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Proposition. *Assume that k is algebraically closed, of characteristic 0. Let L be a solvable Lie algebra, $\dim(L) < \infty$, let V be a vector space, $0 < \dim V < \infty$ and let $\rho : L \longrightarrow End(V)$ be a Lie algebra homomorphism. Then there exists $v \in V - \{0\}$ such that $\rho(x)(v) \in kv$ for all $x \in L$.*

Induction on $\dim(L)$. We can assume $L \neq 0$. Since L is solvable we have $[L, L] \neq L$. Since $L/[L, L]$ is abelian, any hyperplane in it is an ideal; taking its inverse image under $L \longrightarrow L/[L, L]$ we obtain an ideal I of codimension 1 in L with $[L, L] \subset I$. By the induction hypothesis (for I, V instead of L, V) we can find $v' \in V - \{0\}$ and a linear form $\lambda : I \longrightarrow k$ such that $\rho(y)(v') = \lambda(y)v'$ for all $y \in I$. Then $W = \{w \in V | \rho(y)(w) = \lambda(y)w \forall y \in I\} \neq 0$.

Let $w \in W - \{0\}$ and let $x \in L$. For $i \geq 0$ let $W_i = kw + k\rho(x)w + \cdots + k\rho(x)^{i-1}w$. By induction on $i \geq 0$ we see that for any $y \in I$:

(a) $\rho(y)\rho(x)^i w = \lambda(y)\rho(x)^i w \mod W_i$

We can find $n > 0$ so that $\dim W_n = n, W_n = W_{n+1}$. Hence $\rho(x)W_n \subset W_n$. From (a) we see that, if $y \in I$, then $\rho(y)$ acts with respect to the basis

$w, \rho(x)w, \dots, \rho(x)^{n-1}w$

of W_n as an upper triangular matrix with diagonal entries $\lambda(y)$. Hence $\mathrm{tr}(\rho(y), W_n) = n\lambda(y)$. Now $\rho(x)W_n \subset W_n, \rho(y)W_n \subset W_n$. Hence $\rho([x, y])$ acts on W_n as the commutator of two endomorphisms of W_n so that $\mathrm{tr}(\rho([x, y]), W_n) = 0$. Thus $n\lambda([x, y]) = 0$ so that

(b) $\lambda([x, y]) = 0$ (for $x \in L, y \in I$).

Let $w \in W, x \in L$. For $y \in I$ we have

$\rho(y)\rho(x)w = \rho(x)\rho(y)w - \rho([x, y])w = \lambda(y)\rho(x)w - \lambda([x, y])w = \lambda(y)\rho(x)w.$

(We have used (b).) Thus $\rho(x)w \subset W$. We see that

(c) $\rho(x)W \subset W$ for all $x \in L$.

Now choose $z \in L - I$. We have $L = I \oplus kz$. Since $\rho(z)W \subset W$, we can find $v \in W - \{0\}$ such that $\rho(z)v \in kv$. Since $\rho(x)(v) \in kv$ for all $x \in I$ we deduce that $\rho(x)(v) \in kv$ for all $x \in L$.

Corollary (Lie). *Assume that k is algebraically closed, of characteristic 0. Let L be a solvable Lie algebra, $\dim(L) < \infty$, let V be a vector space, $\dim V < \infty$ and let $\rho : L \longrightarrow End(V)$ be a Lie algebra homomorphism. Then there exists a sequence of vector subspaces $0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n = V$, $\dim V_i = i$ such that $\rho(x)V_i \subset V_i$ for $i \in [0, n]$.*

Induction on $\dim(V)$ using the previous proposition.

Corollary. *Assume that k is algebraically closed, of characteristic 0. Let L be a solvable Lie algebra, $\dim(L) < \infty$. Then there exists a sequence of ideals $0 = L_0 \subset L_1 \subset L_2 \subset \ldots \subset L_n = L$ of L, $\dim L_i = i$.*

Apply the previous corollary with $V = L$.

Corollary. *Assume that k is algebraically closed, of characteristic 0. Let L be a solvable Lie algebra, $\dim(L) < \infty$.*
(a) If $x \in [L, L]$ then $ad(x) : L \longrightarrow L$ is nilpotent.
(b) $[L, L]$ is a nilpotent Lie algebra.

