e_{j_s}$. Hence $1 = (e_i, e_i) = (e_i, f)^2 + (e_i, e_{j_1})^2 + \cdots + (e_i, e_{j_s})^2$. Hence $(e_i, e_{j_1})^2 + \cdots +$

$(e_i, e_{j_s})^2 < 1$. Hence $4(e_i, e_{j_1})^2 + \cdots + 4(e_i, e_{j_s})^2 < 4$. The left hand side is the number of edges touching i.

(5) *Assume that $i \neq j$ are joined by three edges. Then i, j form a connected component of our graph.*

Follows from (4).

(6) *Assume that $i_1, i_2, \ldots i_s$ in I are distinct and satisfy: i_1, i_2 are joined by one edge, i_2, i_3 are joined by one edge,..., i_{s-1}, i_s are joined by one edge, but there are no other edges between these. Then $\{e_i | i \notin \{i_1, i_2, \ldots, i_s\} \cup \{e\}$ (where $e = e_{i_1} + e_{i_2} + \cdots + e_{i_s}$) is an admissible set whose graph is obtained from the original graph by collapsing $\{i_1, i_2, \ldots, i_s\}\}$ to a point.*

We have $(e, e) = s - 2(s - 1)\frac{1}{2} = 1$. If $i \notin \{i_1, i_2, \ldots, i_s\}$, then $(e, e_i) = \sum_{p=1}^{s}(e_{i_p}, e_i)$. By (3) there is at most one p in the sum with $(e_{i_p}, e_i) \neq 0$. Thus either $(e, e_i) = 0$ or $(e, e_i) = (e_{i_p}, e_i)$ for some p. The result follows.

A *branch point* is a vertex joined with at least 3 other vertices (hence with exactly 3 vertices, see (4).

(7) *Let I' be a connected component of our graph. (a) I' cannot contain two double edges. (b) I' cannot contain a double edge and a branch point. (c) I' cannot contain two branch points.*

If it did, then droping some vertices (1) and collapsing a subgraph (as in (6)) we see that we may assume that I' has two double edges that are next to each other or a branch point from which a double edge starts or a vertex joined with four other vertices. In each case we have a contradiction with (4).

(8) *Consider a connected component of our graph with set of vertices I'. Then in I':*

(a) any two vertices are joined by 0 or 1 edges and there is no branch point; or

(b) any two vertices are joined by 0 or 1 edges (except one pair that is joined by 2 edges) and there is no branch point; or

(c) any two vertices are joined by 0 or 1 edges and there is one branch point where 3 edges meet; or

(d) there are only two vertices in I' and they are joined by 3 edges.

Follows from (7) and (5).

(9) *In (8)(b), write $I' = \{i_1, \ldots, i_p, j_q, \ldots, j_1\}$ where i_1, i_2 are joined by one edge, i_2, i_3 are joined by one edge,..., i_{p-1}, i_p are joined by one edge, i_p, j_q are joined by two edges, j_q, j_{q-1} are joined by one edge,..., j_2, j_1 are joined by one edge and there are no other edges. Here $p \geq 1, q \geq 1$. Then either $p = 1$ or $q = 1$ or $p = q = 2$.*

Let $e' = \sum_{u=1}^{p} u e_{i_u}$, $e'' = \sum_{u=1}^{q} u e_{j_u}$. Then $(e', e') = \sum_{u=1}^{p} u^2 - \sum_{u=1}^{p-1} u(u + 1) = p(p + 1)/2$. Similarly, $(e'', e'') = q(q + 1)/2$. Now $4(e_p, e_q)^2 = 2$ hence $(e', e'')^2 = p^2 q^2 (e_p, e_q)^2 = p^2 q^2/2$. By the Schwarz inequality we have $(e', e'')^2 < (e', e')(e'', e'')$ hence $p^2 q^2/2 < p(p + 1)q(q + 1)/4$ hence $(p - 1)(q - 1) < 2$. Hence if $p > 1$ and $q > 1$ then $p = q = 2$.

