LIE ALGEBRAS

MIT-Fall 2005

Enveloping algebras

In this section the ground field k is arbitrary.

Let L be a Lie algebra. A pair (U, i) where U is an associative algebra and i is a Lie algebra homomorphism $L \longrightarrow U$ is called a universal enveloping algebra of L if the following holds: if U' is any associative algebra and i' is a Lie algebra homomorphism $L \longrightarrow U'$ then there exists a unique algebra homomorphism $f : U \longrightarrow U'$ such that $i' = fi$.

Lemma. *(a) Let $(U, i), (\tilde{U}, \tilde{i})$ be universal enveloping algebras of L. Then there exists a unique algebra isomorphism $j : U \xrightarrow{\sim} U'$ such that $i = \tilde{i}j$.*

(b) U is generated as an algebra by $i(L)$.

(c) Let L_1, L_2 be Lie algebras. Let $(U_1, i_1), (U_2, i_2)$ be universal enveloping algebras of L_1, L_2. Let $f : L_1 \longrightarrow L_2$ be a Lie algebra homomorphism. Then there exists a unique algebra homomorphism $\tilde{f} : U_1 \longrightarrow U_2$ such that $i_2 f = \tilde{f} i_1$.

(d) Let I be an ideal of L and let $\tilde{I}$ be the ideal of U generated by $i(L)$. Then $i : L \longrightarrow U$ induces a Lie algebra homomorphism $j : L/I \longrightarrow U/\tilde{I}$ and $(U/\tilde{I}, j)$ is a universal enveloping algebras of L/I.

(e) There is a unique algebra anti-automorphism $\pi : U \longrightarrow U$ such that $\pi i = -i$. We have $\pi^2 = 1$.

(f) There is a unique algebra homomorphism $\delta : U \longrightarrow U \otimes U$ such that $\delta(i(a)) = i(a) \otimes 1 + 1 \otimes i(a)$ for all $a \in L$.

(g) If $D : L \longrightarrow L$ is a derivation then there is a unique derivation $D' : U \longrightarrow U$ such that $iD = D'i$.

(a)-(f) are standard. We prove (g). Let U_2 be the algebra of 2×2 matrices with entries in U. Define a linear map $i' : L \longrightarrow U_2$ by

$$a \mapsto \begin{matrix} i(a) & i(D(a)) \\ 0 & i(a) \end{matrix}$$

This is a Lie algebra homomorphism:

$$i'([a,b]) = \begin{matrix} i(a)i(b)-i(b)i(a) & i(D(a))i(b)-i(b)i(D(a))+i(a)i(D(b))-i(D(b))i(a) \\ 0 & i(a)i(b)-i(b)i(a) \end{matrix}$$

$$\begin{matrix} i(a) & i(D(a)) \\ 0 & i(a) \end{matrix} \begin{matrix} i(b) & i(D(b)) \\ 0 & i(b) \end{matrix} - \begin{matrix} i(b) & i(D(b)) \\ 0 & i(b) \end{matrix} \begin{matrix} i(a) & i(D(a)) \\ 0 & i(a) \end{matrix} = i'(a)i'(b) - i'(b)i'(a).$$

Hence there is an algebra homomorphism $j : U \longrightarrow U_2$ such that $i' = ji$. We have $j(x) = \begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}$ for all $x \in U$ where y is uniquely determined by x. Indeed this is true for $x \in i(L)$ and these generate U. We set $y = D'(x)$ where $D' : U \longrightarrow U$. Then D' is a derivation of U such that $iD = D'i$.

Construction of a universal enveloping algebra. Let T be the tensor algebra of L. By definition, $T = T_0 \oplus T_1 \oplus T_2 \oplus \ldots$ where $T_0 = k1, T_1 = L$ and $T_i = L \otimes L \otimes \ldots L$ (i times). The algebra structure is characterized by

$$(x_1 \otimes \ldots \otimes x_i)(y_1 \otimes \ldots \otimes y_j) = x_1 \otimes \ldots \otimes x_i \otimes y_1 \otimes \ldots \otimes y_j.$$

Let K be the ideal of T generated by the elements of form $[a, b] - a \otimes b - b \otimes a$ with $a, b \in L$. Let $U = T/I$. Let $i : L \longrightarrow U$ be the composition of the canonical maps $L \longrightarrow T \longrightarrow U$. We have

$i[a, b] - i(a)i(b) + i(b)i(a) = K -$ coset of $[a, b] - a \otimes b + b \otimes a = K$.
Hence $i : L \longrightarrow U$ is a Lie algebra homomorphism.