Let $0 = L_0 \subset L_1 \subset L_2 \subset \ldots \subset L_n = L$ be as in the previous corollary. Consider a basis $e_1, \ldots, e_n$ of L so that $e_1, \ldots, e_i$ is a basis of L_i for all i. If $x, x' \in L$ then $ad(x), ad(x')$ act in this basis as upper triangular matrices. Hence $[ad(x), ad(x')] = ad([x, x'])$ acts in this basis as an upper triangular matrix with 0 on the diagonal. Thus, if $y \in [L, L]$, $ad(y) : L \longrightarrow L$ acts as an upper triangular matrix with 0 on the diagonal hence it is nilpotent. Hence $ad(y) : [L, L] \longrightarrow [L, L]$ is nilpotent for all $y \in [L, L]$. By Engel, $[L, L]$ is nilpotent.

We now study the "*Jordan decomposition*" of an endomorphism x of a finite dimensional vector space V. (Assume that k is algebraically closed.) We say that x is *semisimple* if there exist $a_1, a_2, \ldots, a_n$ distinct in k such that $(x - a_1)(x - a_2)\ldots(x - a_n) = 0 : V \longrightarrow V$. (Equivalently, V is direct sum of x-eigenspaces.) Then the restriction of x to an x-stable subspace is semisimple and there exists a complementary x-stable subspace. If $x, x' : V \longrightarrow V$ are semisimple and commute then V is sum of the joint eigenspaces of x and x'.

Let $x \in End(V)$. We have $V = \oplus_{\lambda \in k} V_\lambda$ where V_λ is the generalized λ-eigenspace of x. Let $x_s : V \longrightarrow V$ be defined by $x_s = \lambda$ on V_λ. Let $x_n : V \longrightarrow V$ be defined by $x_n = x - \lambda$ on V_λ. Then x_s is semisimple, x_n is nilpotent and

$x = x_s + x_n$, $x_s x_n = x_n x_s$.

(This is the Jordan decomposition of x.) Clearly, if $t \in End(V)$ commutes with x then t commutes with x_s and with x_n.

Conversely, assume that $s, n \in End(V)$,$x = s + n$, $sn = ns$ and s is semisimple, n is nilpotent. We show that $s = x_s, n = x_n$. Indeed, s commutes with $s + n = x$ hence with x_s and x_n. Since s, x_s are commuting semisimple, we have that $s - x_s$ is semisimple. Since n, x_n are commuting nilpotent, we have that $n - x_n$ is nilpotent.Now $s - x_s = -n + x_n$ is both semisimple and nilpotent hence is 0. Thus, $s = x_s, n = x_n$.

Lemma. *Let $x \in End(V)$. Let $A \subset B$ be subspaces of V. Assume that $x(B) \subset A$. Then $x_s(B) \subset A, x_n(B) \subset A$.*

The proof is routine.

Lemma. *Let $x \in End(V)$. Write $x = x_s + x_n$ as above.*
(a) If x is semisimple then $ad(x) : End(V) \longrightarrow End(V)$ is semisimple.

(b) If x is nilpotent then $ad(x) : End(V) \to End(V)$ is nilpotent.

(c) $ad(x) = ad(x_s) + ad(x_n)$ is the Jordan decomposition of $ad(x) : End(V) \to End(V)$.

(a),(b) are immediate. Now (c) follows from (a),(b) since $ad(x_s), ad(x_n)$ satisfy the characterization above of the semisimple and nilpotent part of $ad(x)$. (Note that $[ad(x_s), ad(x_n)] = ad([x_s, x_n]) = 0$.)

From now on, k is assumed to be algebraically closed, of characteristic 0.

Lemma. *Let V be a finite dimensional vector space. Let $A \subset B$ be subspaces of $End(V)$. Let $M = \{x \in End(V) | [x, B] \subset A\}$. Assume that $x \in M$ satisfies $\mathrm{tr}(xy) = 0$ for all $y \in M$. Then x is nilpotent.*

Let $x = s + n$ be the Jordan decomposition of x. Let (v_i) be a basis of V such that $sv_i = a_i v_i, a_i \in k$. Let $e_{ij} \in End(V)$ be defined by $e_{ij}(v_j) = v_i$ and $e_{ij}(v_k) = 0$ for $k \neq j$. Then $ad(s)e_{ij} = (a_i - a_j)e_{ij}$.