(10) *In (8)(c) write $I' = \{i_1, \ldots, i_{p-1}; j_1, \ldots, j_{q-1}, k_1, \ldots, k_{r-1}, l\}$ where i_1, i_2 are joined by one edge, i_2, i_3 are joined by one edge,..., i_{p-2}, i_{p-1} are joined by one*

edge; j_1, j_2 are joined by one edge, j_2, j_3 are joined by one edge,..., j_{q-2}, j_{q-1} are joined by one edge; k_1, k_2 are joined by one edge, k_2, k_3 are joined by one edge,..., k_{p-2}, k_{p-1} are joined by one edge, i_{p-1}, l are joined by one edge, j_{q-1}, l are joined by one edge, k_{r-1}, l are joined by one edge, and there are no other edges. Here $p \geq 2, q \geq 2, r \geq 2$. Then up to a permutation, (p, q, r) is $(p, 2, 2)$ or $(3, 3, 2)$ or $(4, 3, 2)$ or $(5, 3, 2)$.

Let $e = \sum_{u=1}^{p-1} ue_{i_u}$, $e' = \sum_{u=1}^{q-1} ue_{j_u}$, $e'' = \sum_{u=1}^{q-1} ue_{k_u}$. As in (9) we have $(e, e) = p(p-1)/2$, $(e', e') = q(q-1)/2$, $(e'', e'') = r(r-1)/2$. Note that e, e', e'' are orthogonal to each other. Let $f \in \mathbf{R}e_l + \mathbf{R}e + \mathbf{R}e' + \mathbf{R}e''$ such that $(f, f) = 1$, $(f, e) = (f, e') = (f, e'') = 0$. Clearly, $(e_l, f) \neq 0$. We have

$$e_l = (e_l, f)f + \frac{(e_l, e)}{(e, e)}e + \frac{(e_l, e')}{(e', e')}e' + \frac{(e_l, e'')}{(e'', e'')}e''.$$

Hence

$$1 = (e_l, e_l) = (e_l, f)^2 + \frac{(e_l, e)^2}{(e, e)} + \frac{(e_l, e')^2}{(e', e')} + \frac{(e_l, e'')^2}{(e'', e'')}.$$

Hence

$$\frac{(e_l, e)^2}{(e, e)} + \frac{(e_l, e')^2}{(e', e')} + \frac{(e_l, e'')^2}{(e'', e'')} < 1.$$

Now $(e_l, e) = -(p-1)/2$, $(e_l, e') = -(q-1)/2$, $(e_l, e'') = -(r-1)/2$. Hence

$$\frac{(p-1)^2/4}{p(p-1)/2} + \frac{(q-1)^2/4}{q(q-1)/2} + \frac{(r-1)^2/4}{r(r-1)/2} < 1,$$

$$1 - \frac{1}{p} + 1 - \frac{1}{q} + 1 - \frac{1}{r} < 2,$$

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

We may assume that $\frac{1}{p} \leq \frac{1}{q} \leq \frac{1}{r}$. Then $3\frac{1}{r} > 1$ hence $r < 3$. Since $r \geq 2$ we have $r = 2$. Then $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$. Hence $2\frac{1}{q} > \frac{1}{2}$ and $q < 4$. Since $q \geq 2$ we have $q = 2$ or $q = 3$. If $q = 2$ we are done. If $q = 3$ we have $\frac{1}{p} > \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ hence $p < 6$. Since $p \geq 2$ we have $p \in \{2, 3, 4, 5\}$ as required.

Irreducible admissible sets. An admissible set $\mathbf{E}, (,), (e_i)_{i \in I}$ is said to be *irreducible* if $(e_i)_{i \in I}$ is an $\mathbf{R}$-basis of $\mathbf{E}$ and if there is no partition $I = I' \cup I''$ with $I' \neq \emptyset, I'' \neq \emptyset$ such that (e_i, e_j) for all $i \in I', j \in I''$ (the last condition is equivalent to the condition that the associated graph is connected). Two irreducible admissible sets $\mathbf{E}, (,), (e_i)_{i \in I}$ and $\mathbf{E}', (,)', (e'_j)_{j \in J}$ are said to be isomorphic if there exists an isomorphism of vector spaces $\mathbf{E} \xrightarrow{\sim} \mathbf{E}'$ carrying $(,)$ to $(,)'$ and the basis (e_i) to the basis (e'_j). (An equivalent condition is that there exists a bijection $\sigma : I \xrightarrow{\sim} J$ such that $(e_i, e_{i'}) = (e'_{\sigma(i)}, e'_{\sigma(i')})$ for all $i, i' \in I$.)