Proposition. *(U, i) is a universal enveloping algebra of L.*

Let $\{u_j | j \in J\}$ be a basis of the vector space L. The monomials $u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_n}$ (where $j_1, j_2, \ldots, j_n \in J$) form a basis of T_n. We assume that J is ordered. Define

$index(u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_n}) = \sum_{i<k} \eta_{ik}$
where $\eta_{ik} = 0$ if $j_i \leq j_k$ and $\eta_{ik} = 1$ if $j_i > j_k$. We have $index(u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_n}) = 0$ if and only if $j_1 \leq j_2 \leq \ldots j_n$. In this case the monomial is said to be standard. We regard 1 as a standard monomial. Assume now that $j_k > j_{k+1}$; then

$$\begin{aligned} &index(u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_n}) \\ &= 1 + index(u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_{k+1}} \otimes u_{j_k} \otimes \ldots \otimes u_{j_n}). \end{aligned}$$

Lemma 1. *Every element $x \in T$ is congruent modulo K to a linear combination of standard monomials.*

We may assume that x is a monomial. We may assume that x has degree $n > 0$ and index p and that the result is true for monomials of degree $< n$ or for monomials of degree n and index $< p$. Assume $x = u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_n}$ is not standard and suppose $j_k > j_{k+1}$. We have

$$\begin{aligned} &u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_n} \\ &= u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_{k+1}} \otimes u_{j_k} \otimes \ldots \otimes u_{j_n} \\ &+ u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes (u_{j_k} \otimes u_{j_{k+1}} - u_{j_{k+1}} \otimes u_{j_k}) \otimes \ldots \otimes u_{j_n} \\ &= u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_{k+1}} \otimes u_{j_k} \otimes \ldots \otimes u_{j_n} \\ &+ u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes [u_{j_k}, u_{j_{k+1}}] \otimes \ldots \otimes u_{j_n} \mod K. \end{aligned}$$

The result follows from the induction hypothesis.

We now introduce the vector space P_n with basis $u_{i_1} u_{i_2} \ldots u_{i_n}$ indexed by the various $i_1 \leq i_2 \leq \ldots i_n$ in J. Let $P = P_0 \oplus P_1 \oplus P_2 \oplus \ldots$.

Lemma 2. *There exists a linear map $\sigma : T \to P$ such that*

(a) $\sigma(u_{i_1} \otimes u_{i_2} \otimes \ldots \otimes u_{i_n}) = u_{i_1} u_{i_2} \ldots u_{i_n}$ *if* $i_1 \le i_2 \le \ldots i_n$,

(b) $\sigma(u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_n} - u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_{k+1}} \otimes u_{j_k} \otimes \ldots \otimes u_{j_n}) = \sigma(u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes [u_{j_k}, u_{j_{k+1}}] \otimes \ldots \otimes u_{j_n})$

for any $j_1, j_2, \ldots, j_n \in J$ *and any* k.

Let $T_{n,j}$ be the subspace of T_n spanned by the monomials of degree n and index $\le j$. Define $\sigma(1) = 1$. Assume that σ is already defined on $T_0 \oplus T_1 \oplus \ldots \oplus T_{n-1}$ and it satisfies (a),(b) for monomials of degree $< n$. We extend σ linearly to $T_0 \oplus T_1 \oplus \ldots \oplus T_{n-1} \oplus T_{n,0}$ by requiring that $\sigma(u_{i_1} \otimes u_{i_2} \otimes \ldots \otimes u_{i_n}) = u_{i_1} u_{i_2} \ldots u_{i_n}$ for a standard monomial of degree n. Now assume that $i \ge 1$ and that σ has already been defined on $T_0 \oplus T_1 \oplus \ldots \oplus T_{n-1} \oplus T_{n,i-1}$ so that (a),(b) is satisfied for monomials of degree of degree $< n-1$ or for monomials of degree n and index $< i$. Now let $u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_n}$ be of index i. Suppose that $j_k > j_{k+1}$. We set

$(*)$ $\sigma(u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_n}) = \sigma(u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_{k+1}} \otimes u_{j_k} \otimes \ldots \otimes u_{j_n} + \sigma(u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes [u_{j_k}, u_{j_{k+1}}] \otimes \ldots \otimes u_{j_n})$.