Let E be the $\mathbf{Q}$-vector subspace of k spanned by the a_i. Let $f : E \to \mathbf{Q}$ be a linear form. Let $y \in End(V)$ be such that $yv_i = f(a_i)v_i$ for all i. Then $ad(y)e_{ij} = (f(a_i) - f(a_j))e_{ij}$. We can find $P(T) \in k[T]$ such that $P(0) = 0$, $P(a_i - a_j) = f(a_i) - f(a_j)$ for all i, j. We have $ad(y) = P(ad(s))$. Now $ad(x)(B) \subset A$. Hence its semisimple part $ad(s)$ maps B into A. Since $ad(y) = P(ad(s))$ and $P(0) = 0$ we see that $ad(y)(B) \subset A$ that is $y \in M$. By assumption, $tr(xy) = 0$, hence $\sum_i a_i f(a_i) = 0$. Applying f to this identity in E gives $\sum_i f(a_i)^2 = 0$. Since $f(a_i) \in \mathbf{Q}$ we have $f(a_i) = 0$. Thus, $f = 0$. Since this holds for any f we have $E = 0$. Hence $a_i = 0$ for all i and x is nilpotent.

Remark. *Let V be a vector space, $\dim V < \infty$. Let $x, y, z \in End(V)$. We have $\mathrm{tr}([x, y]z) = \mathrm{tr}([y, z]x)$.*

We have $\mathrm{tr}([x, y]z) = \mathrm{tr}([y, z]x) = \mathrm{tr}(xyz - yxz - yzx + zyx) = 0$.

Theorem(Cartan). *Let V be as above. Let L be a Lie subalgebra of $End(V)$. Assume that $\mathrm{tr}(xy) = 0$ for all $x \in [L, L], y \in L$. Then L is solvable.*

Apply the previous lemma with $A = [L, L], B = L$ so that
$M = \{z \in End(V) | [z, L] \subset [L, L]\}$.
We have $L \subset M$. Let $x' \in [L, L], z \in M$. We show that
(a) $\mathrm{tr}(x'z) = 0$.
We may assume that $x' = [x, y]$ where $x, y \in L$. Then $\mathrm{tr}(x'z) = \mathrm{tr}([x, y]z) = \mathrm{tr}([y, z]x)$. This is 0 since $[y, z] \in [L, L]$ (by the definition of M) and $x \in L$ (see our assumption).

Now (a) shows that the previous lemma is applicable. We see that x' is nilpotent. Hence $ad(x') : End(V) \to End(V)$ is nilpotent. Hence $ad(x') : [L, L] \to [L, L]$ is nilpotent for any $x' \in [L, L]$. By Engel, $[L, L]$ is nilpotent. This clearly implies that L is solvable. The theorem is proved.

Corollary. *Let k be as above. Let L be a Lie algebra,* $\dim(L) < \infty$. *Assume that* $\mathrm{tr}(ad(x)ad(y)) = 0$ *for all* $x \in [L, L], y \in L$. *Then L is solvable.*

Apply the theorem to $(ad(L), L)$ instead of (L, V). Here $ad(L)$ is the image of $ad : L \longrightarrow End(L)$. We see that $ad(L)$ is solvable. Now $ad(L) = L/Z(L)$ and $Z(L)$ is solvable. We deduce that L is solvable.

Definition. Let L be a Lie algebra, $\dim(L) < \infty$. For $x, y \in L$ we set

$\kappa(x, y) = \mathrm{tr}(ad(x)ad(y))$

(*Killing form*). This form is symmetric and $\kappa([x, y], z) = \kappa(x, [y, z])$. The radical of κ, that is, $\{x \in L | \kappa(x, y) = 0 \forall y \in L\}$ is an ideal of L.

Lemma. *Let I be an ideal of L. Let κ (resp. κ_I) be the Killing form of L (resp. I). Then κ_I is the restriction of κ to $I \times I$.*

Let $x, y \in I$. Then $ad(x)ad(y) : L \longrightarrow L$ maps L into I hence its trace on L is equal to the trace of the restriction to I.

Theorem. *Let L be as above. Then L is semisimple if and only if κ is non-degenerate.*

Let $R = rad(L)$. Let S be the radical of κ. For any $x \in S, y \in L$ we have $\mathrm{tr}(ad(x)ad(y)) = 0$. In particular this holds for $y \in [S, S]$. By Cartan's theorem the image of S under $ad : S \longrightarrow End(L)$ is solvable. Since the kernel of this map is contained in $Z(L)$ (so is abelian) it follows that S is solvable. Since S is an ideal, we have

$S \subset R$.