The results above give a classification of irreducible admissible sets up to isomorphism. Namely any irreducible admissible set must be one of the following

Type A_n: as in (8)(a)) with n vertices, $n \geq 1$.

Type BC_n: as in (9) with $n \geq 2, p = 1$.

Type D_n: as in (10) with $(p, q, r) = (p, 2, 2), p \geq 2, n = p + 2$.

Type E_6: as in (10) with $(p, q, r) = (3, 3, 2)$.

Type E_7: as in (10) with $(p, q, r) = (4, 3, 2)$.

Type E_8: as in (10) with $(p, q, r) = (5, 3, 2)$.

Type F_4 as in (9) with $p = q = 2$.

Type G_2 as in (5).

Note that we have not shown the existence of an irreducible admissible set of one of the types above. To do this we must verify the positive definiteness of the symmetric bilinear form defined by the graph in each of the cases above. This can be done directly. We will deduce it by constructing in each case an appropriate root system.

Construction of root systems. Let $E = \mathbf{Q}^n$. Let $e_1, \ldots, e_n$ be the standard basis and let $(,) : E \times E \to \mathbf{Q}$ be the symmetric bilinear form such that $(e_i, e_j) = \delta_{ij}$.

Type A. Assume that $n \geq 2$. Let $E' = \{\sum_i c_i e_i \in E | \sum_i c_i = 0\}$. Let $R = \{\alpha \in \mathbf{Z}^n \cap E' | (\alpha, \alpha) = 2\}$. This consists of the vectors $\pm(e_i - e_j)$ where $i < j$. It is a root system. Let E'^+ be the set of all $\sum_i c_i e_i \in E' - \{0\}$ such that the first non-zero c_i is > 0. This is a "linear order". The corresponding set R^+ consists of $e_i - e_j$ where $i < j$. The corresponding set of simple roots is

$$e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n.$$
Dividing the simple roots by their length $\sqrt{2}$ we get an admissible set of type A_{n-1}.

Type C. Assume that $n \geq 2$. Let $R = \{\alpha \in \mathbf{Z}^n \cap E | (\alpha, \alpha) = 1 \text{ or} (\alpha, \alpha) = 2\}$. This consists of the vectors $\pm e_i$ and $\pm e_i \pm e_j$ where $i < j$. It is a root system. Let E^+ be the set of all $\sum_i c_i e_i \in E - \{0\}$ such that the first non-zero c_i is > 0. This is a "linear order". The corresponding set R^+ consists of e_i and $e_i \pm e_j$ where $i < j$. The corresponding set of simple roots is

$$e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n.$$
Dividing the simple roots by their length ($\sqrt{2}$ or 1) we get an admissible set of type BC_n.

Type B. The root system dual to the previous one consists of the vectors $\pm 2e_i$ and $\pm e_i \pm e_j$ where $i < j$. A set of simple roots is $e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, 2e_n$.

Dividing the simple roots by their length ($\sqrt{2}$ or 1) we get an admissible set of type BC_n.

Type D. Assume that $n \geq 2$. Let $R = \{\alpha \in \mathbf{Z}^n \cap E | (\alpha, \alpha) = 2\}$. This consists of the vectors $\pm e_i \pm e_j$ where $i < j$. It is a root system. Let E^+ be the set of all $\sum_i c_i e_i \in E - \{0\}$ such that the first non-zero c_i is > 0. This is a "linear order". The corresponding set R^+ consists of $e_i \pm e_j$ where $i < j$. The corresponding set of simple roots is

$$e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_{n-1} + e_n.$$
Dividing the simple roots by their length $\sqrt{2}$ we get an admissible set of type D_n.

