LIE ALGEBRAS,3

$\mathfrak{sl}_2(k)$-MODULES

Let $L = \mathfrak{sl}_2(k)$. A basis is given by

$$e = \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}, f = \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}, g = \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}.$$

We have $[e,f] = h, [h,e] = 2e, [h,f] = -2f$. Thus, h is semisimple. Since L is simple, it is semisimple. Let V be an L-module, $\dim V < \infty$. Then $h : V \longrightarrow V$ is semisimple. Thus $V = \oplus_{\lambda \in k} V_\lambda$ where $V_\lambda = \{v \in V | hv = \lambda v\}$.

If $v \in V_\lambda$ then $ev \in V_{\lambda+2}, fv \in V_{\lambda-2}$.

Assume now that V is irreducible. We can find $v_0 \in V - \{0\}$ such that $v_0 \in V_\lambda$, $ev_0 = 0$. Set $v_{-1} = 0, v_n = \frac{f^n}{n!} v_0$, $n \in \mathbf{N}$. We have

(a) $hv_n = (\lambda - 2n)v_n$ for $n \geq -1$

(b) $fv_n = (n+1)v_{n+1}$ for $n \geq -1$

(c) $ev_n = (\lambda - n + 1)v_{n-1}$ for $n \geq 0$.

(c) is shown by induction on n. For $n = 0$ it is clear. Assuming $n \geq 1$,

$$\begin{aligned} ev_n &= n^{-1} efv_{n-1} = n^{-1} hv_{n-1} + n^{-1} fev_{n-1} \\ &= n^{-1}(\lambda - 2n + 2)v_{n-1} + n^{-1} f(\lambda - n + 2)v_{n-2} \\ &= n^{-1}(\lambda - 2n + 2)v_{n-1} + n^{-1}(\lambda - n + 2)(n-1)v_{n-1} = (\lambda - n + 1)v_{n-1}. \end{aligned}$$

By (a), the non-zero v_n are linearly independent. Since $\dim V < \infty$, there exists $m \geq 0$ such that $v_0, v_1, \ldots, v_m$ are $\neq 0$ and $v_{m+1} = 0$. Then $v_{m+2} = v_{m+3} = \cdots = 0$. Now $v_0, v_1, \ldots, v_m$ form a basis of an L-submodule which must be the whole of V. Now (c) with $n = m+1$ gives $0 = (\lambda - m)v_n$ hence $\lambda = n$. Thus the action of e, f, h in the basis $v_0, v_1, \ldots, v_m$ is

$hv_n = (m - 2n)v_n$ for $n \in [0, m]$

$fv_n = (n+1)v_{n+1}$ for $n \in [0, m]$

$ev_n = (m - n + 1)v_{n-1}$ for $n \in [0, m]$

with the convention $v_{-1} = 0, v_{m+1} = 0$.

Conversely, given $m \geq 0$ we can define an L-module structure on an $m+1$ dimensional vector space with basis $v_0, v_1, \ldots, v_m$ by the formulas above. Thus we have a 1-1 correspondence between the set of isomorphism classes of irreducible L-modules and the set $\mathbf{N}$.

Now let V be any finite dimensional L-module. Then:

(a) the eigenvalues of $h : V \longrightarrow V$ are integers; the multiplicity of the eigenvalue a equals that of $-a$.

(b) If $h : V \longrightarrow V$ has an eigenvalue in $2\mathbf{Z}$ then it has an eigenvalue 0.

(c) If $h : V \longrightarrow V$ has an eigenvalue in $2\mathbf{Z}+1$ then it has an eigenvalue 1.

Indeed, by Weyl, we are reduced to the case where V is irreducible; in that case we use the explicit description of L given above.

Roots

Let L be a semisimple Lie algebra $\neq 0$. A subalgebra T of L is said to be *toral* if any element of T is semisimple in L.