Thus, if $R = 0$ then $S = 0$. Assume now that $S = 0$. Let I be an abelian ideal of L. Let $x \in I, y \in L$. Then $ad(x)ad(y)$ maps L into I and $(ad(x)ad(y))^2$ maps L into $[I, I] = 0$. Thus $ad(x)ad(y)$ is nilpotent hence $0 = \mathrm{tr}(ad(x)ad(y))$. Thus $I \subset S$. Hence $I = 0$. Now let I' be a solvable non-abelian ideal of L. Then $[I', I'] \neq 0$. Since $[I', I']$ is nilpotent (corollary of Lie's theorem) we must have $Z([I', I']) \neq 0$. Now $Z([I', I'])$ is an abelian ideal of L hence it is 0 by an earlier part of the argument. Contradiction. We see that any solvable ideal of L is 0. Hence $R = 0$.

If $L_1, L_2, \dots, L_t$ are Lie algebras then $L_1 \oplus L_2 \oplus \dots \oplus L_t$ is a Lie algebra with bracket

$[(l_1, l_2, \dots, l_t), (l'_1, l'_2, \dots, l'_t)] = ([l_1, l'_1], [l_2, l'_2], \dots, [l_t, l'_t])$.

Clearly each L_i is an ideal of $L_1 \oplus L_2 \oplus \dots \oplus L_t$.

Proposition. *Let L be as above. Assume that L is semisimple.*

(a) There exist ideals $L_1, L_2, \dots, L_t$ of L which are simple as Lie algebras such that $L = L_1 \oplus L_2 \oplus \dots \oplus L_t$ (as a Lie algebra).

(b) Let I be a simple ideal of L. Then $I = L_i$ for some i.

(c) If $L_1, L_2, \dots, L_t$ are simple finite dimensional Lie algebras then $L = L_1 \oplus L_2 \oplus \dots \oplus L_t$ is a semisimple Lie algebra.

Let I be an ideal of L. Let $I^\perp = \{x \in L | \kappa(x, y) = 0 \forall y \in I\}$. If $x \in I^\perp$ and $z \in L$ then $[z, x] \in I^\perp$. Indeed for $y \in I$, $\kappa([z, x], y) = \kappa(x, [y, z]) = 0$ since $[y, z] \in I$.

Thus, $I^\perp$ is an ideal. Now the Killing form of the ideal $I \cap I^\perp$ is the restriction of that of L hence is 0. By Cartan, $I \cap I^\perp$ is solvable. Since L is semisimple and $I \cap I^\perp$ is a solvable ideal of L we have $I \cap I^\perp = 0$. Since $\dim I + \dim I^\perp = \dim L$ (by the non-degeneracy of the Killing form) we see that $L = I \oplus I^\perp$.

We prove (a) by induction on $\dim L$. If $L = 0$ there is nothing to prove. Assume now that $L \neq 0$. Since L is semisimple, $\neq 0$ it is not solvable hence non-abelian. If L has no ideal other than $0, L$ then L is simple and we are done. If L has an ideal I other than $0, L$ then $L = I \oplus I^\perp$ and $I, I^\perp$ are semisimple. We apply the induction hypothesis to I and $I^\perp$. The result for L follows.

We prove (b). Let I be a simple ideal of L. Then $[I, L]$ is an ideal of I. If it were 0 we would have $I \subset Z(L) = 0$, absurd. Hence $[I, L] = I$. Now $[I, L] = [I, L_1] \oplus [I, L_2] \oplus \ldots \oplus [I, L_t]$ hence all summands except one are 0 and one, say $[I, L_1]$, is $[I, L] = I$. Then $I = [I, L_1] \subset L_1$ hence $I \subset L_1$. Since L_1 is simple we have $I = L_1$.

We prove (c). The Killing form of L is direct sum of the Killing forms of the various L_i hence is non-degenerate.

Corollary. *Let L be as in the previous proposition. Then*
(a) $L = [L, L]$;
(b) any ideal of L is a direct sum of simple ideals of L;
(c) any ideal of L is semisimple;
(d) any quotient Lie algebra of L is semisimple.

For (a) we may assume that L is simple. Then the result is clear. We prove (b). If I is an ideal of L then $I \oplus I' = L$ for some ideal I'. Since the Killing form of L is the direct sum of those of I and I', we see that the Killing form of I is non-degenerate hence I is a semisimple Lie algebra. Similarly I' is a semisimple Lie algebra. Hence I, I' are direct sums of simple Lie algebras. The summands are the simple summands of L hence are ideals of L. This proves (b),(c),(d).