Type E_8. Assume now that $n = 8$. Let L be the subgroup of E consisting of all $x = \sum_i c_i e_i$ with $2c_i \in \mathbf{Z}$, $c_i - c_j \in \mathbf{Z}$ for all i, j and $\sum_i c_i \in 2\mathbf{Z}$. Then L is the subgroup of E generated by $\pm e_i \pm e_j$ where $i < j$ and by $\frac{1}{2} \sum_{i=1}^{8} e_i$. For $\alpha, \beta \in L$ we have $(\alpha, \beta) \in \mathbf{Z}$. Let $R = \{\alpha \in L | (\alpha, \alpha) = 2\}$. This consists of the vectors $\pm e_i \pm e_j$ where $i < j$ and $\frac{1}{2} \sum_{i=1}^{8} (-1)^{\nu_i} e_i$ where $\sum_i \nu_i$ is even. This is a root system. It contains $4\binom{8}{2} + 2^7 = 240$ roots. Let $\rho = \sum_{i=1}^{7} (i-1)e_i + 23e_8 \in L$. Let $E' = \{x \in E | (x, \rho) = 0\}$. Let $E' = E'^+ \cup E'^- \cup \{0\}$ be a "linear order" for

E'. Define a linear order $E = E^+ \cup E^- \cup \{0\}$ by $E^+ = \{x \in E|(x,\rho) > 0\} \cup E'^+$, $E^- = \{x \in E|(x,\rho) < 0\} \cup E'^-$. Now $R \cap E' = \emptyset$. Indeed, $(\pm e_i \pm e_j, \rho) \neq 0$ and

$(\frac{1}{2}\sum_{i=1}^{8}(-1)^{\nu_i}e_i, \rho) = \sum_{i=1}^{6}i(-1)^{\nu_{i+1}} + 23(-1)^{\nu_8}$

and this is $\neq 0$ since $\sum_{i=1}^{6}i = 6.7/2 = 21 < 23$. Thus, $R = R^+ \cup R^-$ where $R^+ = \{\alpha \in R|(\alpha,\rho) > 0\}$, $R^- = \{\alpha \in R|(\alpha,\rho) < 0\}$ are the sets of "positive" or "negative" roots. Now R^+ consists of $\pm e_i + e_j$ for $i < j$ and of $\frac{1}{2}(\sum_{i=1}^{7}(-1)^{\nu_i}e_i + e_8)$ where $\sum_i \nu_i$ is even. For any $\alpha \in R$ we have $(\alpha,\rho) \in \mathbf{Z}$. Consider the roots

$\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$,

$\alpha_2 = e_1 + e_2, \alpha_3 = e_2 - e_1, \alpha_4 = e_3 - e_2, \alpha_5 = e_4 - e_3, \alpha_6 = e_5 - e_4, \alpha_7 = e_6 - e_5, \alpha_8 = e_7 - e_6$.

These roots have $(,\rho) = 1$; thus, $(,\rho)$ is minimal. Hence these must form a set of simple roots. The inner products of two simple roots are as follows:

$(\alpha_4,\alpha_5) = (\alpha_5,\alpha_6) = (\alpha_6,\alpha_7) = (\alpha_7,\alpha_8) = (\alpha_4,\alpha_2) = (\alpha_4,\alpha_3) = (\alpha_3,\alpha_1) = -1$,

all other (α_i,α_j) with $i \neq j$ are 0. Hence the vectors $\alpha_i/\sqrt{2}$ form an admissible set of type E_8.

Type E_7. In $E = \mathbf{Q}^8$ (as above) we consider the hyperplane E' orthogonal to $e_7 + e_8$. Thus, $E' = \{\sum_i c_i e_i | c_7 + c_8 = 0\}$. Let $R' = R \cap E'$. If $\alpha \in R'$ then $r_\alpha(E') = E'$. Hence $r_\alpha(R') = R'$. Then R' is a root system since R' generates E': $\alpha_1, \alpha_2, \ldots, \alpha_7$ belong to E'. These form a set of simple roots for R'. Now R^+ consists of $\pm e_i + e_j$ for $i < j \leq 6$, $-e_7 + e_8$ and of $\frac{1}{2}(\sum_{i=1}^{6}(-1)^{\nu_i}e_i - e_7 + e_8)$ where $\sum_i \nu_i$ is odd. The vectors $\alpha_i/\sqrt{2}$ $(i \leq 7)$ form an admissible set of type E_7.