Lemma. *If T is toral then T is abelian.*

Let $x \in T$. Assume that $ad(x) : T \longrightarrow T$ has some eigenvalue $a \neq 0$. Thus $[x, y] = ay$ for some $y \in T - \{0\}$. Now $ad(y) : L \longrightarrow L$ is semisimple hence $ad(y) : T \longrightarrow T$ is semisimple hence $x = \sum_j u_j$ where $u_j \in T$ are eigenvectors of $ad(y) : T \longrightarrow T$ with corresponding eigenvalue λ_j. Hence $ad(y)x = \sum_{j;\lambda_j\neq 0} \lambda_j u_j$. But $ad(y)x = -ay$. But y is in the 0-eigenspace of $ad(y)$ and $\sum_{j;\lambda_j\neq 0} \lambda_j u_j = -ay$ is a contradiction. Thus, all eigenvalues of $ad(x) : T \longrightarrow T$ are 0. Now $ad(x) : L \longrightarrow L$ is semisimple hence $ad(x) : T \longrightarrow T$ is semisimple hence $ad(x) : T \longrightarrow T$ is 0. The lemma follows.

Let H be a maximal toral subalgebra of L. Now $\{ad(h) : L \longrightarrow L | h \in H\}$ is a family of commuting semisimple endomorphisms of L. Hence $L = \oplus_\alpha L_\alpha$ where α runs over the dual space H^* of H and $L_\alpha = \{x \in L | [h, x] = \alpha(h)x \forall h \in H\}$. Now $L_0 = \{x \in L | [h, x] = 0 \forall h \in H\}$. and $H \subset L_0$ by the lemma. We say that $\alpha \in H^*$ is a *root* or $\alpha \in R$ if $\alpha \neq 0$ and $L_\alpha \neq 0$. We have $L = L_0 \oplus \oplus_{\alpha\in R} L_\alpha$ (root decomposition or Cartan decomposition).

Lemma. *(a) For any $\alpha, \beta \in H^*$ we have $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$.*

(b) If $x \in L_\alpha, \alpha \neq 0$ then $ad(x)$ is nilpotent.

(c) If $\alpha, \beta \in H^*, \alpha + \beta \neq 0$ then $\kappa(L_\alpha, L_\beta) = 0$.

(a) Let $x \in L_\alpha, y \in L_\beta$. For $h \in H$ we have

$[h, [x, y]] = [[h, x], y]+[x, [h, y]] = \alpha(h)[x, y]+\beta(h)[x, y] = (\alpha+\beta)(h)[x, y]$ hence $[x, y] \in L_{\alpha+\beta}$.

(b) For any $\beta \in H^*$ we have $n\alpha+\beta \notin R$ for large n hence using (a), $ad(x)^n L_\beta = 0$. Now (b) follows.

(c) We can find $h \in H$ with $(\alpha + \beta)(h) \neq 0$. Let $x \in L_\alpha$, $y \in L_\beta$. We have $\kappa([h, x], y) = \kappa([y, h], x)$ hence $\alpha(h)\kappa(x, y) = -\beta(h)\kappa(x, y)$. Thus $(\alpha + \beta)(h)\kappa(x, y) = 0$ and $\kappa(x, y) = 0$.

Lemma. *The restriction of κ to L_0 is non-singular.*

Proposition. *$L_0 = H$.*

We show:

(a) *If* $x \in L_0$ *and* $x = s+n$ *is a Jordan decomposition in* L *then* $s \in L_0, n \in L_0$.

We have $ad(x)H \subset \{0\}$ hence $ad(x)_s H \subset \{0\}$,$ad(x)_n H \subset \{0\}$, hence $ad(s)H \subset \{0\}$,$ad(n)H \subset \{0\}$, hence $s \in L_0, n \in L_0$.

(b) *If* $x \in L_0$ *is semisimple in* L *then* $x \in H$.

From the assumption, $H + kx$ is a toral algebra hence it is H by the maximality of H. Hence $x \in H$.

(c) *The restriction of* κ *to* H *is non-singular.*

Assume that $h \in H$ and $\kappa(h, H) = 0$. Let $x = s + n \in L_0$ be as in (a). Then $s \in L_0, n \in L_0$. By (b) we have $s \in H$ hence $\kappa(h, s) = 0$. Now $ad(n) : L \to L$ is nilpotent and $ad(n), ad(h)$ commute hence $ad(h)ad(n)$ is nilpotent hence $\text{tr}(ad(h)ad(n), L) = 0$. Thus $\kappa(h, n) = 0$. Hence $\kappa(h, x) = 0$. Thus $\kappa(h, L_0) = 0$. Since $\kappa|_{L_0}$ is non-singular, we have $h = 0$.

