LIE ALGEBRAS 1

Definition of a Lie algebra

k is a fixed field. Let L be a k-vector space (or vector space). We say that L is a *k-algebra* (or algebra) if we are given a bilinear map $L \times L \longrightarrow L$ denoted $(x, y) \mapsto xy$. If L, L' are algebras, an algebra *homomorphism* $f : L \longrightarrow L'$ is a linear map such that $f(xy) = f(x)f(y)$ for all x, y. We say that f is an algebra *isomorphism* if it is an algebra homomorphism and a vector space isomorphism. In this case $f^{-1} : L' \longrightarrow L$ is an algebra isomorphism. If L is an algebra, a subset $L' \subset L$ is a *subalgebra* if it is a vector subspace and $x \in L', y \in L' \implies xy \in L'$. Then L' is itself an algebra in an obvious way. If L is an algebra, a subset $I \subset L$ is an *ideal* if it is a vector subspace; $x \in L, y \in I \implies xy \in I$; $x \in I, y \in L \implies xy \in I$. Then I is a subalgebra. Moreover, the quotient vector space V/I is an algebra with multiplication $(x + I)(y + I = xy + I$. (Check that this is well defined.) The canonical map $L \longrightarrow L/I$ is an algebra homomorphism with kernel I. Conversely, if $f : L \longrightarrow L'$ is any algebra homomorphism then $\ker(f) = f^{-1}(0)$ is an ideal of L.

We say that the algebra L is a Lie algebra if

(a) $xx = 0$ for all x;

(b) $x(yz) + y(zx) + z(xy) = 0$ for all x, y, z. (*Jacobi identity*).

Note that (a) implies

(a') $xy = -yx$ for all x, y.

Indeed $xy + yx = (x + y)(x + y) - xx - yy = 0$.

Traditionally in a Lie algebra one writes $[x, y]$ instead of xy and one calls $[x, y]$ the *bracket*.

Example (a). Let A be an algebra. Assume that $(xy)z = x(yz)$, that is, the algebra A is associative. Define a new algebra structure on A by $[x, y] = xy - yx$. This makes A into a Lie algebra. We check the Jacobi identity:

$$\begin{aligned}&[x, [y, z]] + [y, [z, x]] + [z, [x, y]] \\ &= x(yz - zy) - (yz - zy)x + y(zx - xz) - (zx - xz)y + z(xy - yx) - (xy - yx)z = 0. \blacksquare\end{aligned}$$

Example (b). Let V be a vector space. Let $End(V)$ be the vector space of endomorphisms of V (linear maps $V \longrightarrow V$). This is an associative algebra where the product xy is the composition of endomorphisms $(xy)(v) = x(y(v))$ for $v \in V$. By example (a), $End(V)$ is a Lie algebra with bracket $[x, y] = xy - yx$. This Lie algebra is also denoted by $\mathfrak{gl}(V)$.

Example (c). Let V be as in (b). Assume that $\dim(V) < \infty$. Let $L = \{x \in End(V) | \mathrm{tr}(x) = 0\}$. This is a (Lie) subalgebra of $End(V)$. Indeed, if $x, y \in End(V) = \mathfrak{gl}(V)$ then $\mathrm{tr}[x, y] = \mathrm{tr}(xy - yx) = 0$ hence $[x, y] \in L$. Thus L is even an ideal of $\mathfrak{gl}(V)$. The Lie algebra L is also denoted by $\mathfrak{sl}(V)$.

Example (d). Let V be as in (b). Let $(,) : V \times V \longrightarrow k$ be a bilinear map. Let $L = \{x \in End(V) | (x(v), v') + (v, x(v')) = 0 \forall v, v' \in V\}$. We show that L is a Lie subalgebra of $\mathfrak{gl}(V)$. Indeed, if $x, y \in L$ and $v, v' \in V$ we have

$$\begin{aligned}&([x,y](v),v') + (v,[x,y](v'))\\&= (x(y(v)),v') - (y(x(v),v') + (v,x(y(v'))) - (v,y(x(v'))) = 0.\end{aligned}$$

Example (e). Let V be as in (c). Assume that we are given a sequence of vector subspaces $0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n = V$, $\dim V_i = i$. Let

$\mathfrak{t} = \{x \in End(V) | x(V_i) \subset V_i \text{ for } i \in [0, n]\}$

$\mathfrak{n} = \{x \in End(V) | x(V_i) \subset V_{i-1} \text{ for } i \in [1, n]\}$.