Type E_6. In $E = \mathbf{Q}^8$ (as above) we consider the subspace E' orthogonal to $e_7 + e_8, e_6 + e_7 + 2e_8$. Let $R'' = R \cap E''$. Then R'' is a root system since R'' generates E'': $\alpha_1, \alpha_2, \ldots, \alpha_6$ belong to E''. These form a set of simple roots for R'. Now R^+ consists of $\pm e_i + e_j$ for $i < j \leq 5$, $-e_7 + e_8$ and of $\frac{1}{2}(\sum_{i=1}^{6}(-1)^{\nu_i}e_i - e_6 - e_7 + e_8)$ where $\sum_i \nu_i$ is even. The vectors $\alpha_i/\sqrt{2}$ $(i \leq 6)$ form an admissible set of type E_6.

Type F_4. Assume now that $n = 4$. Let L be the subgroup of E consisting of all $x = \sum_i c_i e_i$ with $2c_i \in \mathbf{Z}$ and $c_i - c_j \in \mathbf{Z}$ for all i, j. Then L is the subgroup of E generated by the e_i and by $\frac{1}{2}\sum_{i=1}^{4}e_i$. For $\alpha, \beta \in L$ we have $(\alpha,\beta) \in \frac{1}{2}\mathbf{Z}$. Let $R = \{\alpha \in L|(\alpha,\alpha) = 1 \text{ or } 2\}$. This consists of the vectors $\pm e_i, \pm e_i \pm e_j$ where $i \neq j$ and $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$. We show that for $\alpha, \beta \in R$ we have $2(\alpha,\beta)/(\alpha,\alpha) \in \mathbf{Z}$. If $\alpha = \pm e_i$ or $\alpha = \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$ then $(\alpha,\alpha) = 1$ and $(\alpha,\beta) \in \frac{1}{2}\mathbf{Z}$ hence $2(\alpha,\beta)/(\alpha,\alpha) \in \mathbf{Z}$. If $\alpha = \pm e_i \pm e_j$ we have $(\alpha,\alpha) = 2$ and $(\alpha,\beta) \in \mathbf{Z}$ so that $2(\alpha,\beta)/(\alpha,\alpha) \in \mathbf{Z}$. We see that R is a root system. The number of roots is $8 + 4\binom{4}{2} + 2^4 = 48$. Consider the "linear order" on E where E^+ consists of all $\sum_{i=1}^{4}c_i e_i$ in $E - \{0\}$ such that the first non-zero c_i is > 0. The corresponding R^+ consists of e_i, $e_i \pm e_j$ where $i < j$ and $\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$.

Let $\rho = \frac{1}{2}(11e_1 + 5e_2 + 3e_3 + e_4) \in L$. For any $\alpha \in R$ we have $(\alpha,\rho) \neq 0$. Hence $R^+ = \{\alpha \in R|(\alpha,\rho) > 0\}$ is a "set of positive roots". Now R^+ consists of e_i, $e_i \pm e_j$, $i < j$ and $\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$. The inner products (α,ρ) for $\alpha \in R^+$ are as follows:

$(e_4, \rho) = 1/2, (e_2 - e_3, \rho) = 1, (e_3 - e_4, \rho) = 1, (\frac{1}{2}(e_1 - e_2 - e_3 - e_4), \rho) = 1/2, (\frac{1}{2}(e_1 - e_2 - e_3 + e_4), \rho) = 1$
and $(\alpha, \rho) > 1$ for all other α. It follows that $\alpha_3 = e_4, \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$
are simple roots and two of $\alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha = \frac{1}{2}(e_1 - e_2 - e_3 + e_4)$
are simple roots. Now α is not a simple root since it is not linearly independent
from α_3, α_4. Thus the simple roots are $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. The inner products of two
simple roots are as follows:
$(\alpha_1, \alpha_2) = (\alpha_2, \alpha_3) = -1, (\alpha_3, \alpha_4) = -1/2,$
all other (α_i, α_j) with $i \neq j$ are 0. Hence $\alpha_1/\sqrt{2}, \alpha_2/\sqrt{2}, \alpha_3, \alpha_4$ form an admissible
set of type F_4.