Proposition. *Let L be as in the previous proposition. Let L' be the image of $ad : L \to D = Der(L)$. Then $L = L' = D$.*

Since L is semisimple we have $Z(L) = 0$ hence $L = L'$. Thus, L' has a non-degenerate Killing form $\kappa_{L'}$. Let $\delta \in D, x \in L$. We have $[\delta, ad(x)] = ad(\delta(x)) : L \to L$. (Equivalently $\delta([x, y]) - [x, \delta(y)] = [\delta(x), y]$ for $y \in L$.) Thus L' is an ideal of D. Hence $\kappa_{L'}$ is the restriction of κ_D, the Killing form of D. Let $I = \{\delta \in D | \kappa_D(\delta, L') = 0\}$. Then $I \cap L' = 0$ by the nondegeneracy of $\kappa_{L'}$. Now I, L' are ideals of D hence $[I, L'] = 0$. Hence if $\delta \in I$ and $x \in L$, we have $ad(\delta(x)) = [\delta, ad(x)] = 0$. Since $ad : L \to End(L)$ is injective, it follows that $\delta(x) = 0$ for all $\delta \in I, x \in L$ hence $\delta = 0$ for all $\delta \in I$ hence $I = 0$. Equivalently, the map $D \to \mathrm{Hom}(L', k)$ given by $\delta \mapsto [l' \mapsto \kappa_D(\delta, l')]$ is injective. Thus $\dim D \leq \dim L'$. Since $L' \subset D$ we must have $L' = D$.

Modules over a Lie algebra

Let L be a Lie algebra. A *module* over L (or an L-module) is a vector space

V with a bilinear map $L \times V \longrightarrow V$ denoted $(x,v) \mapsto xv$ such that $[x,y]v = x(y(v)) - y(x(v))$ for all $x,y \in L, v \in V$. To give an L-module structure on a vector space is the same as to give a Lie algebra homomorphism (or *representation*) $\phi : L \longrightarrow End(V)$. (We define ϕ by $\phi(x)(v) = xv$.)

An L-submodule of an L-module V is a vector subspace V' of V such that $x \in L, v \in V' \implies xv \in V'$. Then V' and V/V' are naturally L-modules. If V,V' are L-modules, a homomorphism of L-modules $f : L \longrightarrow L'$ is a linear map satisfying $f(xv) = xf(v)$ for all $x \in, v \in V$. For such f, the kernel and image of f are L-submodules of V,V' respectively.

An L-module V is said to be *irreducible* if $V \neq 0$ and there is no L-submodule of V other than 0 and V. An L-module V is said to be *completely reducible* if for any L-submodule V' of V there exists an L-submodule V'' of V such that $V = V' \oplus V''$.

If V is an irreducible L-module of finite dimension and $f : V \longrightarrow V$ is a linear map such that $xf(v) = f(xv)$ for all $x \in L, v \in V$ then $f = a1$ where $a \in k$. (*Schur's lemma*).
Indeed for some $a \in k$, we have $V' = \{v \in V | fv = av\} \neq 0$. By irreducibility we have $V' = V$.

The Casimir element. Let I' be an ideal of the semisimple Lie algebra L. Let V be an L-module such that the restriction of $L \longrightarrow End(V)$ to I' is injective. We show that $(,) : I' \times I' \longrightarrow k$ by $(x,y) = \mathrm{tr}(xy,V)$ is non-degenerate. Let $S = \{x \in I' | (x,y) = 0 \forall y \in I'\}$. This is an ideal of L. By Cartan's criterion (with S,V instead of L,V) we see that S is solvable hence $S = 0$.

Let $e_1, \dots, e_n$ be a basis of I' and let $e'_1, \dots, e'_n$ be the dual basis of I' with respect to $(,)$.

The *Casimir element* $c : V \longrightarrow V$ is defined by $c(v) = \sum_i e_i e'_i v$. It is independent of the choice of (e_i): assume that (f_i) is another basis of I' and (f'_i) is the dual basis. We have $f_i = \sum_j a_{ij} e_j$, $f'_i = \sum_j a'_{ij} e'_j$ where $a_{ij}, a'_{ij} \in k$. Now $\sum_i f_i f'_i v = \sum_{i,j,j'} a_{ij} a'_{ij'} e_j e'_{j'} v$. It suffices to show that $\sum_i a_{ij} a'_{ij'} = \delta_{jj'}$ that is $A^t A' = 1$. We have

$\delta_{cd} = (f'_c, f_d) = (\sum_i a'_{ci} e'_i, \sum_i a_{di} e_i)) = \sum_i a'_{ci} a_{di}$
hence $A' A^t = 1$ hence $A^t A' = 1$ as desired.