(d) L_0 *is nilpotent.*

By Engel it is enough to show that, if $x \in L_0$ then $ad(x) : L_0 \to L_0$ is nilpotent. Write $x = s + n$ as in (a). Now $ad(s) : L_0 \to L_0$ is 0 since $s \in H$ (by (b)). Also $ad(n) : L \to L$ is nilpotent hence $ad(x) = ad(n) : L_0 \to L_0$ is nilpotent.

(e) $[L_0, L_0] \cap H = 0$.

Let $x \in [L_0, L_0] \cap H$. Write $x = \sum_i [x_i, y_i]$ where $x_i, y_i \in L_0$. If $h \in H$ we have $\kappa(h, x) = \sum_i \kappa(h, [x_i, y_i]) = \sum_i \kappa(x_i, [y_i, h]) = 0$ since $[y_i, h] = 0$. Thus $\kappa(H, x) = 0$. Since $x \in H$ we see from (c) that $x = 0$.

(f) $[L_0, L_0] = 0$.

Otherwise, we have $[L_0, L_0] \neq 0$. Since L_0 is nilpotent and $[L_0, L_0]$ is a non-zero ideal, we then have $[L_0, L_0] \cap centre(L_0) \neq 0$ (by a corollary of Engel). Let $x \in [L_0, L_0] \cap centre(L_0), x \neq 0$. Write $x = s + n$ as in (a). Since $ad(x)(L_0) \subset 0$ we have $ad(x)_n(L_0) \subset 0$ hence $ad(n)(L_0) \subset 0$ hence $n \in centre(L_0)$. Hence for any $x' \in L_0$, $ad(x'), ad(n) : L \to L$ commute and $ad(n)$ is nilpotent hence $ad(x')ad(n) : L \to L$ is nilpotent hence $\text{tr}(ad(x')ad(n), L) = 0$ hence $\kappa(x', n) = 0$. Thus $\kappa(L_0, n) = 0$. Since $\kappa|_{L_0}$ is non-singular we have $n = 0$. Thus $x = s \in H$ (see (b)). Hence $x \in [L_0, L_0] \cap H$ which is 0 by (e). Hence $x = 0$ a contradiction.

(g) *If* $x \in L_0$ *is nilpotent then* $x = 0$.

For all $y \in L_0$, $ad(x), ad(y)$ commute and $ad(x)$ is nilpotent hence $ad(x)ad(y) : L \to L$ is nilpotent hence $\text{tr}(ad(x)ad(y), L) = 0$. Hence $\kappa(x, y) = 0$. Hence $\kappa(x, L_0) = 0$. Since $\kappa|_{L_0}$ is non-singular we have $x = 0$.

We can now prove the proposition. Let $x \in L_0$. Write $x = s+n$ as in (a). Then $s \in L_0, n \in L_0$. By (g) we have $n = 0$. By (b) we have $s \in H$. Hence $x \in H$. The proposition is proved.

Properties of roots.

Let $\xi \in H^*$. Since $\kappa|_H$ is non-singular there exists a unique element $t_\xi \in H$ such that $\xi(h) = \kappa(t_\xi, h)$ for all $h \in H$. Now $\xi \mapsto t_\xi$ is an isomorphism $H^* \xrightarrow{\sim} H$.

(a) R *spans the vector space* H^*.

If not, we can find $h \in H, h \neq 0$ so that $\alpha(h) = 0$ for all $\alpha \in R$. Then $[h, L_\alpha] = 0$ for all $\alpha \in R$. Also $[h, L_0] = 0$ since $L_0 = H$ is abelian. Hence $[h, L] = 0$ so that $h \in Z(L)$. But $Z(L) = 0$ since L is semisimple. Thus $h = 0$, contradiction.

(b) *If* $\alpha \in R$ *then* $-\alpha \in R$.