Type G_2. Assume that $n = 3$. Let $E' = \{c_1 e_1 + c_2 e_2 + c_3 e_3 \in E | c_1 + c_2 + c_3 = 0\}$.
Let $R = \{\alpha \in \mathbf{Z}^3 \cap E' | (\alpha, \alpha) = 2 \text{ or } (\alpha, \alpha) = 6\}$. Then R consists of
$\pm(e_1 - e_2), \pm(e_2 - e_3), \pm(e_1 - e_3), \pm(2e_1 - e_2 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2)$.
R is a root system. We can take as set of simple roots:
$\alpha_1 = e_1 - e_2, \alpha_2 = -2e_1 + e_2 + e_3$.
The positive roots are: $\alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2$. Dividing
the simple roots by their length ($\sqrt{3}$ or 1) we get an admissible set of type G_2.

Classification of irreducible root systems. A root system $E, (,), R$ is said to be
irreducible if there is no partition $R = R' \cup R''$ with $R' \neq \emptyset, R'' \neq \emptyset$ such that
$(R, R') = 0$. Let Π be a set of simple roots. We show that

(a) *$E, (,), R$ is irreducible if and only if there is no partition $\Pi = \Pi' \cup \Pi''$ with
$\Pi' \neq \emptyset, \Pi'' \neq \emptyset$ such that $(\Pi', \Pi'') = 0$.*

Assume first that there exists a partition $R = R' \cup R''$ as above. Let $\Pi' = \Pi \cap R', \Pi'' = \Pi \cap R''$. We must show that $\Pi' \neq \emptyset, \Pi'' \neq \emptyset$. If $\Pi' = \emptyset$ we have
$\Pi \in R''$ hence $(\Pi, R') = 0$ hence $(E, R') = 0$ hence $R' \subset \{0\}$ absurd.

Conversely, assume that there exists a partition $\Pi = \Pi' \cup \Pi''$ with $\Pi' \neq \emptyset, \Pi'' \neq \emptyset$
such that with $(\Pi', \Pi'') = 0$. Let E' be the span of Π' and let E'' be the span of
Π''. Let $\alpha \in R$. We show that $\alpha \in E' \cup E''$ by induction on the minimum k such
that $\alpha = r_{i_1} r_{i_2} \ldots r_{i_k} \alpha_j$ with $i_1, \ldots, i_k, j \in I$. For $k = 0$ there is nothing to prove.
Assume that $k \geq 1$. then $\alpha = r_i \alpha'$ where $i \in I$ and $\alpha' \in R \cap (E' \cup E'')$. Assume
for example that $\alpha' \in E'$. Clearly, $r_i(E') \subset E'$. Hence $\alpha \in E'$. This completes
the induction. We see that $R = R' \cup R''$ where $R' = R \cap E', R'' = R \cap E''$. Since
$(E', E'') = 0$, we have $(R', R'') = 0$. This proves (a).

Two irreducible root systems $E, (,), R$ and $E', (,)', R'$ are said to be *isomorphic*
if there exists an isomorphism of vector spaces $\sigma : E \xrightarrow{\sim} E'$ that carries R onto R'
and $(,)$ to $a(,)'$ for some $a \in \mathbf{Q}_{>0}$.

Claim. Assume that R is a root system in E with respect to $(,)$ (positive
definite). Then $(,)$ is uniquely determined by R up to scalar in $\mathbf{Q}_{>0}$. In particular,
in the previous definition the fact that σ carries $(,)$ to $a(,)'$ for some $a \in \mathbf{Q}_{>0}$ is
automatically true and can be dropped.

We first prove the following statement.