This makes sense. We show that $(*)$ is independent of the choice of the pair $j_k > j_{k+1}$. Assume that we have another pair $j_l > j_{l+1}$. There are two cases: (1) $l > k+1$, (2) $l = k+1$.

Case (1). We set $u_{j_k} = u, u_{j_{k+1}} = v, u_{j_l} = w, u_{j_{l+1}} = t$. By the first definition

$$\begin{aligned}
&\sigma(\ldots u \otimes v \otimes \ldots \otimes w \otimes t \ldots) \\
&= \sigma(\ldots v \otimes u \otimes \ldots \otimes w \otimes t \cdots + \ldots [u, v] \otimes \ldots \otimes w \otimes t \ldots) \\
&= \sigma(\ldots v \otimes u \otimes \ldots \otimes t \otimes w \cdots + \ldots v \otimes u \otimes \ldots \otimes [w, t] \ldots \\
&+ \ldots [u, v] \otimes \ldots \otimes t \otimes w \cdots + \ldots [u, v] \otimes \ldots \otimes [w, t] \ldots).
\end{aligned}$$

The second definition leads to the same expression.

Case (2). We set $u_{j_k} = u, u_{j_{k+1}} = v = u_{j_l}, u_{j_{l+1}} = w$. By the first definition

$$\begin{aligned}
&\sigma(\ldots u \otimes v \otimes w \ldots) = \sigma(\ldots v \otimes u \otimes w \cdots + \ldots [u, v] \otimes w \ldots) \\
&= \sigma(\ldots v \otimes w \otimes u \cdots + \ldots v \otimes [u, w] \cdots + \ldots [u, v] \otimes w \ldots) \\
&= \sigma(\ldots w \otimes v \otimes u \cdots + \ldots [v, w] \otimes u \cdots + \ldots v \otimes [u, w] \cdots + \ldots [u, v] \otimes w \ldots).
\end{aligned}$$

By the second definition

$$\begin{aligned}
&\sigma(\ldots u \otimes v \otimes w \ldots) = \sigma(\ldots u \otimes w \otimes v \cdots + \ldots u \otimes [v, w] \ldots) \\
&= \sigma(\ldots w \otimes u \otimes v \cdots + \ldots [u, w] \otimes v \cdots + \ldots u \otimes [v, w] \ldots) \\
&= \sigma(\ldots w \otimes v \otimes u \cdots + \ldots w \otimes [u, v] \cdots + \ldots [u, w] \otimes v \cdots + \ldots u \otimes [v, w] \ldots).
\end{aligned}$$

Thus we are reduced to proving

$\sigma(\ldots [v, w] \otimes u \cdots + \ldots v \otimes [u, w] \cdots + \ldots [u, v] \otimes w \ldots) = \sigma(\ldots w \otimes [u, v] \cdots + \ldots [u, w] \otimes v \cdots + \ldots u \otimes [v, w] \ldots)$

or equivalently $\sigma(\ldots [[v, w], u] \cdots + \ldots [v, [u, w]] \cdots + \ldots [[u, v], w] \ldots) = 0$

which follows from $[[v, w], u] + [v, [u, w]] + [[u, v], w] = 0$. The lemma is proved.

Theorem (Poincaré-Birkhoff-Witt. *The standard monomials form a basis of $U = T/K$.*

By lemma 1 the standard monomials span U. Now K is spanned by elements of the form

$u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_n} - u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_{k+1}} \otimes u_{j_k} \otimes \ldots \otimes u_{j_n} - u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes [u_{j_k}, u_{j_{k+1}}] \otimes \ldots \otimes u_{j_n}$

hence $\sigma(K) = 0$ and σ induces a linear map $U \longrightarrow P$. This linear map takes the standard monomials to linearly independent elements of P. Hence the standard monomials are linearly independent in U.