Assume that $-\alpha \notin R$. Then $L_{-\alpha} = 0$. Hence $\kappa(L_\alpha, L_\beta) = 0$ for any $\beta \in H^*$ hence $\kappa(L_\alpha, L) = 0$. Since κ is non-singular we have $L_\alpha = 0$, absurd.

(c) *If* $\alpha \in R, x \in L_\alpha, y \in L_{-\alpha}$ *then* $[x, y] = \kappa(x, y)t_\alpha$.

Let $h \in H$. We have $\kappa(h, [x, y]) = \kappa(y, [h, x]) = \alpha(h)\kappa(y, x) = \kappa(t_\alpha, h)\kappa(x, y)$ hence $\kappa(h, [x, y] - \kappa(x, y)t_\alpha) = 0$. Thus $\kappa([x, y] - \kappa(x, y)t_\alpha, H) = 0$. Since $[x, y] - \kappa(x, y)t_\alpha \in H$ and κ_H is non-singular, we have $[x, y] - \kappa(x, y)t_\alpha = 0$.

(d) *Let* $\alpha \in R$ *and let* $x \in L_\alpha - \{0\} \neq 0$. *There exists* $y \in L_{-\alpha}$ *such that* $\kappa(x, y) \neq 0$.

Assume that $\kappa(x, L_{-\alpha}) = 0$. Then $\kappa(x, L_\beta) = 0$ for any $\beta \in H^*$ hence $\kappa(x, L) = 0$ hence $x = 0$ absurd.

(e) *Let* $\alpha \in R$. *We have* $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$.

The equality comes from the definition of t_α. Assume that $\alpha(t_\alpha) = 0$. Then $[t_\alpha, L_\alpha] = 0, [t_\alpha, L_{-\alpha}] = 0$. Let x, y be as in (d). We can assume that $\kappa(x, y) = 1$. Then $[x, y] = t_\alpha$. Let $S = kx + ky + kt_\alpha$, a Lie subalgebra of L. We have $[S, S] = kt_\alpha, [kt_\alpha, kt_\alpha] = 0$ hence S is solvable. By Lie's theorem for $ad : S \to End(L)$ we see that $ad(x') : L \to L$ is nilpotent for any $x' \in [S, S]$. In particular $ad(t_\alpha) : L \to L$ is nilpotent. Since $t_\alpha \in H$ and all elements of H are semisimple, we see that $ad(t_\alpha) : L \to L$ is also semisimple hence is 0. Thus $t_\alpha \in Z(L) = 0$. This contradicts $t_\alpha \neq 0$.

(f) *Let* $\alpha \in R$. *Let* $x \in L_\alpha, x \neq 0$. *We can find* $y \in L_{-\alpha}$ *such that, setting* $h = [x, y] \in H$ *we have* $[h, x] = 2x, [h, y] = -2y$.

By (d),(e) we can find $y \in L_{-\alpha}$ such that $\kappa(x, y) = 2/\alpha(t_\alpha)$. Then $h = 2t_\alpha/\alpha(t_\alpha)$. Hence

$[h, x] = (2/\alpha(t_\alpha))[t_\alpha, x] = (2/\alpha(t_\alpha))\alpha(t_\alpha)x = 2x,$

$[h, y] = (2/\alpha(t_\alpha))[t_\alpha, y] = (2/\alpha(t_\alpha))(-\alpha(t_\alpha)y) = -2y.$

(g) *Let* $\alpha \in R$. *Let* x, y, h *be as in (f). Then* $S = kx+ky+kh$ *is a Lie subalgebra of* L *and* $e \to x, f \to y, h \to h$ *is an isomorphism of Lie algebras* $\mathfrak{sl}_2(k) \xrightarrow{\sim} S$.

This is clear.

(h) *Let* $\alpha \in R$. *Let* $h_\alpha = 2t_\alpha/\alpha(t_\alpha)$ *(see (f)). We have* $h_\alpha = -h_{-\alpha}$.

It suffices to show that $t_\alpha = -t_{-\alpha}$. Since $\kappa|_H$ is non-singular it suffices to show that, for any $h \in H$ we have $\kappa(h, t_\alpha) = -\kappa(h, t_{-\alpha})$ or that $\alpha(h) = -(-\alpha(h))$. This is clear.