*Let R be a finite set of vectors that span E. Assume that $r_\alpha(R) = R$ for all
$\alpha \in R$. Let r be a reflection in E such that $r(R) = R$ and such that $r(\alpha) = -\alpha$*

for some $\alpha \in R$. Then $r = r_\alpha$.

Let $s = rr_\alpha^{-1}$. Then $s(R) = R$. Since R is finite, there exists $N \geq 1$ such that s^N is the identity permutation of R (we can take $N = (\sharp R)!$). Since R spans E we have $s^N = 1 : E \to E$. Now s acts as 1 on $\mathbf{Q}\alpha$ and on $E/\mathbf{Q}\alpha$. Thus all eigenvalues of s are 1. Combined with $s^N = 1$ this implies $s = 1$. Thus $r = r_\alpha$.

We can now start the proof of the claim. Assume that R is also a root system with respect to a second positive definite bilinear form $(,)'$. We must show that $(,) = a(,)'$ for some $a \in \mathbf{Q}_{>0}$. For $\alpha \in R$ we denote by $r_\alpha : E \to E$ the reflection defined in terms of $(,)$ and by $r'_\alpha : E \to E$ the reflection defined in terms of $(,)'$. By the statement above we have $r_\alpha = r'_\alpha$. Hence the Weyl group W defined in terms of $R, (,)$ is the same as that defined in terms of $R, (,)'$. Thus, both $(,), (,)'$ are preserved by W. We define $f : E \to E$ by $(e, e') = (f(e), e')'$ for any $e, e' \in E$. This is a linear map. Let $\alpha \in R$. Since $r_\alpha = r'_\alpha$ we have $(\alpha, x)/(\alpha, \alpha) = (\alpha, x)'/(\alpha, \alpha)'$ for all $x \in E$ hence $f(\alpha) = \frac{(\alpha, \alpha)}{(\alpha, \alpha)'}\alpha$. In particular, if α is fixed then $E' = \{e \in E | f(e) = \frac{(\alpha, \alpha)}{(\alpha, \alpha)'}e\}$ is $\neq 0$ (it contains α. This vector space is W-stable. Let E'' be the orthogonal complement of E' with respect to $(,)$. Then $E = E' \oplus E''$. Let $\beta \in R$. We show that $\beta \in E' \cup E''$. If $(\beta, E') = 0$ then $\beta \in E''$. If $(\beta, E') \neq 0$ then choose $x \in E'$ with $(\beta, x) \neq 0$. Now $r_\beta(E') = E'$ hence $r_\beta(x) - x = 2\frac{(x, \beta)}{(\beta, \beta)}\beta \in E'$ hence $\beta \in E'$. Set $R' = R \cap E', R'' = R \cap E''$. Then $R = R' \cup R'', (R', R'') = 0, \alpha \in R'$. By irreducibility of R we have $R = R'$. Since $R \subset E'$ and R spans E we have $E' = E$. Thus $f(e) = \frac{(\alpha, \alpha)}{(\alpha, \alpha)'}e$ for all $e \in E$. Hence $(e, e') = \frac{(\alpha, \alpha)}{(\alpha, \alpha)'}(e, e')'$ for any $e, e' \in E$. Our claim is established.

By attaching to a root system the set of simple roots divided by their length, we obtain a map

Φ : iso. classes of irr. root systems $\longrightarrow$ iso. classes of irr. adm. sets.

By the classification of irreducible admissible sets and the construction of root systems, Φ is surjective. We now study the fibre of Φ at an admissible set.

Let $(\mathbf{E}, (), (e_i)$ is an irreducible admissible set. Consider the set S consisting of all R, Π where Π is a basis of $\mathbf{E}$ such that (e_i) consists of the vectors in Π divided by their length and R is a root system in $[\Pi] = \sum_{\alpha \in \Pi} \mathbf{Q}\alpha$. Note that R is completely determined by Π.

Let R, Π and R', Π' be two elements of S.

Let $\alpha \neq \beta \in \Pi$. Then $\alpha = c\alpha', \beta = d\beta'$ where $c, d \neq 0$ and $\alpha' \neq \beta' \in \Pi'$. We have

(c) $\frac{4(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)} = \frac{4(\alpha', \beta')'^2}{(\alpha', \alpha')'(\beta', \beta')'}$.