Corollary. *The map $i : L \longrightarrow U$ is injective.*

Free Lie algebra. Let X be a set. The free Lie algebra generated by X is a pair (F, i) where F is a Lie algebra and $i : X \longrightarrow F$ is a map such that, if $i' : X \longrightarrow F'$ is a map of X into a Lie algebra, there is a unique Lie algebra homomorphism $j : F \longrightarrow F'$ such that $i' = ji$. We show the existence of (F, i). Let V be the vector space with basis X. Let T be the tensor algebra of V. Let F be the Lie subalgebra of T generated by X. Then i is the obvious imbedding $X \subset F$. Let $i' : X \longrightarrow F'$ be a map into a Lie algebra. This extends to a linear map $V \longrightarrow F'$. Let $h : F' \longrightarrow U'$ be the enveloping algebra of F'. The composition $V \longrightarrow F' \xrightarrow{h} U'$ extends to an algebra homomorphism $T \longrightarrow U'$ and this restricts to a Lie algebra homomorphism $a : F \longrightarrow U'$. Now $a(X) \subset h(F')$. Since F is generated by X as a Lie algebra, and $h(F')$ is a Lie subalgebra, we see that $a(F) \subset h(F')$. Since h is injective (by the PBW theorem) there exists a unique homomorphism of Lie algebras $j : F \longrightarrow F'$ such that $F \xrightarrow{a} U'$ ie equal to $F \xrightarrow{j} F' \xrightarrow{h} U'$. This shows that (F, i) is the free Lie algebra generated by X.

$\mathfrak{sl}_2(k)$-MODULES

Let $L = \mathfrak{sl}_2(k)$. A basis is given by

$$e = \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}, f = \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}, g = \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}.$$

We have $[e, f] = h, [h, e] = 2e, [h, f] = -2f$. Thus, h is semisimple. Since L is simple, it is semisimple. Let V be an L-module, $\dim V < \infty$. Then $h : V \longrightarrow V$ is semisimple. Thus $V = \oplus_{\lambda \in k} V_\lambda$ where $V_\lambda = \{v \in V | hv = \lambda v\}$.

If $v \in V_\lambda$ then $ev \in V_{\lambda+2}, fv \in V_{\lambda-2}$.

Assume now that V is irreducible. We can find $v_0 \in V - \{0\}$ such that $v_0 \in V_\lambda$, $ev_0 = 0$. Set $v_{-1} = 0, v_n = \frac{f^n}{n!} v_0$, $n \in \mathbf{N}$. We have

(a) $hv_n = (\lambda - 2n)v_n$ for $n \geq -1$

(b) $fv_n = (n + 1)v_{n+1}$ for $n \geq -1$

(c) $ev_n = (\lambda - n + 1)v_{n-1}$ for $n \geq 0$.
(c) is shown by induction on n. For $n = 0$ it is clear. Assuming $n \geq 1$,

$$\begin{aligned} ev_n &= n^{-1}efv_{n-1} = n^{-1}hv_{n-1} + n^{-1}fev_{n-1} \\ &= n^{-1}(\lambda - 2n + 2)v_{n-1} + n^{-1}f(\lambda - n + 2)v_{n-2} \\ &= n^{-1}(\lambda - 2n + 2)v_{n-1} + n^{-1}(\lambda - n + 2)(n-1)v_{n-1} = (\lambda - n + 1)v_{n-1}. \end{aligned}$$

By (a), the non-zero v_n are linearly independent. Since $\dim V < \infty$, there exists $m \geq 0$ such that $v_0, v_1, \dots, v_m$ are $\neq 0$ and $v_{m+1} = 0$. Then $v_{m+2} = v_{m+3} = \cdots = 0$. Now $v_0, v_1, \dots, v_m$ form a basis of an L-submodule which must be the whole of V. Now (c) with $n = m + 1$ gives $0 = (\lambda - m)v_n$ hence $\lambda = n$. Thus the action of e, f, h in the basis $v_0, v_1, \dots, v_m$ is
$hv_n = (m - 2n)v_n$ for $n \in [0, m]$
$fv_n = (n + 1)v_{n+1}$ for $n \in [0, m]$
$ev_n = (m - n + 1)v_{n-1}$ for $n \in [0, m]$
with the convention $v_{-1} = 0, v_{m+1} = 0$.