(i) *Let* $\alpha \in R$. *Then* $2\alpha \notin R$.

Let x, y, h, S be as in (g). Let $M = \oplus_{c \in k} L_{c\alpha}$ is an S-module under ad and h acts on $L_{c\alpha}$ as multiplication by $c\alpha(h) = c\alpha(2t_\alpha)/\alpha(t_\alpha) = 2c$. By representation theory of $\mathfrak{sl}_2(k)$ (see below) the eigenvalues of $h : M \to M$ are integers. Hence $M = \oplus_{c \in (1/2)\mathbf{Z}} L_{c\alpha}$. Now $S + H$ is an S-submodule of M. By Weyl, there exists an S-submodule M' of M such that $M = (S + H) \oplus M'$. Now the 0-eigenspace of $h : M \to M$ is $L_0 = H$ hence it is contained in $S + H$. Thus the 0-eigenspace of $h : M' \to M'$ is 0. Hence $h : M' \to M'$ does not have eigenvalues in $2\mathbf{Z}$. The eigenvalues of $h : S + H \to S + H$ are $0, 2, -2$. We see that 4 is not an eigenvalue of $h : M \to M$.

If we had $2\alpha \in R$ then a non-zero-vector in $L_{2\alpha}$ would be an eigenvector of $h : M \to M$ with eigenvalue 4, contradiction.

(j) *Let $\alpha \in R$. Then $\alpha/2 \notin R$.*

If we had $\alpha/2 \in R$ then applying (i) to $\alpha/2$ we would deduce that $\alpha \notin R$, contradiction.

(k) *In (i) we have $M' = 0$.*

From (j) we see that $L_{\alpha/2} = 0$ hence the 1-eigenspace of $h : M \to M$ is 0. Thus $h : M' \to M'$ has no eigenvalue 1 (nor 0, see (i)). Hence $h : M' \to M'$ has no odd or even eigenvalues. Hence $M' = 0$.

(l) *Let $\alpha \in R$. We have $\dim L_\alpha = 1$. Moreover $c\alpha \in R, c \in k$ implies $c \in \{1, -1\}$.*

Let x, y, h, S be as in (g). By (k) we have $\oplus_{c \in k} L_{c\alpha} = S + H$. The result follows.

(m) *Let $\alpha, \beta \in R$, $\beta \neq \pm\alpha$. Let $h_\alpha = 2t_\alpha/\alpha(t_\alpha)$. Then $\beta(h_\alpha) \in \mathbf{Z}$ and $\{n \in \mathbf{Z} | \beta + n\alpha \in R\}$ is of the form $\{-r, -r+1, \dots, 0, \dots, q-1, q\}$ where $-r \leq 0 \leq q$.*

Let x, y, h, S be as in (g). Then $h = h_\alpha$. Let $K = \oplus_{n \in \mathbf{Z}} L_{\beta + n\alpha} \subset L$. This is an $S = \mathfrak{sl}_2(k)$-module under ad such that any eigenvalue of h on $L_{\beta+n\alpha}$ is $\beta(h_\alpha) + 2n$. For $n = 0$ the eigenvalue is $\beta(h_\alpha)$ and it has multiplicity > 0 hence $\beta(h_\alpha) \in \mathbf{Z}$. We see that all eigenvalues have multiplicity one and they all have the same parity. It follows that the S-module K is simple. (See below.) The result follows.

(n) *Let $\alpha, \beta \in R$, $\beta \neq \pm\alpha$. Let $h_\alpha = 2t_\alpha/\alpha(t_\alpha)$. We have $\beta - \beta(h_\alpha)\alpha \in R$.*

By (m), we have $\beta(h_\alpha) - 2r = -(\beta(h_\alpha) + 2q)$ that is $\beta(h_\alpha) = r - q$ and we must show that $-r \leq -\beta(h_\alpha) \leq q$ that is $-r \leq -r + q \leq q$. This is clear.