Assume first that (c) is 1. Then $(\alpha, \alpha) = (\beta, \beta)$ and $(\alpha', \alpha')' = (\beta', \beta')'$. Hence $c^2 = d^2$ hence $c = \pm d$. Since $(\alpha, \beta) < 0, (\alpha', \beta')' < 0$ we have $cd > 0$ hence $c = d$.

Assume next that (c) is 2. Then $(\alpha, \alpha) = 2^{\pm 1}(\beta, \beta)$ and $(\alpha', \alpha')' = 2^{\pm 1}(\beta', \beta')'$. If $(\alpha, \alpha) = 2(\beta, \beta)$ then $(c\alpha', c\alpha')' = (\alpha, \alpha) = 2(\beta, \beta) = 2(d\beta', d\beta')$ hence $(\alpha', \alpha')' = 2d^2/c^2(d\beta', d\beta')$ hence $2d^2/c^2 \in \{2, 2^{-1}\}$ hence $d/c \in \{1, 2^{-1}\}$ (as before we have $cd > 0$). Similarly, if $(\alpha, \alpha) = 2^{-1}(\beta, \beta)$ then $d/c \in \{1, 2\}$.

Assume next that (c) is 3. Then $(\alpha, \alpha) = 3^{\pm 1}(\beta, \beta)$ and as before we see that: if $(\alpha, \alpha) = 3(\beta, \beta)$ then $d/c \in \{1, 3^{-1}\}$; if $(\alpha, \alpha) = 3^{-1}(\beta, \beta)$ then $d/c \in \{1, 3\}$.

If the graph has no double or triple edges then we see that $\Pi = c\Pi'$. Hence $c^{-1}Id$ takes Π to Π', R to R' and $(,)$ to a multiple of $(,)'$. Hence $[\Pi], (,), R$ and $[\Pi'], (,)', R'$ are isomorphic root systems.

Assume now that the graph has a d-fold edge, $d = 2$ or 3. Then $\Pi = \Pi_1 \cup \Pi_2$, $\Pi' = \Pi'_1 \cup \Pi'_2$ where

the vectors in Π_1 have length squared N_1,

the vectors in Π_2 have length squared N_2,

the vectors in Π'_1 have length squared N'_1,

the vectors in Π'_2 have length squared N'_2.

Also $N_2 = d^{\pm 1}N_1$, $N'_2 = d^{\pm 1}N'_1$. As earlier we have (after possible renumbering): $\Pi_1 = c_1\Pi'_1, \Pi_2 = c_2\Pi'_2$ where

$c_2/c_1 \in \{1, d^{-1}\}$ if $N_1 = dN_2$; $c_2/c_1 \in \{1, d\}$ if $N_2 = dN_1$.

If $c_2 = c_1 = c$ then $\Pi = c\Pi'$. Hence $c^{-1}Id$ takes Π to Π', R to R' and $(,)$ to a multiple of $(,)'$. Hence $[\Pi], (,), R$ and $[\Pi'], (,)', R'$ are isomorphic root systems.

If $c_2 = dc_1$, we consider the dual root system $\check{R}, \check{\Pi}$. Then

$\check{\Pi} = \{\frac{2}{N_1}\alpha | \alpha \in \Pi_1\} \cup \{\frac{2}{N_2}\alpha | \alpha \in \Pi_2\}$.

Hence $\check{\Pi}_k = \frac{2}{N_k}c_k\Pi'_k$. We have $\frac{2}{N_2}c_2/(\frac{2}{N_1}c_1) = 1$ and the earlier argument shows that $[\check{\Pi}], (,), \check{R}$ and $[\Pi'], (,)', R'$ are isomorphic root systems. Thus we see that in any case the fibre of Φ consists of a root system and its dual. Now the root system of type G_2 is isomorphic to its dual. The same is true for the root system of type F_4 or B_2. But for $n \geq 3$, the root system of type B_n and C_n are not isomorphic.

Thus the irreducible root systems (up to isomorphism) are classified by the list $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$.