For any $x \in L, v \in V$ we have

(a) $xc(v) = c(xv)$.
Indeed, we must show that $\sum_i x e_i e'_i v = \sum_i e_i e'_i x v$. Write $[x, e_i] = \sum_j a_{ij} e_j$, $[x, e'_i] = \sum_j a'_{ij} e'_j$. We have

$$a_{ij} = ([x, e_i], e'_j) = -([e_i, x], e'_j) = -([x, e'_j], e_i) = -(\sum_h a'_{jh} e'_h, e_i) = -a'_{ji},$$

$$\begin{aligned}&\sum_i x e_i e'_i v - \sum_i e_i e'_i x v = \sum_i [x, e_i] e'_i v + \sum_i e_i [x, e'_i] v\\&= \sum_{i,j} a_{ij} e_j e'_i v + \sum_{i,j} a'_{ij} e_i e'_j v = \sum_{i,j} a_{ij} e_j e'_i v + \sum_{i,j} a'_{ji} e_j e'_i v\\&= \sum_{i,j} (a_{ij} + a'_{ji}) e_j e'_i v = 0.\end{aligned}$$

Assume now that V is irreducible and that $I' \neq 0$. We show that

(b) $c : V \longrightarrow V$ is a bijection.

Using (a) and Schur's lemma we see that $c = a1$ where a is a scalar. We have $a \dim V = \operatorname{tr}(c, V) = \sum_i \operatorname{tr}(e_i e'_i, V) = \sum_i (e_i, e'_i) = \dim I' \neq 0$. Hence $a \neq 0$ and (b) is proved.

Lemma. *Let L be a Lie algebra. Let M, N be L-modules. Then* $\operatorname{Hom}(V, W)$ *may be regarded as an L-module by $(xf)(v) = x(f(v)) - f(xv)$. (Here $f \in$* $\operatorname{Hom}(V, W), v \in V$*.)*

$$\begin{aligned}&([x, x']f)(v) = [x, x'](f(v)) - f([x, x']v)\\&= x(x'(f(v)) - x'(x(f(v)) - f(xx'v)) + f(x'xv),\end{aligned}$$

$$\begin{aligned}&(x(x'(f)))(v) - (x'(x(f)))(v) = x((x'f)(v)) - (x'f)(xv) - x'((xf)(v)) - (xf)(x'v)\\&= x(x'(fv) - f(x'v)) - x'(f(xv)) + f(x'xv) - x'(x(fv) - f(xv)) + x(f(x'v))\\&- f(xx'v) = x(x'(fv)) - x(f(x'v)) - x'(f(xv)) + f(x'xv) - x'(x(fv))\\&+ x'(f(xv)) + x(f(x'v)) - f(xx'v) = x(x'(fv)) - x'(x(fv)) - f(xx'v) + f(x'xv)\end{aligned}$$

hence $[x, x']f = x(x'(f)) - (x'(x(f))$.

Lemma. *Let L be a semisimple Lie algebra. Let $\phi : L \longrightarrow End(V)$ be a Lie algebra homomorphism, $\dim V < \infty$. Then $\operatorname{tr}(\phi(x), V) = 0$ for any $x \in L$. Hence if $\dim V = 1$ then $xV = 0$ for $x \in L$.*

We know already that $L = [L, L]$. Hence we may assume that $x = [x', x'']$, $x', x'' \in L$. Then $\operatorname{tr}\phi(x) = \operatorname{tr}(\phi(x')\phi(x'') - \phi(x'')\phi(x')) = 0$.

Lemma. *Let L be a semisimple Lie algebra. The following two conditions are equivalent:*

(a) Any finite dimensional L-module is completely reducible.

(b) Given a finite dimensional L-module V and a codimension 1 subspace W of V such that $xV \subset W$ for all $x \in L$, there exists a line $\mathcal{L}$ in V complementary to W such that $x\mathcal{L} = 0$ for all $x \in L$.