Then $\mathfrak{t}$ is a Lie subalgebra of $End(V)$ and $\mathfrak{n}$ is a Lie subalgebra of $\mathfrak{t}$.

Example (f). Let A be an algebra. A *derivation* of A is a linear map $d : A \longrightarrow A$ such that $d(xy) = d(x)y + xd(y) = 0$ for all x, y. (Leibniz rule). Let $Der(A)$ be the set of all derivations of A. This is a vector subspace of $End(A)$. We show that $Der(A)$ is a Lie subalgebra of $End(A)$. Indeed, let $d, d' \in Der(A)$ and $x, y \in A$. We have

$$\begin{aligned}&[d,d'](xy) = d(d'(xy)) - d'(d(xy)) = d(d'(x)y + xd'(y)) - d'(d(x)y + xd(y))\\&= d(d'(x))y + d'(x)d(y) + d(x)d'(y) + xd(d'(y))\\&- d'(d(x))y - d(x)d'(y) - d'(x)d(y) - xd'(d(y))\\&= d(d'(x))y - d'(d(x))y + xd(d'(y)) - xd'(d(y)) = [d,d'](x)y + x[d,d'](y)\end{aligned}$$

hence $[d, d'] \in Der(A)$.

Example (g). Let L be a Lie algebra. By (f), $Der(L)$ is again a Lie algebra. If $x \in L$ then $ad(x) : L \longrightarrow L$, $y \mapsto [x, y]$ is a derivation of L, that is, $ad(x)[y, y'] = [ad(x)(y), y'] + [y, ad(x)(y')]$ for all y, y'. (Equivalently, $[x, [y, y']] = [[x, y], y'] + [y, [x, y']]$ which follows from Jacobi's identity.) The map $x \mapsto ad(x)$ is a Lie algebra homomorphism $L \longrightarrow Der(L)$, that is, it is linear (obviously) and, for $x, x', y \in L$ we have $ad([x, x'])(y) = ad(x)(ad(x')y) - ad(x')(ad(x)y)$. (Equivalently, $[[x, x'], y] = [x, [x', y]] - [x', [x, y]]$, which again follows from Jacobi's identity.)

Example (h). Let V be as in (a). Define an algebra structure on A on V by $xy = 0$ for all x, y. This is a Lie algebra. (An *abelian* Lie algebra.)

Example (i). Let L be a Lie algebra. Let $[L, L]$ be the vector subspace of L spanned by $\{[x, y] | x \in L, y \in L\}$. This is an ideal of L. One calls $[L, L]$ the *derived* algebra of L.

Example (j). Let L be a Lie algebra. Let $Z_L = \{x \in L | [x, y] = 0 \forall y \in L\}$. This is an (abelian) subalgebra of L called the *centre* of L.

Example (k). Let L be a Lie algebra. Let K be a vector subspace of L. Then $N_L(K) = \{x \in L | [x, y] \in K \forall y \in K\}$ is a Lie subalgebra of L. Indeed, if $x, x' \in N_L(K)$ and $y \in K$ we have $[[x, x'], y] = [[x, y], x'] + [x, [x', y]] \in [x', K] + [x, K] \subset K + K \subset K$. One calls $N_L(K)$ the normalizer of K in L.

Example (l). Let $L = \mathfrak{sl}(V)$ where V is a 2-dimensional vector space. Identify L with the set of 2×2 matrices with trace 0. A basis is given by

$$e = \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}, f = \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}, g = \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}.$$

We have $[e, f] = h, [h, e] = 2e, [h, f] = -2f$.

A Lie algebra L is said to be *simple* if it is not abelian and L has no ideals other than $0, L$.

We show that L in Ex.(l) is simple. (We assume $2^{-1} \in k$.) Let I be an ideal other than 0. We show that $I = L$. Let $x = ae + bf + ch \in I - \{0\}$. Now

$[e, [e, x]] = [e, bh - 2e] = -2be \in I, [f, [f, x]] = [f, ah + 2cf] = 2af \in I$.