(o) *If $\alpha, \beta, \alpha + \beta \in R$ then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.*

Since $2\alpha \notin R$ we have $\beta \neq \pm\alpha$. Consider the irreducible $S = \mathfrak{sl}_2(k)$-module K in (m). With notations in (m) we have $L_\beta = (r-q)$-eigenspace of $h : K \to K$ and $L_{\alpha+\beta} = (r - q + 2)$-eigenspace of $h : K \to K$. It is enough to show that $e \in \mathfrak{sl}_2(k)$ maps the j-eigenspace of $h : K \to K$ onto the $(j+2)$-eigenspace (if both these eigenspaces are 1-dimensional). This follows from the explicit description of simple $\mathfrak{sl}_2(k)$-modules (see below).

(p) *The smallest Lie subalgebra L' of L that contains L_α for all $\alpha \in R$ is L itself.*

It suffices to show that L' contains H. From (a) it follows that $\{t_\alpha | \alpha \in R\}$ spans H as a vector space. Hence it is enough to show that for $\alpha \in R$ we have $t_\alpha \in L'$. But by (c),(d) we have $t_\alpha \in [L_\alpha, L_{-\alpha}]$.

Rationality.

Define $(,) : H^* \times H^* \to k$ to be the symmetric bilinear form $(\xi, \xi') = \kappa(t_\xi, t_{\xi'}) = \sum_{\alpha \in R} \alpha(t_\xi)\alpha(t_{\xi'})$. This form is non-singular. For $\alpha \in R$ we have $(\xi, \alpha) = \kappa(t_\xi, t_\alpha) = \alpha(t_\xi)$. Hence $(\xi, \xi') = \sum_{\alpha \in R} (\xi, \alpha)(\xi', \alpha)$.

For $\alpha \in R$ we have $(\alpha, \alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$. For $\alpha, \beta \in R$ we have

$2(\alpha, \beta)/(\alpha, \alpha) = \kappa(2t_\alpha/\kappa(t_\alpha, t_\alpha), t_\beta) = \kappa(h_\alpha, t_\beta) = \beta(h_\alpha) \in \mathbf{Z}$.

Now from $(\beta, \beta) = \sum_{\alpha \in R} (\beta, \alpha)^2$ we deduce $4(\beta, \beta)^{-1} = \sum_{\alpha \in R} (2(\beta, \alpha)/(\beta, \beta))^2 \in \mathbf{Z}$. Thus $(\beta, \beta) \in \mathbf{Q}$ hence $(\alpha, \beta) \in \mathbf{Q}$ for any $\alpha, \beta \in R$.

Let E be the $\mathbf{Q}$-subspace of H^* spanned by R. Let $\alpha_1,\dots,\alpha_n$ be a k-basis of H^* contained in R. We show that $\alpha_1,\dots,\alpha_n$ is a $\mathbf{Q}$-basis of E. Let $\alpha\in R$. We have $\alpha=\sum_{i=1}^n c_i\alpha_i$ with $c_i\in k$. It suffices to show that $c_i\in\mathbf{Q}$ for all i. For any $j\in[1,n]$ we have

$2(\alpha,\alpha_j)/(\alpha_j,\alpha_j)=\sum_{i=1}^n c_i 2(\alpha_i,\alpha_j)/(\alpha_j,\alpha_j)$.

This is a linear system of n equations with n unknowns c_i with non-zero determinant and integer coefficients. Hence $c_i\in\mathbf{Q}$ for all i. Hence E coincides with the $\mathbf{Q}$-subspace of H^* spanned by $\alpha_1,\dots,\alpha_n$.

Let $\xi\in E,\xi\ne 0$. We have $(\xi,\xi)=\sum_{\alpha\in R}(\xi,\alpha)^2$. This is a rational number ≥ 0. If it 0 then $(\xi,\alpha)=0$ for all $\alpha\in R$ hence $\xi=0$. Thus $(,)|_E$ has rational values and is positive definite.

We may summarize the properties of $R\subset E$ and $(,)|_E$ as follows:

R spans E as a $\mathbf{Q}$-vector space, $0\notin R$. If $\alpha\in R$ then $-\alpha\in R$ but $c\alpha\notin R$ if $c\in\mathbf{Q}-\{1,-1\}$. If $\alpha,\beta\in R$ then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}\in\mathbf{Z}$ and $\beta-2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha\in R$.