Clearly (a) $\implies$ (b). (We use the previous lemma.) Assume that (b) holds. Let M be a finite dimensional L-module and let N be an L-submodule, $N \neq 0$. Now $\mathrm{Hom}(M, N)$ is naturally an L-module. Let $V = \{f \in \mathrm{Hom}(M, N) | f|_N = \text{scalar}\}, W = \{f \in \mathrm{Hom}(M, N) | f|_N = 0\}$. Then W has codimension 1 in V and $xV \subset W$ for $x \in L$. By (b) there exists $u \in V$ such that $xu = 0$ for all $x \in L$ and $u|_N$ is a non-zero scalar c. We may assume that $c = 1$. Then u is a projection $M \to N$. The condition $xu = 0$ means that $x : M \to M$ commutes with $u : M \to M$. Hence the kernel of u is a complement of N in M which is x-stable for all $x \in L$. Hence (a) holds.

Lemma. *Let L be a semisimple Lie algebra. Let V be an L-module of finite dimension. Let W be a codimension 1 subspace W of V such that $LV \subset W$. Then there exists a line $\mathcal{L}$ in V complementary to W such that $L\mathcal{L} = 0$.*

Since $\dim(V/W) = 1$, L acts as 0 on the L-module V/W hence $LV \subset W$. Let $I = \{x \in L | xW = 0\}$ (an ideal of L) and let I' be an ideal of L such that $L = I \oplus I'$; then $I' \to End(W)$ is injective; hence $I' \to End(V)$ is also injective. We can form the Casimir element c^V relative to I', V and the Casimir element c^W relative to I', W. The form $(,)$ on I' used to define c^V coincides with the form $(,)$ on I' used to define c^W (since $LV \subset W$). Hence $c^V|_W = c^W$. Also, since $LV \subset W$ we have $c^V V \subset W$.

Assume first that the L-module W is irreducible. If $LW = 0$ then $xyV = 0$ for all $x, y \in L$ (since $xyV \subset xW = 0$) hence $[x, y]V = 0$ for all $x, y \in L$; but $L = [L, L]$ hence $xV = 0$ for all $x \in L$ and there is nothing to prove. If $LW \neq 0$, then $I \neq L$ hence $I' \neq 0$. Since W is irreducible, $c^W : W \to W$ is bijective. Since $c^V|_W = c^W$, $c^V V \subset W$, $\{v \in V | c^V v = 0\}$ is a line complementary to W. This line is an L-submodule since $c^V : V \to V$ commutes with any $x \in L$.

We now treat the general case by induction on $\dim V$. We may assume that the L-module W is not irreducible. Let T be an irreducible L-submodule of W. Let $V' = V/T, W' = W/T$. By the induction hypothesis there exists a line $\mathcal{L}'$ in V' complementary to W' and such that $L\mathcal{L}' = 0$. Let Z be the inverse image of $\mathcal{L}'$ in V. It is an L-module containing T with codimension 1 and $Z \cap W = T$ so that $LZ \subset T$. By the first part of the proof we can find a line $\mathcal{L}$ in Z complementary to T such that $L\mathcal{L} = 0$. If $\mathcal{L} \subset W$ then $\mathcal{L} \subset Z \cap W = T$ absurd. Thus, $\mathcal{L}$ is complementary to W in V.

Theorem (H. Weyl). *Let L be a semisimple Lie algebra. Let V be an L-module of finite dimension. Then V is completely reducible.*

We combine the previous two lemmas.

Derivation of an algebra.

Lemma. *Let V be a finite dimensional algebra over k. Let $x \in Der(V)$. Let $x = s + n$ be the Jordan decomposition of x in $End(V)$. Then $s \in Der(V), n \in Der(V)$.*

For $u, v \in V, a, b \in k$ and $N \geq 0$ we have (induction on N):
$(x - (a+b)1)^N(uv) = \sum_i \binom{n}{i}((x - a1)^{n-i}u)((x - b1)^i v)$.
Hence setting as above $V = \oplus_{\lambda \in k} V_\lambda$ we have $V_a V_b \subset V_{a+b}$. If $u \in V_a, v \in V_b$ then $uv \in V_{a+b}$ hence $s(uv) = (a+b)uv = (su)v + u(sv)$. Then $s(uv) = (su)v + u(sv)$ holds for all $u, v \in V$. Hence $s \in Der(V)$. Since s, x are commuting derivations, $x - s$ must be a derivation.

Abstract Jordan decomposition. Let L be a finite dimensional semisimple Lie algebra. We say that $x \in L$ is *semisimple* if $ad(x) : L \longrightarrow L$ is semisimple. We say that $x \in L$ is *nilpotent* if $ad(x) : L \longrightarrow L$ is nilpotent.