If $a \neq 0$ we deduce $f \in I$ hence $h = [e, f] \in I$ and $e = -2^{-1}[e, h] \in I$; thus $I = L$. Similarly if $b \neq 0$ then $I = L$. Thus we may assume that $a = b = 0$. Then $h \in I$ and $e = -2^{-1}[e, h] \in I, f = 2^{-1}[f, h] \in I$ so that $I = L$.

Let A be an algebra. An automorphism of A is an algebra isomorphism $L \xrightarrow{\sim} L$.

Lemma. *Assume that k has characteristic 0. If $d : A \to A$ is a derivation such that $d^n = 0$ for some $n > 0$ then $e^d = \sum_{n \in \mathbf{N}} \frac{d^n}{n!} : A \to A$ is an automorphism of A.*

From the definition of a derivation we get

$\frac{d^s}{s!}(xy) = \sum_{p+q=s} \frac{d^p}{p!}(x) \frac{d^q}{q!}(y)$

for all $s \geq 0$. (Use induction on s.) For $x, y \in A$ we have

$$e^d(x)e^d(y) = \sum_p \frac{d^p}{p!}(x) \sum_q \frac{d^q}{q!}(y) = \sum_s \sum_{p+q=s} \frac{d^p}{p!}(x) \frac{d^q}{q!}(y) = \sum_s \frac{d^s}{s!}(xy) = e^d(xy).$$

Thus e^d is an algebra homomorphism. We have $e^d e^{-d} = e^{-d} e^d = 1$. Thus e^d is a vector space isomorphism. The lemma is proved.

Solvable, nilpotent Lie algebras

Let L be a Lie algebra. If X, X' are subsets of L, let $[X, X']$ be the subspace spanned by $\{[x, x'] | x \in X, x' \in X'\}$. If I, I' are ideals of L then $[I, I']$ is an ideal of L. Hence

$L^{(0)} = L, L^{(1)} = [L, L], L^{(2)} = [L^{(1)}, L^{(1)}], L^{(3)} = [L^{(2)}, L^{(2)}], \dots$

are ideals of L and $L^{(0)} \supset L^{(1)} \supset L^{(2)} \supset \dots$.

L is said to be *solvable* if $L^{(n)} = 0$ for some $n \geq 1$. For example $\mathfrak{t}$ is a solvable Lie algebra.

L solvable, L' subalgebra $\implies$ L' is solvable.

L solvable, I ideal $\implies$ L/I is solvable.

L Lie algebra, I ideal such that I and L/I are solvable $\implies$ L is solvable.

L Lie algebra, I, J ideals such that I and J are solvable $\implies$ $I+J$ solvable ideal.

It follows that, if $\dim L < \infty$ there exists a unique solvable ideal of L which contains any solvable ideal of L. This is call the radical of L. Notation: $rad(L)$. We say that L is *semisimple* if $rad(L) = 0$. In any case $L/rad(L)$ is semisimple.

For a Lie algebra L,

$L^0 = L, L^1 = [L, L], L^2 = [L, L^1], L^3 = [L, L^2], \dots$
are ideals of L and $L^0 \supset L^1 \supset L^2 \supset \dots$.

L is said to be *nilpotent* if $L^n = 0$ for some $n \geq 1$. For example $\mathfrak{n}$ is a nilpotent Lie algebra. But $\mathfrak{t}$ is not nilpotent although it is solvable. Clearly, $L^{(i)} \subset L^i$. Hence L nilpotent $\implies$ L solvable.

L nilpotent, L' subalgebra $\implies$ L' is nilpotent.

L nilpotent, I ideal $\implies$ L/I is nilpotent.

L Lie algebra, $L/Z(L)$ nilpotent $\implies$ L is nilpotent.

L nilpotent, $L \neq 0 \implies Z(L) \neq 0$.

Lemma. *Let V be a vector space,* $\dim V < \infty$*. Let $x \in \mathfrak{gl}(V)$ be nilpotent. Then $ad(x) : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ is nilpotent.*

For $y \in End(V)$ set $A(y) = xy, B(y) = yx$. Then $ad(x)y = (A - B)(y)$. By assumption, A, B are nilpotent endomorphisms of $End(V)$. They commute hence $A - B$ is a nilpotent endomorphism of $End(V)$. The lemma is proved.