Conversely, given $m \geq 0$ we can define an L-module structure on an $m + 1$ dimensional vector space with basis $v_0, v_1, \dots, v_m$ by the formulas above. Thus we have a 1-1 correspondence between the set of isomorphism classes of irreducible L-modules and the set $\mathbf{N}$.

Now let V be any finite dimensional L-module. Then:
(a) the eigenvalues of $h : V \longrightarrow V$ are integers; the multiplicity of the eigenvalue a equals that of $-a$.
(b) If $h : V \longrightarrow V$ has an eigenvalue in $2\mathbf{Z}$ then it has an eigenvalue 0.
(c) If $h : V \longrightarrow V$ has an eigenvalue in $2\mathbf{Z} + 1$ then it has an eigenvalue 1.
Indeed, by Weyl, we are reduced to the case where V is irreducible; in that case we use the explicit description of L given above.

A property of $\mathfrak{sl}_2$-modules

Let V be a $\mathfrak{sl}_2$-module such that $e : V \longrightarrow V, f : V \longrightarrow V$ are locally nilpotent. Then $\exp(e) : V \longrightarrow V, \exp(-f) : V \longrightarrow V$ are well defined isomorphisms. Hence $\tau = \exp(e)\exp(-f)\exp(e) : V \longrightarrow V$ is a well defined isomorphism. For any integer n let $V_n = \{x \in V | hx = nx\}$. Assume that $V = \oplus_n V_n$

Lemma. $\tau(V_n) \subset V_{-n}$.

Step 1. Assume that V has a basis ξ, η where $e\xi = 0, e\eta = \xi, f\xi = \eta, f\eta = 0, h\xi = \xi, h\eta = -\eta$.

We have $V = V_1 \oplus V_{-1}$ and $\exp(e)\xi = \xi, \exp(e)\eta = \eta = \xi, \exp(-f)\xi = \xi - \eta, \exp(-f)\eta = \eta$. It follows that $\tau(\xi) = -\eta, \tau(\eta) = \xi$. hence the result follows in this case.

Step 2. Assume that the result holds for V and for V'. We show that it holds for $V \otimes V'$ where $x \in \mathfrak{sl}_2$ acts as $x \otimes 1 + 1 \otimes x$.

A simple computation shows that for $x \in \mathfrak{sl}_2$, locally nilpotent, $\exp(x)$ acts on $V \otimes V'$ as $\exp(x) \otimes \exp(x)$. Hence τ acts on $V \otimes V'$ as $\tau \otimes \tau$. The result follows easily.

Step 3. If the result holds for V then it holds for any direct summand of V (as a $\mathfrak{sl}_2$-module).

(Obvious.)

Step 4. The result holds when V is the irreducible module of dimension n.

(Induction on n.) This is obvious for $n = 1$ and is true for $n = 2$ by Step 1. Assume now that $n \geq 3$. Then V is a direct summand of $V' \otimes V''$ where V' is an irreducible module of dimension $n - 1$ and V'' is an irreducible module of dimension 2. By the induction hypothesis, the result holds for V', V'' hence it holds for $V' \otimes V''$ by Step 2 and for V by Step 3.

Step 5. The result holds when $\dim V < \infty$.

Follows from the complete reducibility of V and Step 4.

Step 6. The result holds in general.

Let $x \in V_n$. Let N, N' be such that $e^{N+1}x = 0, f^{N'+1}x = 0$. The subspace of V spanned by $f^i e^j x$ with $0 \leq j \leq N, 0 \leq i \leq N + N'$ is easily seen to be an $\mathfrak{sl}_2$-submodule V'. We have $\dim V' < \infty$. By Step 5 the result holds for V'. Hence $\tau(x) \in V_{-n}$.