Let $x \in L$. Then $ad(x) : L \longrightarrow L$ has a Jordan decomposition $ad(x) = S + N$ as an endomorphism of L. By the previous lemma, S and N are derivations of L hence are of the form $S = ad(s), N = ad(n)$ for well defined $s, n \in L$. Thus s, n are semisimple/nilpotent. Since $ad[s, n] = [ad(s), ad(n)] = [S, N] = 0$ and $ad : L \longrightarrow End(L)$ is injective we see that $[s, n] = 0$. We say that s is the semisimple part of x and n is the nilpotent part of x. Assume that $x = s' + n'$ with s' semisimple, n' nilpotent, $[s', n'] = 0$. Let $S' = ad(s'), N' = ad(n')$. Then $S' + N'$ is the Jordan decomposition of $S + N$ in $End(L)$. Hence $S = S', N = N'$. By the injectivity of ad we have $s = s', n = n'$.

We say that $s + n$ is the *Jordan decomposition* of $x \in L$.

Proposition. *Let L be a semisimple Lie algebra. Assume that $L \subset End(V)$ (as a Lie algebra), $\dim V < \infty$. Let $x \in L$ and let $x = \sigma + \nu$ be its Jordan decomposition in $End(V)$. Then $\sigma \in L, \nu \in L$.*

Let N be the normalizer of L in $End(V)$. For any L-submodule W of V let $L_W = \{y \in End(V) | y(W) \subset W, \mathrm{tr}(y, W) = 0\}$. Since $L = [L, L]$ we have $L \subset L_W$. Let $L' = N \cap \cap_W L_W$. (A subalgebra of N containing L as an ideal.)

Let $A = ad(x) : End(V) \longrightarrow End(V)$. We have $AL \subset L$ hence $A_s(L) \subset L$ that is $ad(\sigma)(L) \subset L$ that is $\sigma \in N$. Similarly, $\sigma \in L_W$ hence $\sigma \in L'$. Since L' is an L-module, there exists by Weyl an L-submodule M of L' such that $L' = L \oplus M$. Since $L' \subset N$ we have $[L, L'] \subset L$ hence $[L, M] = 0$. Let W be any irreducible L-submodule of V. If $y \in M$ then $[L, y] = 0$ hence by Schur, y acts on W as a scalar. But $\mathrm{tr}(y, W) = 0$ since $y \in L_W$. Hence y acts on W as zero. By Weyl, V is a direct sum of irreducible submodules hence $y = 0$. Thus $M = 0, L' = L$. Since $\sigma \in L'$ we have $\sigma \in L$. Now $\nu = x - \sigma \in L$.

Proposition. *Let L be a semisimple Lie algebra. Let $\phi : L \longrightarrow End(V)$ be a Lie algebra homomorphism, $\dim V < \infty$. Let $x \in L$ and let $x = s + n$ be its Jordan decomposition. Then $\phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of the endomorphism $\phi(x)$.*

Let $L' = \phi(L)$. This is a semisimple Lie algebra (it is a quotient of a semisimple

Lie algebra). We show that $\phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x)$ in L'. Since $[\phi(s), \phi(n)] = 0$ it suffices to show that $ad\phi(s) : L' \longrightarrow L'$ is semisimple and $ad\phi(n) : L' \longrightarrow L'$ is nilpotent. But $ad\phi(s) : L' \longrightarrow L'$ is induced by $ad(s) : L \longrightarrow L$ which is semisimple hence it is itself semisimple. Similarly $ad\phi(n) : L' \longrightarrow L'$ is nilpotent. We are reduced to the case where $L \subset End(V)$ and ϕ is the imbedding. Let $x = \sigma + \nu$ be the Jordan decomposition of x in $End(V)$. We must show that $\sigma = s, \nu = n$. By the previous result, $\sigma \in L, \nu \in L$. Also $ad(\sigma) : End(V) \longrightarrow End(V)$ is semisimple, $ad(\nu) : End(V) \longrightarrow End(V)$ is nilpotent hence their restrictions $ad(\sigma) : L \longrightarrow L, ad(\nu) : L \longrightarrow L$ are semisimple, nilpotent. Hence $\sigma \in L$ is semisimple, $\nu \in L$ is nilpotent. Since $[\sigma, \nu] = 0$, $x = \sigma + \nu$ is the Jordan decomposition of x in L.