Proposition. *Let L be a Lie algebra,* $\dim(L) < \infty$*, let V be a vector space,* $0 < \dim V < \infty$ *and let $\rho : L \to End(V)$ be a Lie algebra homomorphism such that $\rho(l) : V \to V$ is nilpotent for all $l \in L$. Then there exists $v \in V - \{0\}$ such that $\rho(l)(v) = 0$ for all $l \in L$.*

Induction on $\dim(L)$. If ρ is not injective, then $\dim(\rho(L)) < \dim(L)$ and the proposition is applicable to $\rho(L)$ instead of L. Hence it holds for L itself. Thus we may assume that ρ is injective or that L is a Lie subalgebra of $End(V)$.

We may assume that $L \neq 0$. Let L' be a Lie subalgebra of L with $L' \neq L$ of maximal possible dimension. By the lemma, if $x \in L'$, then $ad(x) : L \to L$ is nilpotent. Hence it induces a nilpotent linear map $L/L' \to L/L'$. By the induction hypothesis applied to $L', L/L'$ instead of L, V, there exists $x \in L - L'$ such that $[x', x] \in L'$ for all $x' \in L'$. Thus, L' is properly contained in $N_L(L')$. By the choice of L' we have $N_L(L') = L$ hence L' is an ideal of L. Any line in L/L' is a one dimensional subalgebra; its inverse image in L contains L' properly hence is equal to L, by the choice of L'. Thus, $\dim(L/L') = 1$. Hence, if $z \in L - L'$ we have $L = L' \oplus kz$. By the induction hypothesis, $W = \{v \in V | xv = 0 \forall x \in L'\} \neq 0$. Since L' is an ideal, W is stable under L. In particular $z(W) = W$. Since z is a nilpotent endomorphism, we can find $v \in W - \{0\}$ such that $zv = 0$. Since $L'v = 0$, we have $Lv = 0$.

Theorem(Engel). *Let L be a Lie algebra,* $\dim(L) < \infty$*. Assume that $ad(x) : L \longrightarrow L$ is nilpotent for any $x \in L$. Then L is nilpotent.*

Induction on $\dim(L)$. We can assume that $L \neq 0$. By the proposition (with $V = L$) there exists $x \in L - \{0\}$ such that $[L, x] = 0$ that is, $Z(L) \neq 0$. Now the induction hypothesis applies to $L/Z(L)$. Hence $L/Z(L)$ is nilpotent. Hence L is nilpotent.

Remark. Conversely, if L is a nilpotent Lie algebra, then $ad(x) : L \longrightarrow L$ is nilpotent for any $x \in L$. Indeed, clearly $ad(x)L^i \subset L^{i+1}$ hence $ad(x)^n L \subset L^n$ for any n. Now use $L^n = 0$ for large n.

Corollary). *Let L, V, ρ be as in the proposition. Then there exists a sequence of vector subspaces $0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n = V$,* $\dim V_i = i$ *such that $\rho(x)V_i \subset V_{i-1}$ for $i \in [1, n]$.*

Induction on $\dim V$. By the proposition, we can find $v \in V - \{0\}$ such that $\rho(x)v = 0$ for all $x \in L$. Set $V_1 = kv$. Let $W = V/V_1$. If $W = 0$, we are done. If $W \neq 0$ we apply the induction hypothesis to L, W instead of L, V. We find a sequence of vector subspaces $0 = W_0 \subset W_1 \subset W_2 \subset \ldots \subset W_{n-1} = W$, $\dim W_i = i$ such that $\rho(x)W_i \subset W_{i-1}$ for $i \in [1, n-1]$. Let V_i be the inverse image of W_{i-1} under $V \longrightarrow V$ $(i \in [2, n])$. Then $V_1, V_2, \ldots$ have the required properties.

Lemma. *Let L be a nilpotent Lie algebra,* $\dim(L) < \infty$*. Let I be an ideal of L such that $I \neq 0$. Then $I \cap Z(L) \neq 0$.*

Applying the proposition to L, I (instead of L, V) we see that there exists $x \in I$ such that $[L, x] = 0$. Then $x \in I \cap Z(L)$.